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Goodness-of-fit tests for copulas

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Abstract

This paper defines two distribution free goodness-of-fit test statistics for copulas. It states their asymptotic distributions under some composite parametric assumptions in an independent identically distributed framework. A short simulation study is provided to assess their power performances.

AMS 2000 subject classification: 62G; 62P

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1. Introduction

In modern finance and insurance, the identification of dependence structures between assets is becoming one of the main challenges we are faced with. Copulas have been recognized as key tools to analyze dependence structures. They are becoming more and more popular among academics and practitioners because multivariate gaussian random variables do not provide satisfying models.

The copula of a multivariate distribution can be considered as the part describing its dependence structure as a complement to the behavior of each of its margins. One attractive property of copulas is their invariance under strictly increasing transformations of the margins. Actually, the use of copulas allows to solve a difficult problem, namely to find a whole multivariate distribution, by performing two easier tasks. The first step starts by modelling every marginal distribution. The second step consists of estimating a copula,

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which summarizes all the dependencies between margins. However, this second task is still in its infancy for most of multivariate financial series, partly because of the presence of temporal dependencies (serial autocorrelation, time varying heteroskedasticity, particularly) in returns of stock indices, credit spreads, interest rates of various maturities.

Estimation of copulas has been essentially spread out in the context of i.i.d. samples. If the true copula is assumed to belong to a parametric family $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$, consistent and asymptotically normally distributed estimates of the parameter of interest can be obtained through maximum likelihood methods. There are mainly two ways to achieve this: a fully parametric method and a semiparametric method. The first method relies on the assumption of parametric marginal distributions. Each parametric margin is then plugged in the full likelihood and this full likelihood is maximized with respect to θ. Alternatively and without parametric assumptions for margins, the marginal empirical cumulative distribution functions can be plugged in the likelihood. These two commonly used methods are detailed in Genest et al. [20] and Shi and Louis [40]. Hu [27] has proved general conditions for consistency and asymptotic normality of M-estimates in copula models. Chen and Fan [8] have studied such inference issues with β-mixing processes.

Beside these two methods, it is also possible to estimate a copula by some nonparametric methods based on empirical distributions, following Deheuvels [10–12]. The so-called empirical copulas look like usual multivariate empirical cumulative distribution functions. They are highly discontinuous (constant on some data-dependent pavements) and cannot be exploited as graphical device. Recently, smooth estimates of copulas in a time-dependent framework have been proposed in Fermanian and Scaillet [17]. They allow to guess which parametric copula family should be convenient. This intuition needs to be properly verified to be validated. In a statistical sense, it means to lead a goodness-of-fit test on the copula specification. This is our topic.

To be specific, consider an i.i.d. sample of d-dimensional vectors $(\mathbf{X}_i)_{i=1,\ldots,n}$. Denote $\mathbf{X}_i = (X_{i1}, \ldots, X_{id})$. and H, resp. C, the cumulative distribution function, resp. the copula, of $\mathbf{X}$. Our goal is to find a technique to solve the similar GOF problem for copulas, say to distinguish between two assumptions:

$$\mathcal{H}_0 : C = C_0, \text{ against } \mathcal{H}_a : C \neq C_0, \text{ when the zero-assumption is simple, or}$$

$$\mathcal{H}_0 : C \in \mathcal{C}, \text{ against } \mathcal{H}_a : C \notin \mathcal{C}, \text{ when the zero-assumption is composite.}$$

Here, C_0 denotes some known copula, and $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$ is some known parametric family of copulas. The copula is the cdf of $(F_1(X_1), \ldots, F_d(X_d))$.

In a multidimensional framework, it is usually difficult to build distribution free GOF tests. Some more or less satisfying solutions exist. Justel et al. [28] have proposed to use the transformation of Rosenblatt [38] before testing a simple GOF assumption. Several authors have tried to replace an evaluation over a d-dimensional space by a univariate function, by considering some families of subsets in $\mathbb{R}^d$ indexed by a univariate parameter. Then, some Kolmogorov–Smirnov type test statistics are available. See Saunders and Laud [39], Foutz [18] or more recently Polonik [36]. Moreover, Khmaladze [30,31] and especially [32] has transformed the usual empirical process into an asymptotically distribution free empirical process, for simple and composite assumptions. Nonetheless, these techniques are involved or cannot be extended easily to slightly different situations. Actually, the simplest way to build GOF composite tests for multivariate r.v. is to consider multidimensional chi-square tests, as in Pollard [35].

Particularly, it seems to be too difficult to adapt these techniques for copulas. The difficulty is coming from the fact the marginal cdfs' F_j are unknown. Particularly, the chi square test procedures do not work anymore in general, when replacing marginal cdfs' by some usual estimates. For all these reasons, the general problem of GOF test for copulas has not been dealt conveniently by authors. Some of them use the bootstrap procedure to evaluate the limiting distribution of the test statistic (e.g. Andersen et al. [3]). Genest and Rivest [20] solve the problem in the case of archimedean copulas, for which the problem can be reduced to a one dimensional one, for which some standard methods are available. For instance, Frees and Valdez [19] use Q–Q plots to fit the "best" archimedean copula. None of the authors have dealt the case of time-dependent copulas, except Patton [33,34], but the latter author tests all the joint specification and not only the copula itself. Recently, some authors have applied Rosenblatt's transformation (cf. [37]) to the original multivariate series, before testing the copula specification: Breymann et al. [7], Chen et al. [9]. The latter authors compare the smoothed copula density of their transformed r.v. to the uniform density by means of a L^2 criterion, as in Hong and Li [29]. So their methodology is relatively closed to ours (see below the test statistics $\mathcal{T}$). Nonetheless, as we said previously, the use of Rosenblatt's transformation is a tedious preliminary, especially with high dimension variables, and it is model specific. Thus the test methodology is not really distribution-free.

Note that we could build some test procedures based on some estimates of $\mathbf{X}$'s cdf by modelizing the marginal distributions simultaneously. It seems to be a good idea, because some "more or less usual" tests are available to check the GOF of H itself. Nonetheless, it is not our point of view. Indeed, doing so produces tests for the whole specification—the copula and the margins—but not for the dependence structure itself—the copula only. A slightly different point of view could be to test each marginal separately in a first step. If each marginal model is accepted, then a test of the whole multidimensional distribution can be led (by the previously cited methodologies). Nonetheless, such a procedure is heavy, and it is always necessary to deal with a multidimensional GOF test. Moreover, it is always interesting to study dependence in depth first, independently of the specification of margins.

To build a GOF test, a natural way would be to use to asymptotic behavior of the empirical copula process. According to Fermanian et al. [16], we know that the bivariate empirical copula process $n^{1/2}(C_n - C_0)$ tends in law, under the null simple assumption, towards the gaussian process $\mathbb{G}_{C_0}$, where

$$\mathbb{G}_{C_0}(u,v) = \mathbb{B}_{C_0}(u,v) - \partial_1 C_0(u,v)\mathbb{B}_{C_0}(u,1) - \partial_2 C_0(u,v)\mathbb{B}_{C_0}(1,v), \quad (u,v) \in [0,1]^2.$$

We have denoted by $\mathbb{B}_C$ a brownian bridge on $[0,1]^2$, such that

$$E[\mathbb{B}_C(u,v)\mathbb{B}_C(u',v')] = C(u \wedge u', v \wedge v') - C(u,v)C(u',v'),$$

for every $u, u', v, v' \in [0,1]$. Here, $a \wedge b = \inf(a,b)$. Unfortunately, this limiting process is a lot more complicated than with the multidimensional brownian bridge $\mathbb{B}_{C_0}$. For instance, the covariance between $\mathbb{G}_{C_0}(u,v)$ and $\mathbb{G}_{C_0}(u',v')$ is the sum of 18 terms (while there are 2 terms in $\mathbb{B}_{C_0}$'s case). These terms involve C_0 and its partial derivatives. Thus, GOF tests based directly on empirical copula processes C_n seem to be unpractical, except by bootstrapping. Nonetheless, such procedures are computationally intensive. Even if the bootstrapped empirical copula process is weakly convergent (Fermanian et al. [16]), we prefer to propose a more usual test procedure.

A simple chi-square type test procedure is defined in Section 2. Then, a more powerful and more sophisticated test statistics is described in Section 3. The power of these tests is studied by simulations in Section 4. The proofs are postponed in Appendixes A–D.

2. A simple direct chi-square approach

There exists a simple direct way to circumvent the difficulty. Indeed, by smoothing the empirical copula process, we get an estimate of the copula density. The limit of these statistics is far simpler than $\mathbb{G}_{C_0}$. Let us consider first an i.i.d. framework.

For each index i, set the d-dimensional vectors $\mathbf{Y}_i = (F_1(X_{i,1}), \ldots, F_d(X_{i,d}))$ and $\mathbf{Y}_{n,i} = (F_{n,1}(X_{i,1}), \ldots, F_{n,d}(X_{i,d}))$, denoting by F_k and $F_{n,k}$ the true and the empirical kth marginal cdf of $\mathbf{X}$. Obviously, the copula C is the cdf of $\mathbf{Y}_i$. The empirical copula process we consider here is

$$C_n(\mathbf{u}) = \frac{1}{n}\sum_{i=1}^{n}\prod_{k=1}^{d}\mathbf{1}(F_{n,k}(X_{i,k})\leqslant u_k),$$

instead of the “usual” copula process

$$C_n^*(\mathbf{u}) = F_n(F_{n,1}^-(u_1), \ldots, F_{n,d}^-(u_d)), \text{ where } F_{n,k}^-(u) = \inf\{t | F_{n,k}(t)\geqslant u\}.$$

It is easy to verify these two empirical processes differ only by the small quantity n^{-1} at most. Thus, it would not be an hard task to adapt the proofs to C_n^*.

We will assume the law of the vectors $\mathbf{Y}_i$ has a density τ with respect to the Lebesgue measure. By definition the kernel estimator of a copula density τ at point $\mathbf{u}$ is

$$\tau_n(\mathbf{u}) = \frac{1}{h^d}\int K\left(\frac{\mathbf{u}-\mathbf{v}}{h}\right)C_n(d\mathbf{v}) = \frac{1}{nh^d}\sum_{i=1}^{n}K\left(\frac{\mathbf{u}-\mathbf{Y}_{n,i}}{h}\right), \tag{2.1}$$

where K is a d-dimensional kernel and $h = h(n)$ is a bandwidth sequence. More precisely, $\int K = 1, h(n) > 0$, and $h(n) \to 0$ when $n \to \infty$. As usual, we denote $K_h(\cdot) = K(\cdot/h)/h^d$. For convenience, we will assume

Assumption (K). The kernel K is the product of d univariate even compactly supported kernels $K_r, r = 1, \ldots, d$. It is assumed p_K -times continuously differentiable.

