E. FINANCIAL APPLICATIONS

E.I. Life time value

Economic measure of the potential worth of a customer.

E.II. Default, prepayment and credit lifetime

Copulas can be used to model the dependence between default and prepayment events.

E.III. Measuring the risk of credit portfolios

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E.I. Life time value

The Life Time Value (LTV) is an economic measure of the potential worth of a customer.

One of the aim of such a measure is to determine the length of time T a customer will remain active and profitable.

The LTV of an individual corresponds to

$$LTV(t) = \sum_k B(t,t_k)R(t_k)S(t_k)$$

where $t_1, t_2, ..$ correspond to future dates (yearly payments for examples), $B(t,t_k)$ is the discount factor, $R(t_k)$ is the payoff (profitability) function, and $S(t_k)$ the survival function.

LTV is a useful tool in terms of segmentation and customer relationship management (CRM).

A bank might use LTV to decide how much to spend on keeping customers.

A bank might use LTV to decide to link the contract rate of a mortgage with the LTV of a customer.

Note that LTV between customers might be correlated (husband-wife, parents-children), and independence should be assumed with care.

E.II. Default, prepayment and credit lifetime

Modeling mortgage terminations is essential for the bank to value correctly its market price.

Two risks should be modeled: default risk and prepayment risk.

The credit lifetime is linked to these events:

- either the counterparty default at survival time T_D

- or a prepayment occurs at survival time T_P

- or the credit goes to maturity T_M

By postulating parametric distributions for the survival functions, and a copula to link the two, we can easily model the interdependence between default and prepayment.

One usually observes that default and prepayment are negatively dependent once the model is estimated on credit data.

E.III. Measuring the risk of credit portfolios

One of the main issue concerning credit risk is without doubt the modelling of joint default distribution.

Copulas are a suitable tool for such a problem.

Indeed, a default is generally described by a survival function $S(t) = P(T > t)$, which indicates the probability that a security will attain age t.

The survival time T is often called the *time-until-default*.

We consider here the problem of the risk measure of a credit portfolio and the economic capital allocation:

Quote from Basel Committee on Banking Supervision:

The estimated economic capital needed to support a bank's credit risk exposure is generally referred to as its required economic capital for credit risk. The process for determining this amount is analogous to value at risk methods used in allocating economic capital against market risks. Specifically, the economic capital for credit risk is determined so that the estimated probability of unexpected credit loss exhausting economic capital is less than some target insolvency rate.

Capital allocation systems generally assume that it is the role of reserving policies to cover expected credit losses, while it is that of economic capital to cover unexpected credit losses. Thus, required economic capital is the additional amount of capital necessary to achieve the target insolvency rate, over and above that needed for coverage of expected loss.

Let L denote the credit loss variable with distribution F.

The expected loss is then equal to

$$EL = E(L),$$

whereas the unexpected loss is given by:

$$UL = TL - E(L)$$

with TL being a target loss given by a risk measure such as the Value at Risk (VaR)

$$TL = VaR(\alpha) = F^{-1}(\alpha)$$

or the expected shortfall (ES):

$$TL = ES(\alpha) = E[L | L > VaR(\alpha)]$$

at a specified confidence level α.

