Practical 1: Copulas **Quantitative Risk Management**

The aim of this practical is to get familiar with copulas by learning how to simulate, how to illustrate and two techniques to fit them. You may use your favourite software. In-class correction will be on Matlab. All the material can be found in the Chapter 5 of the book reference of Embrechts, P., Frey, R., & McNeil, A. (2005). Quantitative risk management. *Princeton Series in Finance, Princeton, 10* (**QRM**).

Exercise 1: simulation

Prior to simulate data with your software, it is important that you set the seed of the random number generator in order to be able to reproduce the simulations. For each of the following copulas, write a code to generate random numbers for any dimension d and any sample size n:

a) **gaussian copula** using Algorithm 5.9;

b) t **copula** using Algorithm 5.10;

c) **Clayton copula** using Algorithm 5.48;

d) **Gumbel copula** using Algorithm 5.48;

e) **Frank copula** using Algorithm 5.48.

As argument of the function, you should have: the dimension d, the sample size n, the copula parameter θ and the seed.

Exercise 2: illustration

In order to illustrate the copulas, it is better practice to transform the uniform margins into standard normal random variables. Choose a parameter value θ for the copula. Simulate a large number of points ($n = 5000$). By using the inverse cumulative distribution function of the standard normal:

a) illustrate the five copulas of Exercise 1 with $d = 2$;

b) illustrate the five copulas of Exercise 1 with $d = 3$;

c) illustrate the five copulas of Exercise 1 with $d = 10$.

Vary the copula parameter θ to understand how it modifies the shape of the copulas.

Exercise 3: method of moments

With the exception of the t copula (why?), there exists a one-to-one relationship between the copula parameter θ of Exercise 1 and the Kendall's τ (check Theorem 5.36 and Table 5.5). For $d = 2$ and $n = 1000$:

a) Simulate a Gumbel copula with $\theta = 3$. Estimate the tau of Kendall $\hat{\tau}$ on the simulated sample. Estimate $\hat{\theta}$ using the relationship with $\hat{\tau}$.

b) Same as **a)** for a gaussian copula with correlation $\theta = 0.5$.

c) Same as **a)** for a Clayton copula with $\theta = 4$.

d) Same as **a)** for a Frank copula with $\theta = 2$

Exercise 4: maximum likelihood

The method of maximum likelihood is generally more efficient than the method of moments seen in Exercise 3. You will need to code the log-likelihood function. For a bivariate copula, the copula density is given by

$$c(u, v; \theta) = \frac{\partial^2 C(u, v; \theta)}{\partial u \partial v}$$

The log-likelihood is obtained by taking the logarithm of the density. The bivariate Clayton copula is given by:

$$C(u, v; \theta) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}$$

a) Derive the density of the Clayton copula;

b) Write and code the log-likelihood function with arguments the observations u, v and the copula parameter θ;

c) Simulate a Clayton sample as in point **c)** of Exercise 3;

d) Using your software optimizer, fit the Clayton copula $\hat{\theta}$ to the simulated sample.

Exercise 5: fit comparison

For the Clayton copula with parameters $d = 2, n = 1000$ and $\theta = 4$ repeat Exercise 3 and 4 $M = 1000$ times. Remember to change the seed for each repetition. Use a boxplot to compare the estimators obtained by the method of moments and the maximum likelihood.