These assumptions are far from minimal. Particularly, we could consider some multivariate kernels whose support is the whole space $\mathbb{R}^d$, if they tend to zero “sufficiently quickly” when their arguments tend to the infinity (for instance, at an exponential rate, like for the gaussian kernel). Since this speed depends on the behavior of τ, we are rather the simpler assumption (K).

As usual, the bandwidth needs to tend to zero not too quick.

Assumption (B0). When n tends to the infinity, $nh^d \to \infty$, $nh^{4+d} \to 0$ and

$$nh^{3+d/2}/(\ln_2 n)^{3/2} \longrightarrow \infty.$$

We have set $\ln_2 n = \ln(\ln n)$. Assumption (B0) can be weakened easily by assuming (K) with $p_K > 3$ (see details in the proofs).

Moreover, a certain amount of regularity of τ is necessary, for instance

Assumption (T0). $\tau(\mathbf{u}, \theta)$ and its first two derivatives with respect to $\mathbf{u}$ exist and are uniformly continuous on $\mathcal{V}(\mathbf{u}_k) \times \mathcal{V}(\theta_0)$, for every vectors $\mathbf{u}_k$, $k = 1, \ldots, m$, denoting by $\mathcal{V}(\mathbf{u}_k)$ (resp. $\mathcal{V}(\theta_0)$) an open neighborhood of $\mathbf{u}_k$ (resp. θ_0).

In the appendix, we prove:

Theorem 1. *Under* (*K*) *with* $p_K = 3$, ($B0$) *and* ($T0$), *for every m and every vectors* $\mathbf{u}_1, \ldots, \mathbf{u}_m$ *in* $]0, 1[^d$, *such that* $\tau(\mathbf{u}_k) > 0$ *for every k, we have*

$$(nh^d)^{1/2}((\tau_n - \tau)(\mathbf{u}_1), \ldots, (\tau_n - \tau)(\mathbf{u}_m)) \xrightarrow[n\to\infty]{law} \mathcal{N}(0, \Sigma),$$

where Σ *is diagonal, and its k-th diagonal term is* $\int K^2.\tau^2(\mathbf{u}_k)$.

Now, imagine we want to build a procedure for a GOF test with some composite zero assumption. Under $\mathcal{H}_0$, the parametric family is $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$. Assume we have estimated θ consistently by $\hat{\theta}$, and

$$\hat{\theta} - \theta_0 = O_P(n^{-1/2}). \tag{2.2}$$

We denote by $\tau(\cdot, \theta_0)$ (or simpler τ when there is no ambiguity) the "true" underlying copula density. Clearly, $\tau(\mathbf{u}, \hat{\theta}) - \tau(\mathbf{u}, \theta_0)$ tends to zero quicker than $(\tau_n - \tau)(\mathbf{u})$ under (T0) and Eq. (2.2). Thus, a simple GOF test may be

$$\mathcal{S} = \frac{nh^d}{\int K^2} \sum_{k=1}^{m} \frac{(\tau_n(\mathbf{u}_k) - \tau(\mathbf{u}_k, \hat{\theta}))^2}{\tau(\mathbf{u}_k, \hat{\theta})^2}.$$

Corollary 2. *Under the assumptions of Theorem* 1 *and Eq.* (2.2), *if* $\tau(\mathbf{u}, \theta)$ *is continuously differentiable with respect to* θ *in a neighborhood of* θ_0 *for every* $\mathbf{u} \in]0, 1[^d$, *then* $\mathcal{S}$ *tends in law towards a m-dimensional chi-square distribution under the composite zero-assumption* $\mathcal{H}_0$.

We could replace τ by the convolution of K and τ in Theorem 1 and Corollary 2. This allows to remove the assumption $nh^{4+d} \longrightarrow 0$. Indeed, this assumption prevents us from using the usual asymptotically optimal bandwidth that minimizes the asymptotic mean squared error.

The points $(\mathbf{u}_k)_{k=1,\ldots,m}$ are chosen arbitrarily. They could be chosen in some particular areas of the d-dimensional square, where the user seeks a good fit. For instance, for risk management purposes, it would be fruitful to consider some dependencies in the tails. For the particular copula family $\mathcal{C}$, it is necessary to specify these areas.

Clearly, the power of the $\mathcal{S}$ test depends strongly on the choice of the points $(\mathbf{u}_k)_{k=1,\ldots,m}$. This is a bit the same drawback as the choice of cells in the usual GOF chi-square test. Without a priori, it is always possible to choose a uniform grid of the type $(i_1/N, i_2/N, \ldots, i_d/N)$, for every integers $i_1, \ldots, i_d$, $1 \leqslant i_k \leqslant N-1$. Nonetheless, the number m will become very large when the dimension d increases.

More seriously, the power of the test will not be very large surely. Actually, the adequacy of the fit for a finite number of points is not a guarantee for a good adequacy of the whole copula. That is why we propose another test statistics. This statistics will consider the whole underlying distribution potentially, and not only of a finite number of points.

3. The main test

This test is based on the proximity between the smoothed copula density and the estimated parametric density. Under $\mathcal{H}_0$, they will be near each other. To measure such a proximity, we will invoke the L^2 norm. To simplify, denote the estimated parametric $\tau(\cdot, \hat{\theta})$ density by $\hat{\tau}$. Consider the statistics

$$J_n = \int (\tau_n - K_h * \hat{\tau})^2(\mathbf{u})\omega(\mathbf{u})\, d\mathbf{u},$$

where ω is a weight function, viz a measurable function from $[0, 1]^d$ towards $\mathbb{R}^+$. Note that we consider the convolution between the kernel K_h and $\hat{\tau}$ instead of $\hat{\tau}$ itself. This trick allows to remove a bias term in the limiting behavior of J_n (see Fan [13]). Note that the expectation of $\tau_n(u)$ is different from $K_h * \tau(u)$, contrary to the usual i.i.d. density case. This will complicate slightly the proof.

The minimization of the criterion J_n is known to produce consistent estimates in numerous situations. These ideas appear first in the seminal paper of Bickel and Rosenblatt [5]. They are applied in the usual density case for i.i.d. observations. Rosenblatt [38] extended the results in a two-dimensional framework and discusses consistency with respect to several alternatives. Fan [13] extended these works to deal with every choices of the smoothing parameter. The comparison of some nonparametric statistics-especially nonparametric regressions- and their model-dependent equivalents has been formalized in a lot of papers in statistics and econometrics: Härdle and Mammen [26], Zheng [42], Fan and Li [14], among others.

Similar results have been obtained for dependent processes more recently: Fan and Ullah [15], Hjellvik et al. [25], Gouriéroux and Tenreiro [22], e.g. For instance, Aït-Sahalia [2] applies these techniques to find a convenient specification for the dynamics of the short interest rate. Recently, Gouriéroux and Gagliardini [23] use such a criterion to estimate possibly infinite dimensional parameters of a copula function, for instance the univariate function defining an archimedean copula. Instead of for inference purposes, we will use J_n as a test statistics, like in Fan [13].

Let us assume that we have found a convenient estimator of θ.

Assumption (E). There exists $\hat{\theta} \in \mathbb{R}^q$ such that

$$\hat{\theta} - \theta_0 = n^{-1} A(\theta_0)^{-1} \sum_{i=1}^{n} B(\theta_0, \mathbf{Y}_i) + o_P(r_n), \tag{3.1}$$

and r_n tends to zero quicker than $n^{-1/2}(\ln_2 n)^{-1/2}$ when n tends to the infinity. Here, $A(\theta_0)$ denotes a $q \times q$ positive-definite matrix and $B(\theta_0, \mathbf{Y})$ is a q-dimensional random vector. Moreover, $E[B(\theta_0, \mathbf{Y}_i)] = 0$ and $E[\| B(\theta_0, \mathbf{Y}_i)\|^2] < \infty$.

Particularly, under (E), $\hat{\theta} - \theta = O_P(n^{-1/2})$. Typically, $B(\theta, \cdot)$ is a score function. In Section D in Appendix A–D we prove these assumptions are satisfied particularly for the usual semiparametric maximum likelihood estimator whose theoretical properties are detailed in Genest et al. [20] and Chen and Fan [8]. But more general procedures can be used, like *M*-estimators.

Assumption (T). For some open neighborhood $\mathcal{V}(\theta_0)$ of θ_0,
- $\tau(\mathbf{u}, \theta)$ and its first two derivatives with respect to θ exist and are uniformly continuous on $[0, 1]^d \times \mathcal{V}(\theta_0)$, or
- $\tau(\mathbf{u}, \theta)$ and its first two derivatives with respect to θ exist and are uniformly continuous on $[\varepsilon, 1-\varepsilon]^d \times \mathcal{V}(\theta_0)$, for some $\varepsilon > 0$, and the support of ω is included in $[\varepsilon_0, 1-\varepsilon_0]^d$, for some $\varepsilon_0 > \varepsilon$.

When τ and its derivatives with respect of θ are uniformly bounded on $[0, 1]^d \times \mathcal{V}(\theta_0)$, ω can be chosen arbitrarily. Unfortunately, it is not always the case. For instance, by choosing a bivariate gaussian copula density. To avoid technical troubles, we reduce the GOF test to a strict subsample of $[0, 1]^d$, say ω's support.

Assumption (B). $nh^d \longrightarrow \infty$ and $nh^{4+d/2}/\ln_2^2 n \underset{n\to\infty}{\longrightarrow} \infty$.

Actually, the latter condition could be relaxed. It is sufficient to expand K up to higher order terms. We had chosen the order 4 so that condition (B) is not too strong. But, it is possible to exchange some degree of regularity of K against less constraints on the bandwidth.

Theorem 3. *Under assumptions $\mathcal{H}_0$, (T), (E), (B) and (K) with $p_K = 4$, we have*

$$nh^{d/2}\left(J_n - \frac{1}{nh^d}\int K^2(\mathbf{t}).(\tau\omega)(\mathbf{u} - h\mathbf{t})\, d\mathbf{t}\, d\mathbf{u} \right.$$
$$\left. + \frac{1}{nh}\int \tau^2\omega . \sum_{r=1}^{d} \int K_r^2 \right) \underset{n\to\infty}{\overset{law}{\longrightarrow}} \mathcal{N}(0, 2\sigma^2),$$

$$\sigma^2 = \int \tau^2\omega \cdot \int \left\{ \int K(\mathbf{u})K(\mathbf{u} + \mathbf{v})\, d\mathbf{u} \right\}^2 d\mathbf{v}.$$

Thus, a test statistics may be

$$\mathcal{T} = \frac{n^2h^d(J_n - (nh^d)^{-1}\int K^2(\mathbf{t}).(\hat{\tau}\omega)(\mathbf{u}-h\mathbf{t})\,d\mathbf{t}\,d\mathbf{u} + (nh)^{-1}\int \hat{\tau}^2\omega.\sum_{r=1}^d \int K_r^2)^2}{2\int \hat{\tau}^2\omega \cdot \int\{\int K(\mathbf{u})K(\mathbf{u}+\mathbf{v})\,d\mathbf{u}\}^2\,d\mathbf{v}}.$$

Corollary 4. *Under the assumptions of Theorem* 3, *the previous statistics $\mathcal{T}$ tends in law towards a chi-square distribution.*

See the proof in the appendix. Since the kernel K is even, we can replace the second term of the previous numerator by the simpler expression $-(nh^d)^{-1}\int K^2.\int \hat{\tau}\omega$. Moreover, the third term in the numerator can be replaced by

$$\frac{2}{nh}\sum_r \int K(\mathbf{t})K_r(t_r)\hat{\tau}(\mathbf{u}-h\mathbf{t})\hat{\tau}(\mathbf{u})\omega(\mathbf{u})\,dt\,d\mathbf{u} - \frac{1}{nh}\sum_r \int \hat{\tau}^2\omega \cdot \int K_r^2.$$

This expression is a consequence of the proof, and offers surely a better approximation, even if it is a bit more complicated.