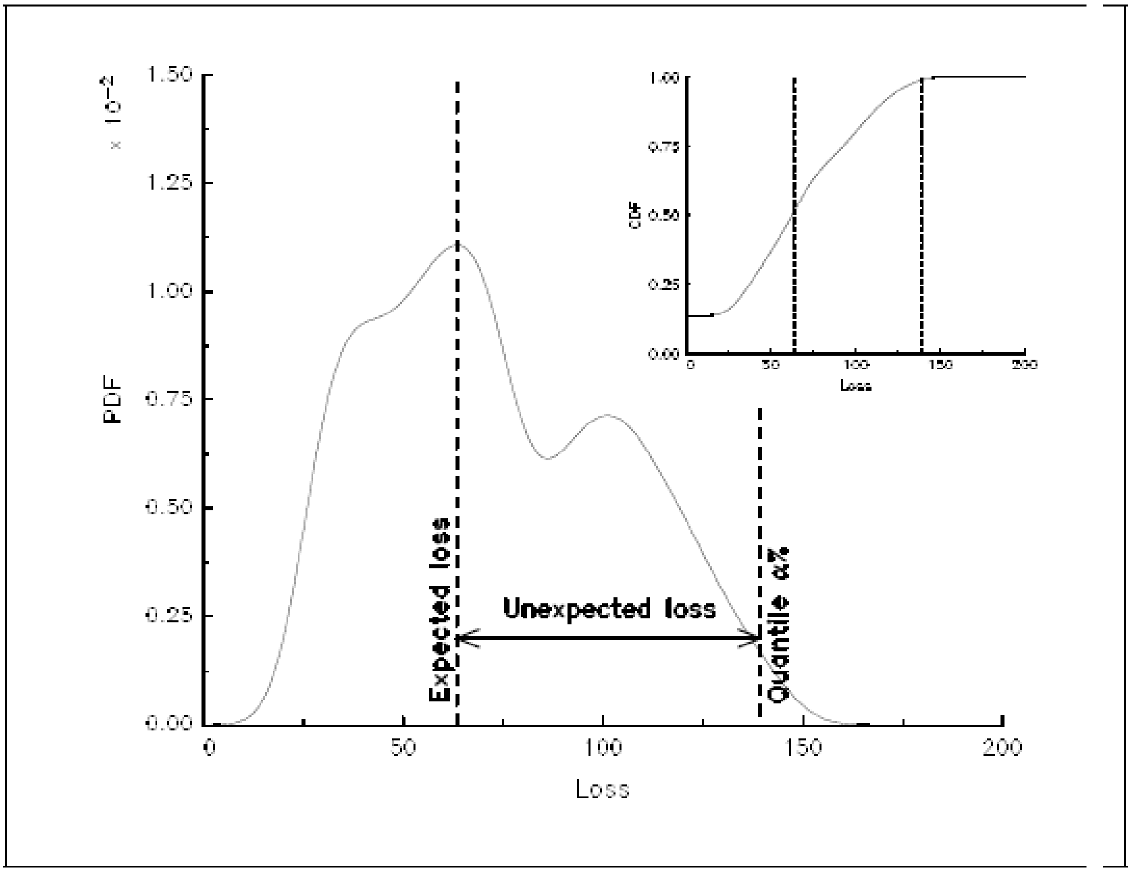

Figure 14: Density of the loss and allocated economic capital

Here $1-\alpha$ can be interpreted as the « target insolvency rate », and depends on the rating target of the bank.

The values of α which are generally used are the following ones.

Rating target	BBB	A	AA	AAA
α	99.75%	99.9%	99.95%	99.97%

The unexpected loss is then used to compute the Capital at Risk or the economic capital after application of some multiplicative coefficients.

Note that the notion of a credit event may vary. The Basel Commitee on Banking Supervision provides two definitions:

- In the default mode (DM) paradigm, "a credit loss arises only if a borrower defaults within the planning horizon".

- In the mark-to-market (MtM) paradigm, "a credit loss can arise in response to deterioration in an asset's credit quality short of default".

Available softwares on the market distinguish themselves:

- by the assumption on the copula: normal (CreditMetrics) or frailty (CreditRisk+),

- by the definition of a credit event: MtM (CreditMetrics) or DM (CreditRisk+),

- by the assumption on the Loss-Given-Default or Recovery Rate: beta distribution or constant,…

E.IV. Pricing Credit Derivatives

Credit derivatives are defined as "derivative securities whose payoff depends on the credit quality of a certain issuer. This credit quality can be measured by the credit rating of the issuer or by the yield spread of his bonds over the yield of a comparable default-free bond".

The default-free discount bond (zero coupon bond) price with maturity T is given at time t by

$$B(t,T) = E_t^Q\left[\exp\left(-\int_t^T r_s ds\right)\right]$$

where Q denotes the risk-neutral probability and r is the instantaneous risk-free rate.

When the interest rate and the survival time of the defaultable discount bond are independent, the price of the defaultable discount bond $\overline{B}(t,T)$ is given by:

$$\begin{aligned}\overline{B}(t,T) &= R \times B(t,T) \\ &\quad + (1-R)B(t,T)S(T)\end{aligned}$$

where R is the constant recovery rate.

If $S(T) = 1$, then $\overline{B}(t,T) = B(t,T)$.

