Master in Wealth Management - GSEM
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Practical 2: GJR-copula **Quantitative Risk Management**

Introduction

The aim of this practical is to model stock returns by taking into account heteroskedasticity as well as the non-normality of innovations with a GJR model, and the possibly nontrivial cross-sectional dependence via a copula model. Specifically, we assume that the stock log-returns follow:

$$x_{i,t} = \mu_i + \varepsilon_{i,t}, \quad i = 1, \dots, D, \quad t = 1, \dots, T \tag{1}$$

where $\varepsilon_{i,t} = \sigma_{i,t} z_{i,t}$, μ_i is the mean, $\sigma_{i,t}^2$ is the time-t conditional variance, and $z_{i,t}$ is an innovation. The variance follows the GJR-GARCH(1,1,1) model given by:

$$\sigma_{i,t}^2 = \omega_i + (\alpha_i + \gamma_i \mathbf{1}_{\{\varepsilon_{i,t-1} < 0\}}) \varepsilon_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2, \tag{2}$$

where the indicator function $\mathbf{1}_{\{x\}}$ equals to 1 if condition x is satisfied and 0 otherwise. The model has the following constraints to satisfy the stationarity and positivity:

for each stock i,

- $\omega_i > 0$
- $\alpha_i, \beta_i \geq 0$
- $\alpha_i + \gamma_i \geq 0$
- $\alpha_i + \gamma_i + \beta_i < 1$.

The innovations $z_{i,t}$ are independent identically distributed (i.i.d.) and follow marginal distribution given by cumulative distribution function (*cdf*) F_{δ_i}. Furthermore, the joint innovations follow copula C_θ :

$$(z_1^T, \dots, z_D^T)^T \sim C_\theta$$

Useful toolboxes

You are encouraged to code everything by yourself, but you might consider employing the following libraries. The libraries are already installed in Jupyter application at the Nuvolos.

- **scipy.stats** does not contain a class for the Hansen's skewed t-distribution. However, you may find all necessary methods in **SkewStudent** class from **skewstudent** toolbox.[1]
- **arch_model** class from **arch.univariate** library[2] has **.fit()** method for estimation of univariate marginals (Exercise 2-a).

[1] `https://github.com/khrapovs/skewstudent`
[2] `https://arch.readthedocs.io/en/latest/univariate/introduction.html`

Exercise 1: simulation

a) **Hand-write** an algorithm that generates a GJR-GARCH as in (1)-(2) using the inverse method: $z_{i,t} = F_{\delta_i}^{-1}(u_{i,t})$, where u is a realization of a uniform random variable $U \sim \mathcal{U}(0,1)$.

b) Using **a)**, write a code to generate (1)-(2) with innovations distributed according to the Student's t-distribution. Use the inverse *cdf* for the Student's t-distribution.

c) As in **b)**, but the innovations follow the Hansen's skewed t-distribution. The probability density function (*pdf*) of the skewed t-distribution is given by:

$$f(z;\,\eta,\lambda) = \begin{cases} bc\left(1+\frac{1}{\eta-2}\left(\frac{bz+a}{1-\lambda}\right)^2\right)^{-\frac{\eta+1}{2}}, & \text{if } z < -a/b \\ bc\left(1+\frac{1}{\eta-2}\left(\frac{bz+a}{1+\lambda}\right)^2\right)^{-\frac{\eta+1}{2}}, & \text{if } z \geq -a/b, \end{cases} \tag{3}$$

where $a = 4\lambda c\frac{\eta-2}{\eta-1}$, $b^2 = 1 + 3\lambda^2 - a^2$, and $c = \frac{\Gamma\left(\frac{\eta+1}{2}\right)}{\sqrt{\pi(\eta-2)}\Gamma\left(\frac{\eta}{2}\right)}$ with $\Gamma(\cdot)$ being the gamma function. The parameters η ($\eta > 2$) and λ ($-1 < \lambda < 1$) are degrees of freedom and a skewness parameter, respectively. *What happens if* $\lambda = 0$*?* Generation of the Hansen's skewed t-random variable with the inverse method is obtained using the following equation

$$z = \begin{cases} \frac{1}{b}\left[(1-\lambda)\sqrt{\frac{\eta-2}{\eta}}F_\eta^{-1}\left(\frac{u}{1-\lambda}\right) - a\right], & \text{if } u < \frac{1-\lambda}{2}, \\ \frac{1}{b}\left[(1+\lambda)\sqrt{\frac{\eta-2}{\eta}}F_\eta^{-1}\left(\frac{u+\lambda}{1+\lambda}\right) - a\right], & \text{if } u \geq \frac{1-\lambda}{2}, \end{cases} \tag{4}$$

where F_η^{-1} is the inverse *cdf* of the Student's t-distribution with η degrees of freedom.

d) Using your code of Practical 1, generate first a Clayton copula of size $n = 1000$ and $D = 2$ with $\theta = 3$, and then generate a GJR-GARCH with the Student's t-innovations using **b)** and plot the two time series side-by-side. What can you tell about the extreme events?

e) Repeat **d)** with the Hansen's t-distribution of point **c)**.

Exercise 2: model fit by IFM

The IFM, or Inference For Margins, is a two-steps maximum likelihood method to fit a copula. Use the sample generated in Exercice 1-**e)**.

a) Fit each marginal x_i ($i = 1, 2$) with the GJR-GARCH defined in (1)-(2) and the Hansen's skewed t-innovations.

b) Compute estimated innovations $\hat{z}_{i,t} = \frac{x_{i,t}-\hat{\mu}_i}{\hat{\sigma}_{i,t}}$ and plot them.

c) Compute the estimated probabilities by using

$$\hat{u} = \begin{cases} (1-\lambda)F_\eta\left(\frac{b\hat{z}+a}{1-\lambda}\sqrt{\frac{\eta}{\eta-2}}\right), & \text{if } \hat{z} < -a/b, \\ (1+\lambda)F_\eta\left(\frac{b\hat{z}+a}{1+\lambda}\sqrt{\frac{\eta}{\eta-2}}\right) - \lambda, & \text{if } \hat{z} \geq -a/b. \end{cases} \tag{5}$$

d) Fit the Clayton copula to the $\hat{u}$'s as in Exercise 4 of Practical 1.

Exercise 3: fit comparison

Repeat Exercise 2 $M = 1000$ times and compare the estimators $\hat{\theta}$ with those obtained in Exercise 5 of Practical 1. Is the variance of $\hat{\theta}$ the same?