Moreover, under some additional regularity assumptions, we could replace $\hat{\tau}$ by τ_n inside $\mathcal{T}$. Indeed, it can be proved the kernel estimator of the density $\tau(\mathbf{u})$ converges uniformly with respect to $\mathbf{u}$ on ω's support at a convenient rate. The proof requires to control the uniform upper bound of the remainder terms $R_k(\mathbf{u})$, $k = 1, 2, 3$ that are defined in the proof of Theorem 1. This can be done by applying lemma B1 in Ai [1], e.g. The details are left to the reader.

Note that our test statistics differs from similar GOF test statistics in an i.i.d. framework with usual density functions (e.g. Fan [13]). Indeed, there is an additional term

$$(nh)^{-1}\int \hat{\tau}^2\omega.\sum_{r=1}^d \int K_r^2,$$

in $\mathcal{T}$. This is the price to work with copulas, and to estimate the margins empirically. Nonetheless, when $d > 2$, this additional term is negligible with respect to

$$(nh^d)^{-1}\int K^2(\mathbf{t}).(\hat{\tau}\omega)(\mathbf{u}-h\mathbf{t})\,d\mathbf{t}\,d\mathbf{u}.$$

4. A short simulation study

To asses the power of our test statistics, we have led a simple analysis by simulation. We generate some samples whose copula is the mixture of a bivariate frank's copula and an independent copula, viz

$$C_{\theta,\alpha}(u,v) = \alpha uv - \frac{(1-\alpha)}{\theta}\ln\left(1 + \frac{(\exp(-\theta u)-1)(\exp(-\theta v)-1)}{\exp(-\theta)-1}\right),$$
$$\theta \neq 0,\ \alpha \in [0,1].$$

More precisely, we generate iid uniform samples $(U_{i,1}, U_{i,2})_{i=1,\ldots,200}$ on $[0, 1]^2$. For every $i = 1, \ldots, 200$, we get

$$X_{i,1} = \Phi^{-1}(U_{i,1}) \text{ and } X_{i,2} = \Phi^{-1}(V_i),$$

where V_i satisfies the equality $\partial_1 C_{\theta,\alpha}(U_{i,1}, V_i) = U_{i,2}$. Thus, the random vectors $(X_{i,1}, X_{i,2})$ have the desired copula.

We compute the test statistics $\mathcal{S}$ and $\mathcal{T}$ with these data sets. The zero assumption is: "the true underlying copula is Frank's". Concerning $\mathcal{S}$, we choose 81 points on the uniform grid $(i/10, j/10)$, $i, j = 1, \ldots, 9$. We use the convolution between K and $\hat{\tau}$ instead of $\hat{\tau}$ itself. Concerning $\mathcal{S}$ and $\mathcal{T}$, the kernel is a sufficiently regular compactly supported kernel, say

$$K(\mathbf{u}) = \left(\frac{15}{16}\right)^2 \prod_{k=1}^{2} (1 - u_k^2)^2 \mathbf{1}(u_k \in [0, 1]).$$

The bandwidths are chosen by the usual Silverman's rule [41]: $\hat{h} = \sqrt{(\sigma_1^2 + \sigma_2^2)/2}n^{-1/6}$, denoting by σ_k^2 the empirical variance of $F_{n,k}$, $k = 1, 2$. Note that these two variances are the same in our case because they depend on the sample size only. The weight function w is chosen as $w(\mathbf{u}) = \mathbf{1}(\mathbf{u} \in [0.01, 0.99])$. The parameters of the copulas are estimated by the usual semiparametric maximum likelihood procedure (see Shi and Louis [40]).

For different values for α and θ, we compute the two test statistics $\mathcal{S}$ and $\mathcal{T}$. We have made 100 replications for 200 points samples: see Table 1. When α is zero, we check the asymptotic level 0.05 is underestimated by all but one case (thus the test is a bit too conservative). When α increases, the percentages of rejection grow, especially for the $\mathcal{T}$ test. The latter seems to be more powerful than $\mathcal{S}$, even if this advantage weakens when the copula is more and more far from the Frank's copula (viz when the zero assumption is more and more false). Moreover, the reported powers are higher and higher when θ is larger and larger, because the corresponding Frank's copula (and its mixture with the independent copula) becomes far away from the independent one.

When α is near 1, note that the power is very weak. In such a case, the underlying copula is "almost" the independent copula. Since the latter belongs to the boundary of the Frank's family (when leaving θ to tend to zero), it is doubtful the estimator $\hat{\theta}$ satisfies assumption (E) (e.g. cf. Andrews [4]). Moreover, the estimation of an "almost discontinuous" function τ near the boundaries of $[0, 1]^2$ induces some non-standard limit laws, especially biases. Even if we have restricted ourselves into the interior of the unit square by the function w, there may be a practical issue with the (small) samples we have considered. Thus, in such a case, the results cannot be easily interpreted.

Globally, these partial results are very convincing. Particularly, with very small sample sizes, the power of the test $\mathcal{T}$ is far from ridiculous even when the proportion of perturbation is weak. Our results seem to be better than those reported by the test 1 proposed by Chen et al. [9]. In the latter case, the powers are near zero when the sample size is not greater than 500 for every level of perturbation (but with a different model). Nonetheless, our results need to be completed by a more in depth simulation study.

Table 1
Percentages of rejection at 5% level with $n = 200$ and 100 replications

% of noise α	Parameter θ	% of rejection (test $\mathcal{S}$)	% of rejection (test $\mathcal{T}$)
	5	0	2
	10	0	0
0.0	15	0	1
	20	0	1
	25	0	8
	5	0	0
	10	0	0
0.1	15	0	7
	20	0	22
	25	0	60
	5	1	1
	10	1	5
0.2	15	3	36
	20	17	80
	25	31	95
	5	3	3
	10	13	21
0.3	15	18	67
	20	57	95
	25	84	100
	5	7	12
	10	19	33
0.5	15	58	71
	20	89	98
	25	95	100
	5	2	1
	10	3	0
0.9	15	6	0
	20	2	2
	25	37	3

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AppendixA.. Proof of Theorem 1

We will prove that the behavior of $\tau_n(\mathbf{u})$ is the same as the behavior of

$$\tau_n^*(\mathbf{u}) = n^{-1} \sum_{i=1}^{n} K_h(\mathbf{u} - \mathbf{Y}_i),$$

for every $\mathbf{u}$. Indeed,

$$\begin{aligned}\tau_n(\mathbf{u}) &= \tau_n^*(\mathbf{u}) + \frac{(-1)}{nh}\sum_{i=1}^{n}(dK)_h(\mathbf{u}-\mathbf{Y}_i)\cdot(\mathbf{Y}_{ni}-\mathbf{Y}_i) \\ &\quad + \frac{1}{2nh^2}\sum_{i=1}^{n}(d^2K)_h(\mathbf{u}-\mathbf{Y}_i)\cdot(\mathbf{Y}_{ni}-\mathbf{Y}_i)^{(2)} \\ &\quad + \frac{(-1)}{6nh^3}\sum_{i=1}^{n}(d^3K)_h(\mathbf{u}-\mathbf{Y}_{ni}^*)\cdot(\mathbf{Y}_{ni}-\mathbf{Y}_i)^{(3)} \\ &= \tau_n^*(\mathbf{u}) + R_1(\mathbf{u}) + R_2(\mathbf{u}) + R_3(\mathbf{u}),\end{aligned}$$

for some random vector $\mathbf{Y}_{ni}^*$ satisfying $\|\mathbf{Y}_{n,i}^* - \mathbf{Y}_i\| \leqslant \|\mathbf{Y}_{n,i} - \mathbf{Y}_i\|$ a.e.

Let us first study $R_1(\mathbf{u})$. Its expectation is $O(n^{-1}h^{-1})$. Moreover,

$$\begin{aligned}E[R_1^2(\mathbf{u})] &= \frac{1}{n^2h^2}\sum_{i,j}E[(dK)_h(\mathbf{u}-\mathbf{Y}_i)\cdot(\mathbf{Y}_{ni}-\mathbf{Y}_i)\cdot(dK)_h(\mathbf{u}-\mathbf{Y}_j)\cdot(\mathbf{Y}_{nj}-\mathbf{Y}_j)] \\ &= \frac{1}{n^4h^2}\sum_{i,j}\sum_{k,l}E[(dK)_h(\mathbf{u}-\mathbf{Y}_i)\cdot(\mathbf{1}(\mathbf{Y}_k\leqslant\mathbf{Y}_i)-\mathbf{Y}_i) \\ &\qquad \cdot(dK)_h(\mathbf{u}-\mathbf{Y}_j)\cdot(\mathbf{1}(\mathbf{Y}_l\leqslant\mathbf{Y}_j)-\mathbf{Y}_j)].\end{aligned}$$