If we set $R = 0$ (loose everything when default), the credit spread is such that

$$CS(t,T) = -\frac{1}{T-t} \ln(\overline{B}(t,T)/B(t,T))$$
$$= \frac{1}{T-t} \Lambda(T)$$

where $\Lambda(T)$ is the hazard function.

The hazard function of the defaultable bond plays an important role for its pricing.

In the case of the pricing of basket credit derivatives the important thing is the joint survival function, and not only the univariate characterization of the survival times.

A. European Default Digital Put option

It pays off 1 at maturity if there has been a default at some time before maturity on a basket of N obligors.

Under independence between the interest rate and the first to default time τ, the price is:

$$DDP(t,T)$$
$$= E_t^Q\left[\exp\left(-\int_t^T r_s ds\right) I(\tau < T)\right]$$
$$= (1 - S_\tau(T))B(t,T)$$
$$= (1 - \overline{C}(S_1(T),\dots,S_N(T)))B(t,T)$$

The price is influenced by the individual survival functions S_j and the dependence between the survival times through $\overline{C}$.

Example:

We consider N defaultable securities with exponential survival times ($\lambda_j = 5\%$).

We assume that the matrix of parameters of the Gaussian copula is of the following form:

$$\begin{pmatrix} 1 & \rho & \cdots & \rho \\ & \ddots & \ddots & \vdots \\ & & \ddots & \rho \\ & & & 1 \end{pmatrix}$$

The interest rate r is constant and is equal to 5%.

The Gaussian copula is nowadays considered as the benchmark copula in the credit derivatives industry, as Black-Scholes formula is for standard option pricing.

We may then compare the DDP premium for various maturities T and parameter values ρ in the case of $N = 2$.

We verify that the higher the correlation the lower the premium.

The relation between the premium and the maturity T is more complex because there are two effects: $1 - S_\tau(T)$ is an increasing function of T, but $B(t,T)$ is a decreasing function of T.

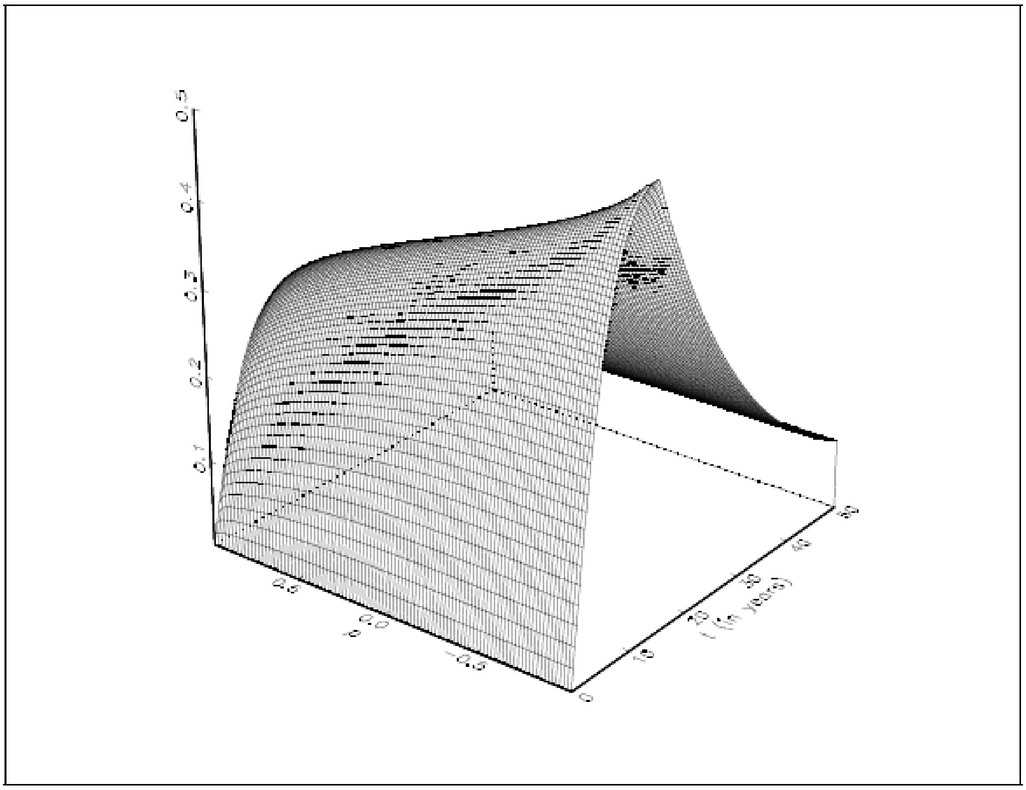

Figure 17: Influence of the parameter ρ on the DDP premium

We may also have a look at the impact of the number of defaultable securities on the premium.