We will denote by $\mathbf{1}(\mathbf{y}\leqslant\mathbf{u})$ a d-dimensional vector whose kth component is $\mathbf{1}(y_k\leqslant u_k)$. The expectations of the summands are zero, except if there are some equalities involving k and l. For instance, assume $k=l\neq i\neq j$. Let us note that

$$\begin{aligned}&E\left[(dK)_h(\mathbf{u}-\mathbf{Y}_j)\cdot(\mathbf{1}(\mathbf{Y}_i\leqslant\mathbf{Y}_j)-\mathbf{Y}_j)|\mathbf{Y}_i=\mathbf{y}_i\right] \\ &\quad = \int (dK)_h(\mathbf{u}-\mathbf{v})\cdot(\mathbf{1}(\mathbf{y}_i\leqslant\mathbf{v})-\mathbf{v})\tau(\mathbf{v})\,d\mathbf{v} \\ &\quad = \sum_{r=1}^{d}\int(\partial_rK)(\mathbf{v})\cdot(\mathbf{1}(y_{i,r}\leqslant u_r-hv_r)-u_r+hv_r)\tau(\mathbf{u}-h\mathbf{v})\,d\mathbf{v} \\ &\quad = \sum_{r=1}^{d}\int(\partial_rK)(\mathbf{v})\cdot(\mathbf{1}(v_r\leqslant(u_r-y_{i,r})/h)-u_r+hv_r)\{\tau(\mathbf{u})+h\psi(\mathbf{u},\mathbf{v})\}\,d\mathbf{v},\end{aligned}$$

where ψ is a bounded compactly supported function, for n sufficiently large. Since we assume K is the product of some univariate kernels K_r, $r=1,\ldots,d$, we get

$$\begin{aligned}&E[(dK)_h(\mathbf{u}-\mathbf{Y}_j)\cdot(\mathbf{1}(\mathbf{Y}_i\leqslant\mathbf{Y}_j)-\mathbf{Y}_j)|\mathbf{Y}_i=\mathbf{y}_i] \\ &\quad = \tau(\mathbf{u})\sum_{r=1}^{d}K_r\left(\frac{u_r-y_{i,r}}{h}\right)+O(h)\cdot\phi(\mathbf{u}),\end{aligned} \tag{A.1}$$

for every couple (i,j) with $i\neq j$, where ϕ is bounded, compactly supported and independent of $\mathbf{y}_i$. Thus, the corresponding term in $E[R_1^2(\mathbf{u})]$ is

$$\frac{1}{nh^2}\int\left\{\tau(\mathbf{u})\sum_r K_r\left(\frac{u_r-y_r}{h}\right)+O(h)\phi(\mathbf{u})\right\}$$

$$\times \left\{ \tau(\mathbf{u}) \sum_s K_s \left(\frac{u_s - y_s}{h} \right) + O(h)\phi(\mathbf{u}) \right\} \tau(\mathbf{y}_1)\tau(\mathbf{y}_2)\, d\mathbf{y}$$

$$= \frac{1}{nh^2} \int \tau^2(\mathbf{u}) \left\{ \sum_{r \neq s} K_r \left(\frac{u_r - y_r}{h} \right) K_s \left(\frac{u_s - y_s}{h} \right) dy_r\, dy_s \right.$$

$$\left. + \sum_r K_r^2 \left(\frac{u_r - y_r}{h} \right) dy_r \right\} + O(n^{-1}h^{-1}) = O(\frac{1}{nh}) = o(n^{-1}h^{-d}),$$

by some usual changes of variables with respect to y_r and y_s. The other equalities between i, j, k and l provide a similar conclusion. Thus, the variance of $R_1(\mathbf{u})$ is $o(n^{-1}h^{-d})$, and $R_1(\mathbf{u}) = o_P(1/\sqrt{nh^d})$.

The study of $R_2(\mathbf{u})$ is similar. We get by the same method $E[R_2(\mathbf{u})] = O(n^{-1}h^{-2})$ and $E[R_2^2(\mathbf{u})] = O(n^{-2}h^{-4})$, hence $R_2(\mathbf{u}) = o_P(1/\sqrt{nh^d})$. Since,

$$\|\mathbf{Y}_{n,i} - \mathbf{Y}_i\|_\infty = O_P\left(\left(\frac{\ln_2 n}{n} \right)^{1/2} \right), \tag{A.2}$$

we deduce directly $R_3(\mathbf{u}) = O_P(h^{-3-d}n^{-3/2}.\ln_2^{3/2} n)$, which is $o_P(n^{-1/2}h^{-d/2})$ if $nh^{3+d/2}/\ln_2^{3/2} n$ tends to the infinity when $n \to \infty$. Thus, under our assumptions,

$$\tau_n(\mathbf{u}) = \tau_n^*(\mathbf{u}) + o_P\left(\frac{1}{\sqrt{nh^d}} \right).$$

Moreover, Bosq and Lecoutre's [6] theorem VIII.2 provides the asymptotic normality of the joint vector $(nh^d)^{1/2}((\tau_n^* - \tau)(\mathbf{u}_1), \ldots, (\tau_n^* - \tau)(\mathbf{u}_m))$. This concludes the proof. □

AppendixB.. Proof of Theorem 3

Clearly,

$$J_n = \int (\tau_n - K_h * \hat{\tau})^2(\mathbf{u})\omega(\mathbf{u})\, d\mathbf{u}$$

$$= \int (\tau_n - E\tau_n)^2\omega + 2\int (\tau_n - E\tau_n)(\mathbf{u}) \cdot (E\tau_n - K_h * \hat{\tau})(\mathbf{u})\omega(\mathbf{u})\, d\mathbf{u}$$

$$+ \int (E\tau_n - K_h * \hat{\tau})^2\omega \equiv \int (\tau_n - E\tau_n)^2\omega + 2J_{\mathrm{I}} + J_{\mathrm{II}}. \tag{B.1}$$

The main term of J_n will be

$$J_n^* = \int (\tau_n - E\tau_n)^2\omega = \frac{1}{n}\int \left(\sum_{i=1}^n K_h(\mathbf{u} - \mathbf{Y}_{n,i}) - EK_h(\mathbf{u} - \mathbf{Y}_{ni}) \right)^2 \omega(\mathbf{u})\, d\mathbf{u}$$

$$= \frac{1}{n^2} \sum_{i,j=1}^n \int \left(K_h(\mathbf{u} - \mathbf{Y}_{n,i}) - EK_h(\mathbf{u} - \mathbf{Y}_{n,i}) \right) \cdot (K_h(\mathbf{u} - \mathbf{Y}_{n,j})$$

$$-EK_h(\mathbf{u} - \mathbf{Y}_{n,j}))\omega(\mathbf{u})\, d\mathbf{u}$$

Thus,

$$J_n^* = \frac{1}{n^2}\sum_{i=1}^n \int a_{n,i}^2 \omega + \frac{2}{n^2}\sum_{i<j}\int a_{n,i}a_{n,j}\omega \equiv J_{n,1}^* + J_{n,2}^*, \tag{B.2}$$

where we have set

$$a_{n,i}(\mathbf{u}) = K_h(\mathbf{u} - \mathbf{Y}_{n,i}) - EK_h(\mathbf{u} - \mathbf{Y}_{n,i}).$$

Intuitively, $a_{n,i}(\mathbf{u})$ is close to $a_i(\mathbf{u}) = K_h(\mathbf{u} - \mathbf{Y}_i) - EK_h(\mathbf{u} - \mathbf{Y}_i)$. For technical reasons, we will need to expand the difference between the two latter terms up to the fourth order, viz

$$a_{n,i}(\mathbf{u}) - a_i(\mathbf{u}) = b_{n,i}(\mathbf{u}) + c_{n,i}(\mathbf{u}) + d_{n,i}(\mathbf{u}) + e_{n,i}(\mathbf{u}),$$

$$b_{n,i}(\mathbf{u}) = \frac{(-1)}{h}\left[(dK)_h(\mathbf{u} - \mathbf{Y}_i)\cdot(\mathbf{Y}_{n,i} - \mathbf{Y}_i) - E(dK)_h(\mathbf{u} - \mathbf{Y}_i)\cdot(\mathbf{Y}_{n,i} - \mathbf{Y}_i)\right],$$

$$c_{n,i}(\mathbf{u}) = \frac{1}{2h^2}\left[(d^2K)_h(\mathbf{u} - \mathbf{Y}_i)\cdot(\mathbf{Y}_{n,i} - \mathbf{Y}_i)^{(2)} - E(d^2K)_h(\mathbf{u} - \mathbf{Y}_i)\cdot(\mathbf{Y}_{n,i} - \mathbf{Y}_i)^{(2)}\right],$$

$$d_{n,i}(\mathbf{u}) = \frac{(-1)}{6h^3}\left[(d^3K)_h(\mathbf{u} - \mathbf{Y}_i)\cdot(\mathbf{Y}_{n,i} - \mathbf{Y}_i)^{(3)} - E(d^3K)_h(\mathbf{u} - \mathbf{Y}_i)\cdot(\mathbf{Y}_{n,i} - \mathbf{Y}_i)^{(3)}\right],$$

$$e_{n,i}(\mathbf{u}) = \frac{1}{24h^4}\left[(d^4K)_h(\mathbf{u} - \mathbf{Y}_{n,i}^*)\cdot(\mathbf{Y}_{n,i} - \mathbf{Y}_i)^{(4)} - E(d^4K)_h(\mathbf{u} - \mathbf{Y}_{n,i}^*)\cdot(\mathbf{Y}_{n,i} - \mathbf{Y}_i)^{(4)}\right],$$

for some $\mathbf{Y}_{n,i}^*$ that lies between $\mathbf{Y}_i$ and $\mathbf{Y}_{n,i}$ a.e. Most of the sums involving the previous terms will be negligible with respect to $1/(nh^{d/2})$.

B.1. Study of $J_{n,2}^*$

Now

$$J_{n,2}^* = \frac{2}{n^2}\sum_{i<j}\int\left[a_i + b_{n,i} + c_{n,i} + d_{n,i} + e_{n,i}\right]\left[a_j + b_{n,j} + c_{n,j} + d_{n,j} + e_{n,j}\right]\omega$$
$$= \frac{2}{n^2}\sum_{i<j}\int a_i a_j \omega + \frac{2}{n^2}\sum_{i<j}\int (a_i b_{n,j} + a_j b_{n,i})\omega + \cdots .$$

From Hall [24], it is known that

$$\frac{nh^{d/2}}{2}\,\frac{1}{n^2}\sum_{i<j}\int a_i a_j \omega \xrightarrow{law} \frac{1}{2\sqrt{2}}\mathcal{N}(0,\sigma^2), \tag{B.3}$$

$$\sigma^2 = \int \tau^2\omega \int \left[\int K(\mathbf{u})K(\mathbf{u}+\mathbf{v})\,d\mathbf{u}\right]^2 d\mathbf{v}.$$

Therefore, the main term of $J^*_{n,2}$ seems to be of order $O(n^{-1}h^{-d/2})$. We will check it by studying the terms of the expansion of $J^*_{n,2}$ successively.