It increases with the number of securities.

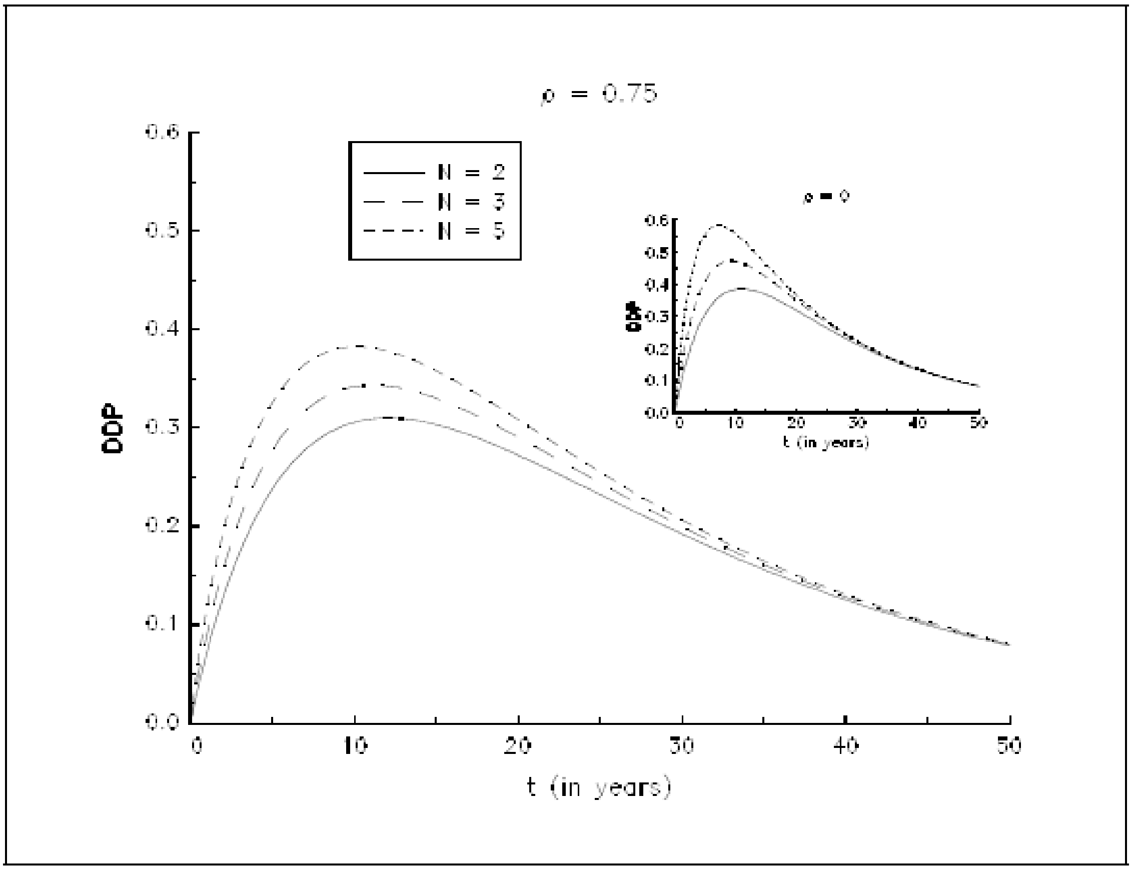

Figure 18: Influence of the number of defaultable securities on the DDP premium

B. Credit Default Swap

The product has the following characteristics:

- there are two counterparties *A* (the default protection seller) and *B* (the buyer) and a bond issuer (or a reference security) *C*;
- the counterparty *B* pays to *A* a fixed leg (or a swap rate) *l* at time T_m ($m = 1,\dots,M$) *until default or maturity* of the CDS;
- *B* receives at default a payment $C(T_m)$ from *A* if the bond issuer *C* defaults;

The payment is generally the difference between the recovery value and the notional value of the bond.