B.1.1. Study of $T_\alpha \equiv 2n^{-2}\sum_{i<j}\int a_i b_{n,j}\omega$

Note that the expectation of T_α is not zero, because some $\mathbf{Y}_i$ appears inside $b_{n,j}$, for every j. For convenience, set

$$b_{n,j}(\mathbf{u}) = \frac{(-1)}{nh}\sum_{k=1}^{n} b_{n,j,k}(\mathbf{u}), \text{ with}$$

$$\begin{aligned} b_{n,j,k}(\mathbf{u}) \,=\, & (dK)_h(\mathbf{u}-\mathbf{Y}_j)\cdot(\mathbf{1}(\mathbf{Y}_k\leqslant \mathbf{Y}_j)-\mathbf{Y}_j) \\ & -E[(dK)_h(\mathbf{u}-\mathbf{Y}_j)\cdot(\mathbf{1}(\mathbf{Y}_k\leqslant \mathbf{Y}_j)-\mathbf{Y}_j)]. \end{aligned}$$

Moreover,

$$\begin{aligned} T_\alpha &= \left(\frac{-2}{n^3h}\right)\sum_{i<j}\sum_{k=1}^{n}\int a_i b_{n,j,k}\omega \\ &= \left(\frac{-2}{n^3h}\right)\left\{\sum_{i<j}\sum_{k\neq i,k\neq j}^{n}\int a_i b_{n,j,k}\omega + \sum_{i<j}\int a_i b_{n,j,i}\omega + \sum_{i<j}\int a_i b_{n,j,j}\omega\right\} \\ &\equiv T_\alpha^{(1)} + T_\alpha^{(2)} + T_\alpha^{(3)}. \end{aligned}$$

First, let us study $T_\alpha^{(3)}$. Its expectation is zero. Its variance is

$$\begin{aligned} & E[(T_\alpha^{(3)})^2] \\ &= \frac{4}{n^6h^2}\sum_{i_1<j_1}\sum_{i_2<j_2}\int E[a_{i_1}(\mathbf{u}_1)b_{n,j_1,j_1}(\mathbf{u}_1)a_{i_2}(\mathbf{u}_2)b_{n,j_2,j_2}(\mathbf{u}_2)]\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2 \\ &= \frac{4}{n^6h^2}\left\{\sum_{i_1<j_1,i_2=i_1,j_2=j_1} + \sum_{i_1<j_1,i_2=j_1,j_2=i_1}\right\} \equiv V_{\alpha,1}^{(3)} + V_{\alpha,2}^{(3)}. \end{aligned}$$

The first of these terms is

$$\begin{aligned} V_{\alpha,1}^{(3)} \,=\, & \frac{4}{n^6h^2}\sum_{i<j}\int \left\{K_h(\mathbf{u}_1-\mathbf{y}_i) - \int K(\mathbf{v})\tau(\mathbf{u}_1-h\mathbf{v})\,d\mathbf{v}\right\} \\ & \cdot \left\{K_h(\mathbf{u}_2-\mathbf{y}_i) - \int K(\mathbf{v})\tau(\mathbf{u}_2-h\mathbf{v})\,d\mathbf{v}\right\} \end{aligned}$$

$$\cdot \left\{(dK)_h(\mathbf{u}_1 - \mathbf{y}_j)\cdot(1-\mathbf{y}_j) - E[(dK)_h(\mathbf{u}_1-\mathbf{Y})\cdot(1-\mathbf{Y})]\right\}$$
$$\cdot \{(dK)_h(\mathbf{u}_2 - \mathbf{y}_j)\cdot(1-\mathbf{y}_j)$$
$$-E[(dK)_h(\mathbf{u}_2-\mathbf{Y})\cdot(1-\mathbf{Y})]\}\tau(\mathbf{y}_i)\tau(\mathbf{y}_j)\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2\,d\mathbf{y}_i\,d\mathbf{y}_j.$$

The "hardest" term among the latter ones is

$$\frac{4}{n^6h^2}\sum_{i<j}\int K_h(\mathbf{u}_1-\mathbf{y}_i)K_h(\mathbf{u}_2-\mathbf{y}_i)(dK)_h(\mathbf{u}_1-\mathbf{y}_j)\cdot(1-\mathbf{y}_j)\cdot(dK)_h(\mathbf{u}_2-\mathbf{y}_j)$$
$$\cdot(1-\mathbf{y}_j)\cdot\tau(\mathbf{y}_i)\tau(\mathbf{y}_j)\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{y}_i\,\mathbf{y}_j\,d\mathbf{u}_1\,d\mathbf{u}_2$$
$$=\frac{4}{n^6h^{2+d}}\sum_{i<j}\int K(\tilde{\mathbf{y}}_i)K(\tilde{\mathbf{u}}_2+\tilde{\mathbf{y}}_i)(dK)(\tilde{\mathbf{y}}_j)\cdot(1-\mathbf{u}_1+h\tilde{\mathbf{y}}_j)$$
$$\cdot(dK)(\tilde{\mathbf{u}}_2+\tilde{\mathbf{y}}_j)\cdot(1-\mathbf{u}_1+h\tilde{\mathbf{y}}_j)\tau(\mathbf{u}_1-h\tilde{\mathbf{y}}_i)\tau(\mathbf{u}_1-h\tilde{\mathbf{y}}_j)$$
$$\times\omega(\mathbf{u}_1)\omega(\mathbf{u}_1+h\tilde{\mathbf{u}}_2)\,d\tilde{\mathbf{y}}_i\,d\tilde{\mathbf{y}}_j\,d\mathbf{u}_1\,d\tilde{\mathbf{u}}_2.$$

Since K is compactly supported, clearly, we can assume every variable belongs to some compact real subset. Thus, the latter term is of order $n^{-4}h^{-2-d}$. It is $o(n^{-2}h^{-d})$ since nh tends to the infinity when n is large. The seven other terms of $V_{\alpha,1}^{(3)}$ can be dealt similarly. Actually, they are even of a weaker order (we win an extra factor h^d). Moreover,

$$V_{\alpha,2}^{(3)} = \frac{4}{n^6h^2}\sum_{i<j}\int\left\{K_h(\mathbf{u}_1-\mathbf{y}_i)-\int K(\mathbf{v})\tau(\mathbf{u}_1-h\mathbf{v})\,d\mathbf{v}\right\}$$
$$\cdot\left\{K_h(\mathbf{u}_2-\mathbf{y}_j)-\int K(\mathbf{v})\tau(\mathbf{u}_2-h\mathbf{v})\,d\mathbf{v}\right\}$$
$$\cdot\{(dK)_h(\mathbf{u}_1-\mathbf{y}_i)\cdot(1-\mathbf{y}_i)-E[(dK)_h(\mathbf{u}_1-\mathbf{Y})\cdot(1-\mathbf{Y})]\}$$
$$\cdot\{(dK)_h(\mathbf{u}_2-\mathbf{y}_j)\cdot(1-\mathbf{y}_j)$$
$$-E[(dK)_h(\mathbf{u}_2-\mathbf{Y})\cdot(1-\mathbf{Y})]\}\,\tau(\mathbf{y}_i)\tau(\mathbf{y}_j)\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2\,d\mathbf{y}_i\,d\mathbf{y}_j.$$

Working exactly like $V_{\alpha,1}^{(3)}$, we can show $V_{\alpha,2}^{(3)} = O(n^{-4}h^{-2-d})$. Thus, we have proved that

$$T_\alpha^{(3)} = o_P\left(\frac{1}{nh^{d/2}}\right).$$

Second, let us study $T_\alpha^{(2)}$. Recall that

$$T_\alpha^{(2)} = \left(\frac{-2}{n^3h}\right)\sum_{i<j}\int a_i(\mathbf{u})(dK)_h(\mathbf{u}-\mathbf{Y}_j)\cdot(\mathbf{1}(\mathbf{Y}_i\leqslant\mathbf{Y}_j)-\mathbf{Y}_j)\omega(\mathbf{u})\,d\mathbf{u}.$$

The expectation of this term is not zero. By applying Eq. (A.1), we obtain

$$E[T_\alpha^{(2)}] = \frac{(-1)}{nh}\left(1-\frac{1}{n}\right)\int E\left[(K_h(\mathbf{u}-\mathbf{Y}_1)-E[K_h(\mathbf{u}-\mathbf{Y})])\right.$$
$$\left.\cdot\left(\tau(\mathbf{u})\sum_{r=1}^d K_r\left(\frac{u_r-Y_{1,r}}{h}\right)+O(h)\phi(\mathbf{u})\right)\right]\omega(\mathbf{u})\,du$$

$$= \frac{(-1)}{nh}\left(1-\frac{1}{n}\right)\sum_{r=1}^{d}\left\{\int (K_h(\mathbf{u}-\mathbf{y})-E[K_h(\mathbf{u}-\mathbf{Y})])K_r\left(\frac{u_r-y_r}{h}\right)\right.$$
$$\left.\cdot\ \tau(\mathbf{y})\tau(\mathbf{u})\omega(\mathbf{u})\,d\mathbf{u}\,d\mathbf{y}+O(h)\right\} = \frac{(-1)}{nh}\sum_{r=1}^{d}\int K_r^2\cdot\int\tau^2\omega + O(n^{-1}).$$

Note we have used the fact that the density of Y_r is uniform on [0, 1].

The order of the expectation of $T_\alpha^{(2)}$ is then $(nh)^{-1}$. Unfortunately, it is not $o(1/nh^{d/2})$ when $d=2$. Nonetheless, its variance will be small enough so that we can consider this term is reduced to its expectation. Indeed,

$$\begin{aligned}
&Var(T_\alpha^{(2)})\\
&= \frac{4}{n^6h^2}\sum_{i_1<j_1,i_2<j_2}\int E[a_{i_1}(\mathbf{u}_1)\cdot(dK)_h(\mathbf{u}_1-\mathbf{Y}_{j_1})\cdot(\mathbf{1}(\mathbf{Y}_{i_1}\leqslant\mathbf{Y}_{j_1})-\mathbf{Y}_{j_1})\\
&\quad\cdot a_{i_2}(\mathbf{u}_2)\cdot(dK)_h(\mathbf{u}_2-\mathbf{Y}_{j_2})\cdot(\mathbf{1}(\mathbf{Y}_{i_2}\leqslant\mathbf{Y}_{j_2})-\mathbf{Y}_{j_2})\\
&\quad -E[a_{i_1}(\mathbf{u}_1)b_{n,j_1,i_1}(\mathbf{u}_1)]\cdot E[a_{i_2}(\mathbf{u}_2)b_{n,j_2,i_2}(\mathbf{u}_2)]]\\
&\quad\times\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2\\
&= \frac{4}{n^6h^2}\left\{\sum_{i_1<j_1,i_2<j_2,i_1=i_2}+\sum_{i_1<j_1,i_2<j_2,i_1=j_2}+\sum_{i_1<j_1,i_2<j_2,j_1=i_2}+\sum_{i_1<j_1,i_2<j_2,j_1=j_2}\right\}\\
&\equiv V_{\alpha,1}^{(2)}+V_{\alpha,2}^{(2)}+V_{\alpha,3}^{(2)}+V_{\alpha,4}^{(2)}.
\end{aligned}$$

Let us study the first of the previous terms.