Usually if A defaults too it is assumed that B receives nothing, i.e. the recovery rate $R(T_m) = 0$ on the payment $C(T_m)$.

Then by invoking no arbitrage argument, namely that the expected present value of the payment in case of default should be equal to the expected present value of the periodic payments, it can be shown that the swap rate should be equal to:

$$l = \frac{\sum_{m=1}^{M} B(t,T_m)C(T_m)\left[S(T_{m-1},T_m) - S(T_m,T_m)\right]}{\sum_{m=1}^{M} B(t,T_m)S(T_{m-1},T_{m-1})}$$

where S is the bivariate survival function of the survival times of A and C.

Example:

The annual payment is $C(T_m) = 1$; the survival times are exponentially distributed with hazard rate $\lambda_A = \lambda_C = 5\%$. The interest rate r is assumed constant.

We may then compare the different values of the leg l in terms of maturities and correlation of the Gaussian copula.

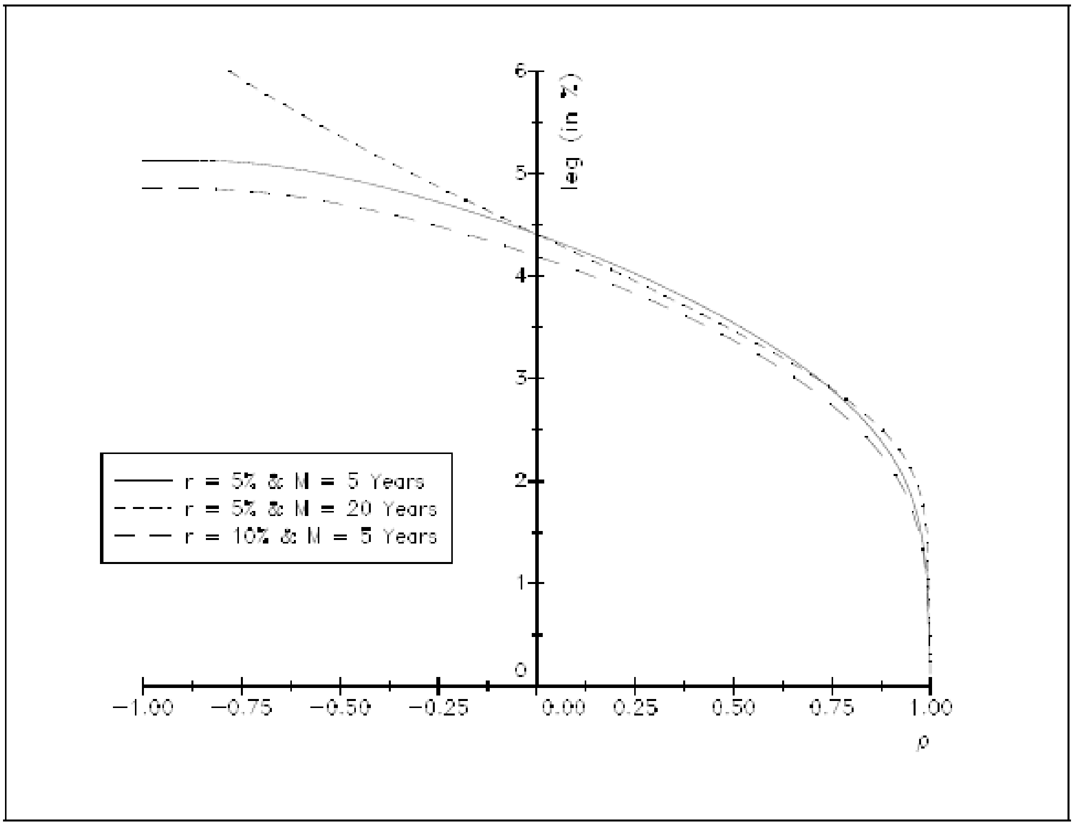

Figure 19: Value of the CDS leg ℓ with $R(t_m) = 0$

C. First-to-Default

A First-to-Default is a contingent claim that pays at the first of N credit events an amount $\varpi(\tau)$,

where $\tau = \min(T_1,\ldots,T_N)$.