$$\begin{aligned}
V_{\alpha,1}^{(2)} &= \frac{4}{n^6h^2}\sum_{i_1<j_1,i_1<j_2}\int\{K_h(\mathbf{u}_1-\mathbf{y}_{i_1})-\int K(\mathbf{t})\tau(\mathbf{u}_1-h\mathbf{t})\,d\mathbf{t}\}\\
&\quad\cdot\left\{K_h(\mathbf{u}_2-\mathbf{y}_{i_1})-\int K(\mathbf{t})\tau(\mathbf{u}_2-h\mathbf{t})\,d\mathbf{t}\right\}\\
&\quad\cdot\{(dK)(\tilde{\mathbf{y}}_{j_1})\cdot(\mathbf{1}(\mathbf{y}_{i_1}\leqslant\mathbf{u}_1-h\tilde{\mathbf{y}}_{j_1})-(\mathbf{u}_1-h\tilde{\mathbf{y}}_{j_1}))\}\\
&\quad\cdot\{(dK)(\tilde{\mathbf{y}}_{j_2})\cdot(\mathbf{1}(\mathbf{y}_{i_1}\leqslant\mathbf{u}_2-h\tilde{\mathbf{y}}_{j_2})-(\mathbf{u}_2-h\tilde{\mathbf{y}}_{j_2}))\}\\
&\quad\times\tau(\mathbf{y}_{i_1})\tau(\mathbf{u}_1-h\tilde{\mathbf{y}}_{j_1})\tau(\mathbf{u}_2-h\tilde{\mathbf{y}}_{j_2})\\
&\quad\cdot\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2\,d\tilde{\mathbf{y}}_{j_1}d\tilde{\mathbf{y}}_{j_2}+O\left(\frac{1}{n^6h^2}\cdot\frac{n^2}{h^d}\right).
\end{aligned}$$

The remainder term corresponds to the case $i_1=i_2$, $j_1=j_2$. The main previous term of $V_{\alpha,1}^{(2)}$ can be expressed as a sum of four terms. The first one involves the factor $K_h(\mathbf{u}_1-\mathbf{y}_{i_1}).K_h(\mathbf{u}_2-\mathbf{y}_{i_1})$. The second (resp. the third) one involves the factor $K_h(\mathbf{u}_1-\mathbf{y}_{i_1})$ (resp. $K_h(\mathbf{u}_2-\mathbf{y}_{i_1})$) only. The last one has no such factor (viz no more denominators h^{-d}). If necessary, we can set one or two changes of variables among $\tilde{\mathbf{y}}_{i_1}=(\mathbf{u}_1-\mathbf{y}_{i_1})/h$, $\tilde{\mathbf{y}}_{i_1}=(\mathbf{u}_2-\mathbf{y}_{i_1})/h$ or $\tilde{\mathbf{u}}_2=(\mathbf{u}_2-\mathbf{u}_1)/h$. It allows to clear all the factors h^{-d}. Thus we get easily,

$$V_{\alpha,1}^{(2)} = O\left(\frac{1}{n^6h^2}\cdot n^3\right) + O\left(\frac{1}{n^4h^{2+d}}\right) = o\left(\frac{1}{n^2h^d}\right), \tag{B.4}$$

since nh tends to the infinity. The three other terms $V_{\alpha,l}^{(2)}, l = 2, 3, 4$ can be dealt similarly, because there exist always four free variables ($\mathbf{u}_1, \mathbf{u}_2$ and three ones among i_1, i_2, j_1, j_2) that can be used for some change of variables. Like previously, all the factors h^{-d} disappear. To conclude, $V_\alpha^{(2)} = O(1/(n^3h^2) + 1/(n^4h^{2+d}))$, and

$$T_\alpha^{(2)} = ET_\alpha^{(2)} + o_P\left(\frac{1}{nh^{d/2}}\right) = \frac{(-1)}{nh}\int \tau^2\omega \cdot \sum_{r=1}^d \int K_r^2 + o_P\left(\frac{1}{nh^{d/2}}\right).$$

Now, let us deal with $T_\alpha^{(1)}$. Recall that

$$T_\alpha^{(1)} = \frac{(-2)}{n^3h}\sum_{i<j}\sum_{k,k\neq i,k\neq j}\int a_i b_{njk}\omega.$$

Clearly, $T_\alpha^{(1)}$ is centered. Moreover, its variance is

$$E[\left(T_\alpha^{(1)}\right)^2] = \frac{4}{n^6h^2}\sum_{i_1<j_1}\sum_{i_2<j_2}\sum_{k_1\neq i_1,j_1}\sum_{k_2\neq i_2,j_2} \times E\int \left(a_{i_1}b_{n,j_1k_1}\right)(\mathbf{u}_1)\cdot(a_{i_2}b_{n,j_2k_2})(\mathbf{u}_2)\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2.$$

A lot of the latter terms are zero. The only nonzero terms appear in the following cases: ($k_1 = i_2$ and $k_2 = i_1$), ($k_1 = k_2$ and $i_1 = i_2$), ($k_1 = i_2$, $k_2 = j_1$ and $i_1 = j_2$), ($k_1 = j_2$, $k_2 = i_1$ and $i_2 = j_1$), ($k_1 = j_2$, $k_2 = j_1$ and $i_1 = i_2$), ($k_1 = k_2$, $i_1 = j_2$ and $i_2 = j_1$).

Thus, the variance of $T_\alpha^{(1)}$ is the sum of six terms, denoted by $V_{\alpha,l}^{(1)}$, $l = 1,\dots,6$. Assuming that there are no other equalities except $k_1 = i_2$ and $k_2 = i_1$, the first variance term is

$$\begin{aligned} V_{\alpha,1}^{(1)} = \frac{4}{n^4h^2}\sum_{i_1,i_2,i_1\neq i_2}\int &\left\{K_h(\mathbf{u}_1 - \mathbf{y}_{i_1}) - \int K(\mathbf{t})\tau(\mathbf{u}_1 - h\mathbf{t})\,d\mathbf{t}\right\}\\ &\cdot\left\{K_h(\mathbf{u}_2 - \mathbf{y}_{i_2}) - \int K(\mathbf{t})\tau(\mathbf{u}_2 - h\mathbf{t})\,d\mathbf{t}\right\}\\ &\cdot\left\{\tau(\mathbf{u}_1)\sum_{r=1}^d K_r\left(\frac{u_{1r} - y_{i_2r}}{h}\right) + O(h)\phi(\mathbf{u}_1)\right\}\\ &\cdot\left\{\tau(\mathbf{u}_2)\sum_{s=1}^d K_r\left(\frac{u_{2s} - y_{i_1s}}{h}\right) + O(h)\phi(\mathbf{u}_2)\right\}\tau(\mathbf{y}_{i_2})\tau(\mathbf{y}_{i_1})\\ &\times\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{y}_{i_1}\,d\mathbf{y}_{i_2}\,d\mathbf{u}_1\,d\mathbf{u}_2. \end{aligned}$$

This sum can be split into 16 other terms. The main one is

$$\frac{4}{n^4h^2}\sum_{r,s}\sum_{i_1,i_2}\int K_h(\mathbf{u}_1 - \mathbf{y}_{i_1})K_h(\mathbf{u}_2 - \mathbf{y}_{i_2})\tau(\mathbf{u}_1)K_r\left(\frac{u_{1r} - y_{i_2r}}{h}\right)$$

$$\cdot\tau(\mathbf{u}_2)K_r\left(\frac{u_{2s}-y_{i_1 s}}{h}\right)\tau(\mathbf{y}_{i_2})\tau(\mathbf{y}_{i_1})\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{y}_{i_1}\,d\mathbf{y}_{i_2}\,d\mathbf{u}_1\,d\mathbf{u}_2$$
$$=\frac{4}{n^4h^2}\sum_{r,s}\sum_{i_1,i_2}\int K(\tilde{\mathbf{y}}_{i_1})K(\tilde{\mathbf{y}}_{i_2})\tau(\mathbf{u}_1)K_r\left(\frac{u_{1r}-u_{2r}+h\tilde{y}_{i_2 r}}{h}\right)$$
$$\cdot\tau(\mathbf{u}_2)K_r\left(\frac{u_{2s}-u_{1s}+h\tilde{y}_{i_1 s}}{h}\right)\tau(\mathbf{u}_2-h\tilde{\mathbf{y}}_{i_2})\tau(\mathbf{u}_1-h\tilde{\mathbf{y}}_{i_1})\omega(\mathbf{u}_1)$$
$$\times\omega(\mathbf{u}_2)\,d\tilde{\mathbf{y}}_{i_1}\,d\tilde{\mathbf{y}}_{i_2}\,d\mathbf{u}_1\,d\mathbf{u}_2$$

If $r\neq s$, set the change of variables $\tilde{u}_{1r}=(u_{1r}-u_{2r})/h$ and $\tilde{u}_{1s}=(u_{1s}-u_{2s})/h$ to get an extra factor h^2. If $r=s$, we obtain only one factor h. Thus, the previous variance term is $O(n^{-4}h^{-2}\cdot n^2\cdot h)=O(n^{-2}h^{-1})$. This is $o(n^{-2}h^{-d})$.

Imagine we have some other equalities between the indices i_1, i_2, j_1, j_2, k_1 and k_2 in $V_\alpha^{(1)}$. For instance $j_1=j_2$. This would not be a problem because we gain a factor n and we can always remove the annoying factor h^{-d} by some change of variables with respect to $\mathbf{u}_1$, $\mathbf{u}_2$ and the variables $\mathbf{y}$. Thus, we get the order $O(n^{-6}h^{-2}\cdot n^3)=o(n^{-2}h^{-d})$.

The 15 other terms that are coming from the expansion of $V_{\alpha,1}^{(1)}$ can be dealt similarly. Thus, $V_{\alpha,1}^{(1)}=o(n^{-2}h^{-d})$.