The copula approach seems to be particularly adequate to price this type of asset (cf. competing risk models).

Again the price *FtoD* is given by the expected discounted value of the paid amount:

$$FtoD(t,T)$$

$$= E_t^Q\left[\varpi(\tau)\exp\left(-\int_t^T r_s ds\right)I[\tau < T]\right]$$

Example :

We consider N defaultable securities with exponential survival times.

We assume that the matrix of parameters of the Gaussian copula is of the following form:

$$\begin{pmatrix} 1 & \rho & \cdots & \rho \\ & \ddots & \ddots & \vdots \\ & & \ddots & \rho \\ & & & 1 \end{pmatrix}$$

The interest rate r is constant and is equal to 5%.

We may then compare the *FtoD* premium for various maturities T, parameter values ρ, and number of obligors N.

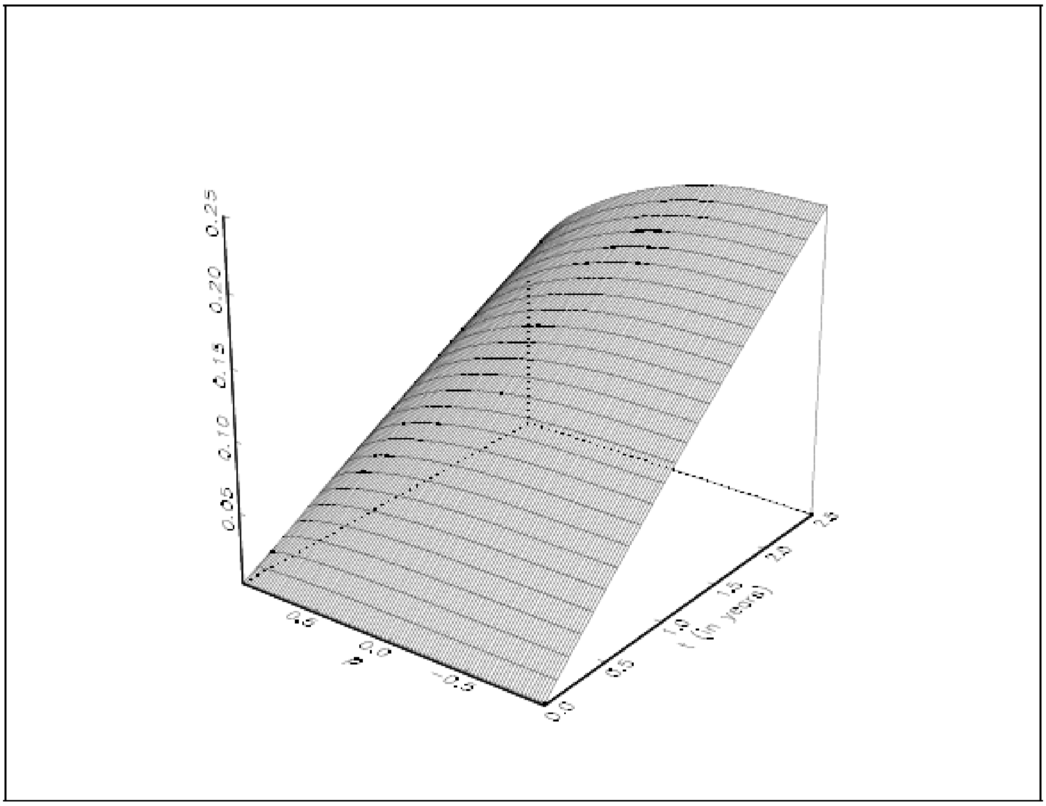

Figure 21: Value of the FtoD premium ($N = 2$, $\lambda_1 = 5\%$ and $\lambda_2 = 5\%$)