Another critical term should be

$$V_{\alpha,2}^{(1)}=\frac{4}{n^6h^2}\sum_{i<j_1}\sum_{i<j_2}\sum_{k,k\neq i,j_1,j_2}\int E[(a_ib_{n,j_1,k})(\mathbf{u}_1)(a_ib_{n,j_2,k})(\mathbf{u}_2)]$$
$$\times\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2.$$

Since k is different from all other indices, this equals

$$\frac{4}{n^6h^2}\sum_{i<j_1}\sum_{i<j_2}\sum_{k}\int$$
$$\left\{K_h(\mathbf{u}_1-\mathbf{y}_i)-\int K(\mathbf{t})\tau(\mathbf{u}_1-h\mathbf{t})\,d\mathbf{t}\right\}\cdot\left\{K_h(\mathbf{u}_2-\mathbf{y}_i)-\int K(\mathbf{t})\tau(\mathbf{u}_2-h\mathbf{t})\,d\mathbf{t}\right\}$$
$$\cdot(dK)_h(\mathbf{u}_1-\mathbf{y}_{j_1})\cdot(\mathbf{1}(\mathbf{y}_k\leqslant\mathbf{y}_{j_1})-\mathbf{y}_{j_1})\cdot(dK)_h(\mathbf{u}_2-\mathbf{y}_{j_2})\cdot(\mathbf{1}(\mathbf{y}_k\leqslant\mathbf{y}_{j_2})-\mathbf{y}_{j_2})$$
$$\cdot\tau(\mathbf{y}_i)\tau(\mathbf{y}_{j_2})\tau(\mathbf{y}_{j_1})\tau(\mathbf{y}_k)\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{y}_i\,d\mathbf{y}_{j_1}\,d\mathbf{y}_{j_2}\,d\mathbf{y}_k\,d\mathbf{u}_1\,d\mathbf{u}_2$$
$$=\frac{4}{n^4h^2}\sum_{i}\sum_{k}\int\left\{K(\tilde{\mathbf{y}}_i)-h^d\int K(\mathbf{t})\tau(\mathbf{u}_1-h\mathbf{t})\,d\mathbf{t}\right\}$$
$$\cdot\left\{K_h(\mathbf{u}_2-\mathbf{u}_1+h\tilde{\mathbf{y}}_i)-\int K(\mathbf{t})\tau(\mathbf{u}_2-h\mathbf{t})\,d\mathbf{t}\right\}$$
$$\cdot\left\{\tau(\mathbf{u}_1)\sum_{r=1}^{d}K_r\left(\frac{u_{1r}-y_{k,r}}{h}\right)+O(h)\phi(\mathbf{u}_1)\right\}$$
$$\cdot\left\{\tau(\mathbf{u}_2)\sum_{s=1}^{d}K_s\left(\frac{u_{2s}-y_{k,s}}{h}\right)+O(h)\phi(\mathbf{u}_2)\right\}$$
$$\cdot\tau(\mathbf{u}_1-h\tilde{\mathbf{y}}_i)\tau(\mathbf{y}_k)\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\tilde{\mathbf{y}}_i\,d\mathbf{y}_k\,d\mathbf{u}_1\,d\mathbf{u}_2.$$

We have assumed there are no additional equalities between i, j_1, j_2. By setting $h\tilde{\mathbf{u}}_2=\mathbf{u}_2-\mathbf{u}_1$, we remove the factor h^{-d}. Moreover, by setting $h\tilde{u}_{1r}=u_{1r}-y_{kr}$, we get an extra factor h. Thus, the term if of order $O(n^{-4}h^{-2}\cdot n^2\cdot h)=o(n^{-2}h^{-d})$. When there are some other equalities between the other indices $i,\, j_1$ and j_2, we gain a factor n even if we lose eventually a factor h^d. In every case, the order of these terms is lower than $n^{-2}h^{-d}$. Therefore, $V^{(1)}_{\alpha,2}=o(n^{-2}h^{-d})$.

All the other terms $V^{(1)}_{\alpha,l}$, $l=3,\dots,6$ are simpler. Indeed, with respect to $V^{(1)}_{\alpha,1}$, there is an additional equality between the indices. At the opposite, it should be harder to remove all the four terms h^{-d}. Actually, it can be done at least three times over four, because there are always two free variables $\mathbf{y}$ (at least), and we have $\mathbf{u}_1$ or $\mathbf{u}_2$ at our disposal too. Thus, all these terms are $O(n^{-6}h^{-2}\cdot n^3h^{-d})=o(n^{-2}h^{-d})$ since nh^2 tends to the infinity.

Therefore, the variance of $T^{(1)}_\alpha$ is negligible with respect to $n^{-2}h^{-d}$ and $T^{(1)}_\alpha = o_P(1/(nh^{d/2}))$. To conclude,

$$T_\alpha=\frac{(-1)}{nh}\int\tau^2\omega\cdot\sum_{r=1}^d\int K_r^2+o_P\left(\frac{1}{nh^{d/2}}\right). \tag{B.5}$$

B.1.2. Study of $T_\beta=2n^{-2}\sum_{i<j}b_{n,i}b_{n,j}\omega$

Note that

$$\begin{aligned}T_\beta = \frac{2}{n^4h^2}\sum_{i<j}\sum_{k,k'}\int &\{(dK)_h(\mathbf{u}-\mathbf{Y}_i)\cdot(\mathbf{1}(\mathbf{Y}_k\leqslant\mathbf{Y}_i)-\mathbf{Y}_i)\\ &-E[(dK)_h(\mathbf{u}-\mathbf{Y}_i)\cdot(\mathbf{1}(\mathbf{Y}_k\leqslant\mathbf{Y}_i)-\mathbf{Y}_i)]\}\\ &\cdot\{(dK)_h(\mathbf{u}-\mathbf{Y}_j)\cdot(\mathbf{1}(\mathbf{Y}_{k'}\leqslant\mathbf{Y}_j)-\mathbf{Y}_j)-E[(dK)_h(\mathbf{u}-\mathbf{Y}_j)\\ &\cdot(\mathbf{1}(\mathbf{Y}_{k'}\leqslant\mathbf{Y}_j)-\mathbf{Y}_j)]\}\omega(\mathbf{u})\,d\mathbf{u}.\end{aligned}$$

The latter term needs to be considered with respect to the potential number of equalities between the indices $i,\, j, k, k'$.

<u>No equalities between $i,\, j, k, k' : T^{(1)}_\beta$</u>

Thus, the expectation of the corresponding term is zero. Moreover, its variance is

$$\begin{aligned}&\frac{4}{n^8h^4}\sum_{i_1<j_1}\sum_{i_2<j_2}\sum_{k_1\neq k_1'\neq i_1,j_1}\sum_{k_2\neq k_2'\neq i_2,j_2}E\int(dK)_h(\mathbf{u}_1-\mathbf{Y}_{i_1})\cdot(\mathbf{1}(\mathbf{Y}_{k_1}\leqslant\mathbf{Y}_{i_1})-\mathbf{Y}_{i_1})\\ &\cdot(dK)_h(\mathbf{u}_1-\mathbf{Y}_{j_1})\cdot(\mathbf{1}(\mathbf{Y}_{k_1'}\leqslant\mathbf{Y}_{j_1})-\mathbf{Y}_{j_1})\cdot(dK)_h(\mathbf{u}_2-\mathbf{Y}_{i_2})\cdot(\mathbf{1}(\mathbf{Y}_{k_2}\leqslant\mathbf{Y}_{i_2})-\mathbf{Y}_{i_2})\\ &\cdot(dK)_h(\mathbf{u}_2-\mathbf{Y}_{j_2})\cdot(\mathbf{1}(\mathbf{Y}_{k_2'}\leqslant\mathbf{Y}_{j_2})-\mathbf{Y}_{j_2})\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2.\end{aligned}$$

The expectations are zero, except if there are some equalities between our eight indices. More precisely, the equalities have to concern all the indices k_1, k_1', k_2, k_2', otherwise the corresponding term is zero. This provides the following cases:

- $k_1=k_2$ and $k_1'=k_2'$,
- $k_1=k_2'$ and $k_1'=k_2$,
- $k_1=i_2, k_2=i_1, k_1'=k_2'$, or their variations,
- $k_1=i_2, k_1'=j_2, k_2=i_1, k_2'=j_1$, or their variations.

The corresponding variances are called $V^{(1)}_{\beta,j}$, $j = 1, \ldots, 4$. Let us deal with the first configuration. It provides the "variance-type" term

$$\begin{aligned}
&\frac{4}{n^8h^4}\sum_{i_1<j_1}\sum_{i_2<j_2}\sum_{k\neq k'} E\int (dK)_h(\mathbf{u}_1-\mathbf{Y}_{i_1})\cdot(\mathbf{1}(\mathbf{Y}_k\leqslant\mathbf{Y}_{i_1})-\mathbf{Y}_{i_1})(dK)_h(\mathbf{u}_1-\mathbf{Y}_{j_1})\\
&\quad\cdot(\mathbf{1}(\mathbf{Y}_{k'}\leqslant\mathbf{Y}_{j_1})-\mathbf{Y}_{j_1})(dK)_h(\mathbf{u}_2-\mathbf{Y}_{i_2})\cdot(\mathbf{1}(\mathbf{Y}_k\leqslant\mathbf{Y}_{i_2})-\mathbf{Y}_{i_2})\\
&\quad\times(dK)_h(\mathbf{u}_2-\mathbf{Y}_{j_2})\cdot(\mathbf{1}(\mathbf{Y}_{k'}\leqslant\mathbf{Y}_{j_2})-\mathbf{Y}_{j_2})\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2\\
&\quad=\frac{4}{n^8h^4}\sum_{i_1<j_1}\sum_{i_2<j_2}\sum_{k\neq k'} E\int\left\{\tau(\mathbf{u}_1)\sum_{r=1}^{d}K_r\left(\frac{u_{1r}-Y_{kr}}{h}\right)+O(h)\phi(\mathbf{u}_1)\right\}\\
&\quad\cdot\left\{\tau(\mathbf{u}_1)\sum_{r'=1}^{d}K_{r'}\left(\frac{u_{1r'}-Y_{k'r'}}{h}\right)+O(h)\phi(\mathbf{u}_1)\right\}\\
&\quad\cdot\left\{\tau(\mathbf{u}_2)\sum_{s=1}^{d}K_s\left(\frac{u_{2s}-Y_{ks}}{h}\right)+O(h)\phi(\mathbf{u}_2)\right\}\\
&\quad\cdot\left\{\tau(\mathbf{u}_2)\sum_{s'=1}^{d}K_{s'}\left(\frac{u_{2s'}-Y_{k's'}}{h}\right)+O(h)\phi(\mathbf{u}_2)\right\}\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2.
\end{aligned}$$

The main member of the previous expansion is

$$\begin{aligned}
&\frac{4}{n^8h^4}\sum_{i_1<j_1}\sum_{i_2<j_2}\sum_{k\neq k'}\sum_{r,r',s,s'} E\int\tau^2(\mathbf{u}_1)K_r\left(\frac{u_{1r}-Y_{kr}}{h}\right)K_{r'}\left(\frac{u_{1r'}-Y_{k'r'}}{h}\right)\\
&\quad\cdot\tau^2(\mathbf{u}_2)K_s\left(\frac{u_{2s}-Y_{ks}}{h}\right)K_{s'}\left(\frac{u_{2s'}-Y_{k's'}}{h}\right)\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2.
\end{aligned}$$

The "worse" situation occurs when $r = s$ and $r' = s'$. In this case, we get

$$\begin{aligned}
&\frac{4}{n^8h^4}\sum_{i_1<j_1}\sum_{i_2<j_2}\sum_{k\neq k'}\int\tau^2(\mathbf{u}_1)K_r\left(\frac{u_{1r}-y_{kr}}{h}\right)K_{r'}\left(\frac{u_{1r'}-y_{k'r'}}{h}\right)\tau^2(\mathbf{u}_2)\\
&\quad\cdot K_r\left(\frac{u_{2r}-y_{kr}}{h}\right)K_{r'}\left(\frac{u_{2r'}-y_{k'r'}}{h}\right)\tau_r(y_{kr})\tau_{r'}(y_{k'r'})\\
&\quad\times\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2\,dy_{kr}\,dy_{k'r'}\\
&=\frac{4}{n^8h^2}\sum_{i_1<j_1}\sum_{i_2<j_2}\sum_{k\neq k'}\int\tau^2(\mathbf{u}_1)K_r(\tilde{y}_{kr})K_{r'}(\tilde{y}_{k'r'})\tau^2(\mathbf{u}_2)K_r\left(\frac{u_{2r}-u_{1r}+h\tilde{y}_{kr}}{h}\right)\\
&\quad\cdot K_{s'}\left(\frac{u_{2r'}-u_{1r'}+h\tilde{y}_{k'r'}}{h}\right)\tau_r(u_{1r}-h\tilde{y}_{kr})\tau_{r'}(u_{1r'}-h\tilde{y}_{k'r'})\\
&\quad\times\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2\,d\tilde{y}_{kr}\,d\tilde{y}_{k'r'}.
\end{aligned}$$

By setting $h\tilde{u}_{2r} = u_{2r} - u_{1r}$, we get an extra factor h. The previous variance term is then $O(n^{-8}h^{-1} \cdot n^6) = o(n^2 h^{-d})$. Thus, $V_{\beta,1}^{(1)} = o(n^{-2}h^{-d})$.