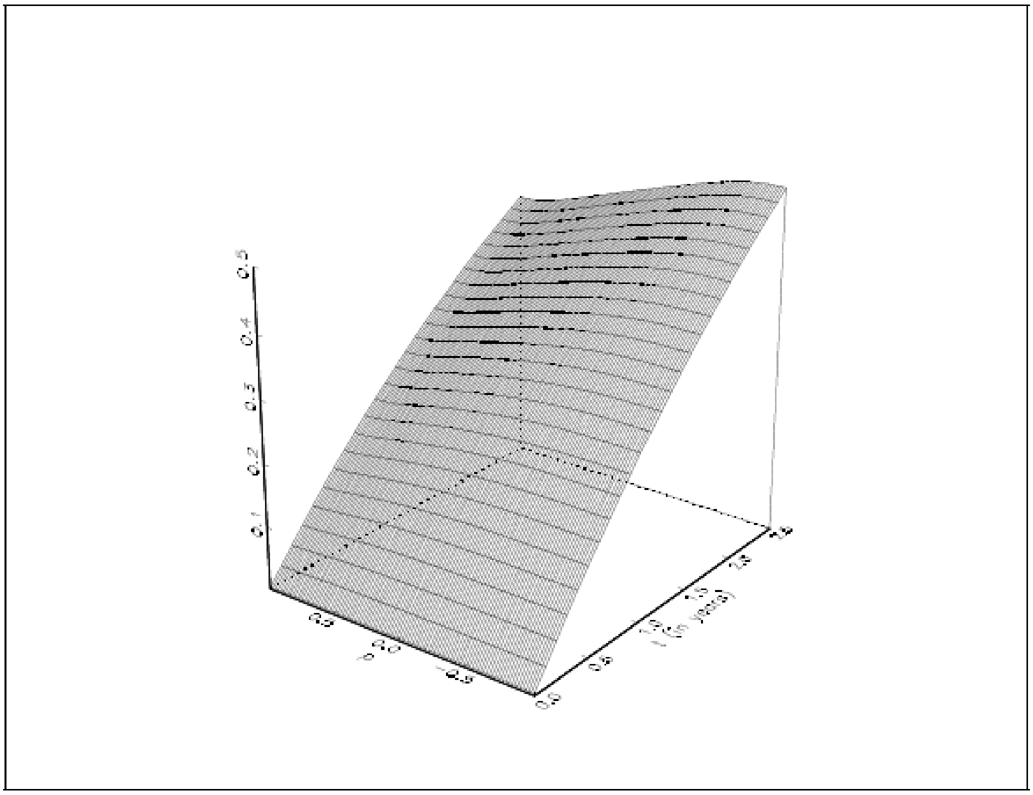

Figure 22: Value of the FtoD premium ($N = 2$, $\lambda_1 = 5\%$ and $\lambda_2 = 25\%$)

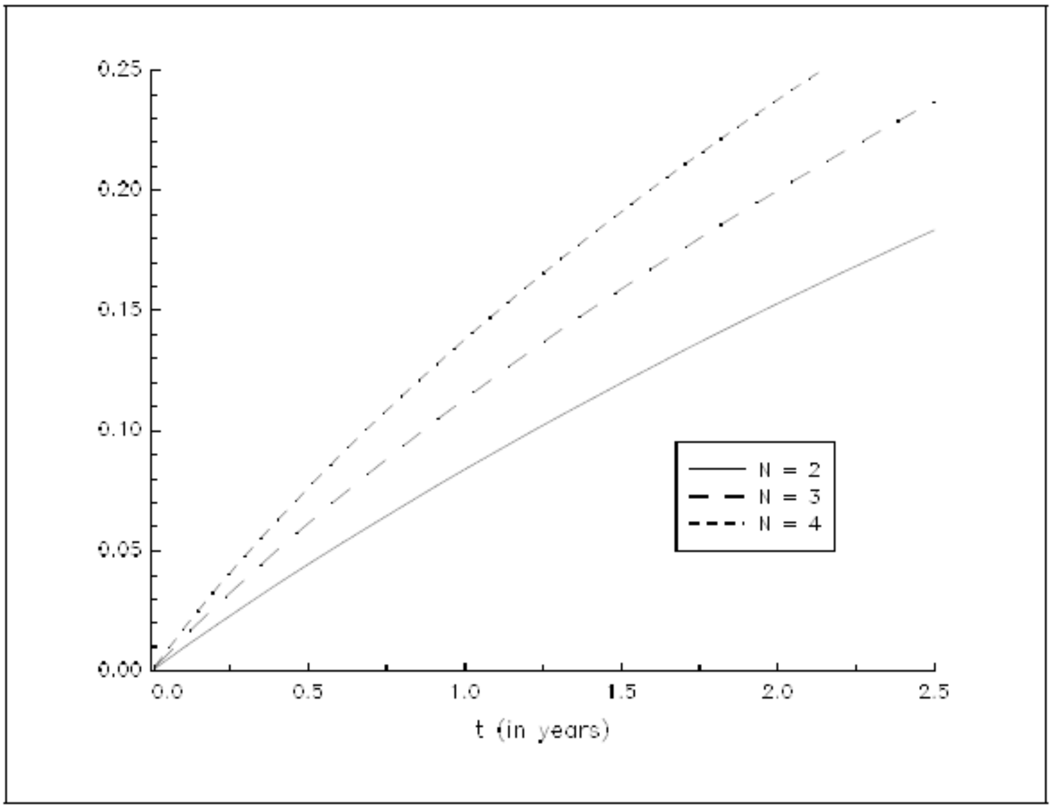

Figure 23: Value of the FtoD premium for $\rho = 0.5$

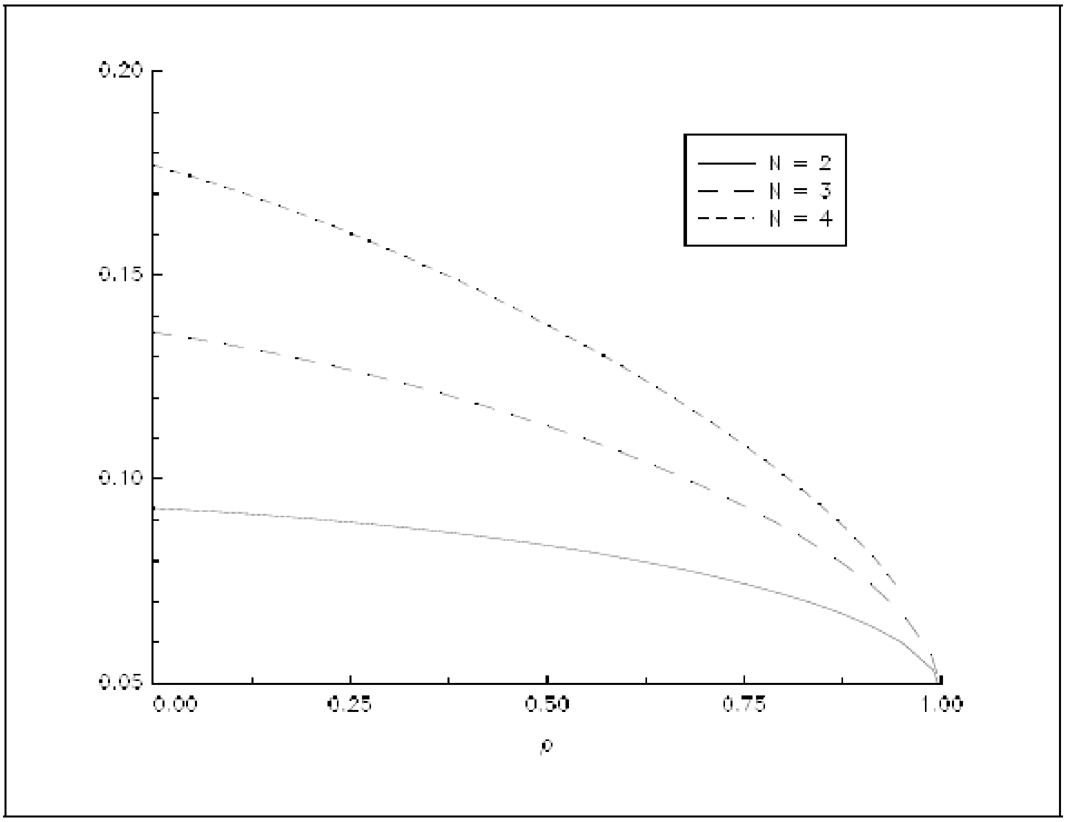

Figure 24: Value of the FtoD premium for $t = 1$

E.V. Conclusions

The above examples show the importance of the dependence function in pricing credit derivatives.

The given illustrations exhibit a monotone dependence.

When the copula is neither PQD nor NQD, its impact on prices could be further complex.

Here, we have only looked at simple credit derivatives.

In the case of more complicated products, the price can be critically dependent on the survival copula. This is for example the case of CDO (collateralized debt obligations).