The variance term $V_{\beta,2}^{(1)}$ corresponding to the case $k_1 = k_2'$ and $k_1' = k_2$ can be dealt exactly as $V_{\beta,1}^{(1)}$. The third one, $V_{\beta,3}^{(1)}$, is

$$\frac{4}{n^8h^4}\sum_{i_1<j_1}\sum_{i_2<j_2}\sum_{k} E\int (dK)_h(\mathbf{u}_1 - \mathbf{Y}_{i_1}) \cdot (\mathbf{1}(\mathbf{Y}_{i_2} \leqslant \mathbf{Y}_{i_1}) - \mathbf{Y}_{i_1})(dK)_h(\mathbf{u}_1 - \mathbf{Y}_{j_1}) \cdot (\mathbf{1}(\mathbf{Y}_k \leqslant \mathbf{Y}_{j_1}) - \mathbf{Y}_{j_1})(dK)_h(\mathbf{u}_2 - \mathbf{Y}_{i_2}) \cdot (\mathbf{1}(\mathbf{Y}_{i_1} \leqslant \mathbf{Y}_{i_2}) - \mathbf{Y}_{i_2}) \cdot (dK)_h(\mathbf{u}_2 - \mathbf{Y}_{j_2}) \cdot (\mathbf{1}(\mathbf{Y}_k \leqslant \mathbf{Y}_{j_2}) - \mathbf{Y}_{j_2})\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\, d\mathbf{u}_1\, d\mathbf{u}_2.$$

It can be bounded easily: $V_{\beta,3}^{(1)} = O(1/(n^8h^4) \cdot n^5) = o(1/(n^2h^d))$, since nh^2 tends to the infinity when n is large.

$V_{\beta,4}^{(1)}$ and the other variance terms that are obtained by adding some equalities between the indices can be dealt similarly. All of them provide negligible terms. To conclude,

$$T_\beta^{(1)} = o_P\left(\frac{1}{nh^{d/2}}\right).$$

Only the equality $k = k'$: $T_\beta^{(2)}$

We get

$$T_\beta^{(2)} = \frac{2}{n^4h^2}\sum_{i<j}\sum_{k\neq i,j}\int (dK)_h(\mathbf{u} - \mathbf{Y}_i) \cdot (\mathbf{1}(\mathbf{Y}_k \leqslant \mathbf{Y}_i) - \mathbf{Y}_i)(dK)_h(\mathbf{u} - \mathbf{Y}_j) \cdot (\mathbf{1}(\mathbf{Y}_k \leqslant \mathbf{Y}_j) - \mathbf{Y}_j)\omega(\mathbf{u})\, d\mathbf{u}.$$

Its expectation is nonzero. More precisely,

$$\begin{aligned} ET_\beta^{(2)} &= \frac{2}{n^4h^2}\sum_{i<j}\sum_{k\neq i,j} E\int \left\{\tau(\mathbf{u})\sum_{r=1}^{d} K_r\left(\frac{u_r - Y_{kr}}{h}\right) + O(h)\phi(\mathbf{u})\right\} \\ &\quad \cdot \left\{\tau(\mathbf{u})\sum_{s=1}^{d} K_s\left(\frac{u_s - Y_{ks}}{h}\right) + O(h)\phi(\mathbf{u})\right\}\omega(\mathbf{u})\, d\mathbf{u} \\ &= \frac{2}{n^4h^2}\left\{\sum_{r\neq s}\sum_{i<j}\sum_{k\neq i,j} + \sum_{r=s}\sum_{i<j}\sum_{k\neq i,j}\right\} + O(n^{-2}) \\ &\equiv E_{\beta,1}^{(2)} + E_{\beta,2}^{(2)} + O(n^{-2}). \end{aligned}$$

By setting $h\tilde{y}_{kr} = u_r - y_{kr}$ and $h\tilde{y}_{ks} = u_s - y_{ks}$, we get easily

$$E_{\beta,1}^{(2)} = O\left(\frac{1}{n^4h^2} \cdot n^3 \cdot h^2\right) = O(n^{-2}).$$

Concerning $E^{(2)}_{\beta,1}$, one change of variables only is possible. It provides

$$E^{(2)}_{\beta,2} = \frac{2}{n^4h^2}\sum_{i<j}\sum_{k\neq i,j}\sum_{r=1}^{d}\int \tau^2(\mathbf{u})K_r^2\left(\frac{u_r-y_{kr}}{h}\right)\omega(\mathbf{u})\mathbf{1}(y_{kr}\in[0,1])\,d\mathbf{u}\,dy_{kr}$$
$$= \frac{2}{n^4h}\cdot\frac{n(n-1)}{2}\cdot(n-2)\left\{\sum_{r=1}^{d}\int K_r^2\cdot\int\tau^2\omega\right\}$$
$$= \frac{1}{nh}\sum_r\int K_r^2\int\tau^2\omega + o(\frac{1}{nh}),$$

for n sufficiently large. Therefore, the expectation of $T^{(2)}_\beta$ is not $o(n^{-1}h^{-d/2})$ (in the case $d=2$). Let us deal now with its variance. To lighten the notations, we set

$$e_0(\mathbf{u}) = E[(dK)_h(\mathbf{u}-\mathbf{Y}_1)\cdot(\mathbf{1}(\mathbf{Y}_3\leqslant\mathbf{Y}_1)-\mathbf{Y}_1)\cdot(dK)_h(\mathbf{u}-\mathbf{Y}_2)\cdot(\mathbf{1}(\mathbf{Y}_3\leqslant\mathbf{Y}_2)-\mathbf{Y}_2)].$$

Therefore,

$$Var(T^{(2)}_\beta) = \frac{4}{n^8h^4}\sum_{i_1<j_1,i_2<j_2}\sum_{k_1\neq i_1,j_1}\sum_{k_2\neq i_2,j_2} E\int\{(dK)_h(\mathbf{u}_1-\mathbf{Y}_{i_1})\cdot(\mathbf{1}(\mathbf{Y}_{k_1}\leqslant\mathbf{Y}_{i_1})-\mathbf{Y}_{i_1})\cdot(dK)_h(\mathbf{u}_1-\mathbf{Y}_{j_1})\cdot(\mathbf{1}(\mathbf{Y}_{k_1}\leqslant\mathbf{Y}_{j_1})-\mathbf{Y}_{j_1})-e_0(\mathbf{u}_1)\}\cdot\{(dK)_h(\mathbf{u}_2-\mathbf{Y}_{i_2})\cdot(\mathbf{1}(\mathbf{Y}_{k_2}\leqslant\mathbf{Y}_{i_2})-\mathbf{Y}_{i_2})\cdot(dK)_h(\mathbf{u}_2-\mathbf{Y}_{j_2})\cdot(\mathbf{1}(\mathbf{Y}_{k_2}\leqslant\mathbf{Y}_{j_2})-\mathbf{Y}_{j_2})-e_0(\mathbf{u}_2)\})\times\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2.$$

When there are no equalities between the indices $i_1, j_1, k_1, i_2, j_2, k_2$, the corresponding expectation is zero. At the opposite, there could be one, two or three equalities between them. In every case, it is always possible to make some changes of variables with respect to $\mathbf{y}_{i_1}$ and $\mathbf{y}_{j_1}$. Moreover, it is possible to set $h\tilde{\mathbf{u}}_2 = \mathbf{u}_2 - \mathbf{u}_1$, as previously. Thus, it is easy to check that

$$Var(T^{(2)}_\beta) = O\left(\frac{1}{n^8h^4}\cdot(n^5+n^4h^{-d})\right) = o(n^{-2}h^{-d}).$$

Thus,

$$T^{(2)}_\beta = \frac{1}{nh}\sum_r\int K_r^2\int\tau^2\omega + o_P(n^{-1}h^{-d/2}).$$

Only the equality $k=i$ or j (or $k'=i$ or j): $T^{(3)}_\beta$

The expectation is zero and the variance can be dealt exactly as in the latter case.

Two equalities, or more, between the indices: $T^{(4)}_\beta$

To fix the ideas, imagine there are two equalities between our four indices. It means $i=k$ and $j=k'$, or the reverse. It is obvious to bound the expectation of $T^{(4)}_\beta$ by

$O(n^{-4}h^{-2}\cdot n^2)=o(n^{-1}h^{-d/2})$. Moreover, the variance is clearly $O(n^{-8}h^{-4}\cdot n^4\cdot h^{-d})$, by the same calculations as previously. Thus, $T_\beta^{(4)}$ is negligible with respect to $n^{-1}h^{-d/2}$, in probability.

To conclude,

$$T_\beta=\frac{1}{nh}\sum_r\int K_r^2\int\tau^2\omega+o_P(n^{-1}h^{-d/2}). \tag{B.6}$$

B.1.3. Study of $2n^{-2}\sum_{i<j}a_ic_{n,j}\omega$ *and* $2n^{-2}\sum_{i<j}a_id_{n,j}\omega$

To deal with these two terms simultaneously, denote

$$\begin{aligned}T_{\gamma,m} = &\frac{2}{n^{2+m}h^m}\sum_{i<j}\sum_{k_1,\ldots,k_m}\int\{K_h(\mathbf{u}-\mathbf{Y}_i)-EK_h(\mathbf{u}-\mathbf{Y}_i)\}\\ &\cdot\{(d^mK)_h(\mathbf{u}-\mathbf{Y}_j)\cdot(\mathbf{1}(\mathbf{Y}_{k_1}\leqslant\mathbf{Y}_j)-\mathbf{Y}_j)\cdots(\mathbf{1}(\mathbf{Y}_{k_m}\leqslant\mathbf{Y}_j)-\mathbf{Y}_j)\\ &-E[(d^mK)_h(\mathbf{u}-\mathbf{Y}_j)\cdot(\mathbf{1}(\mathbf{Y}_{k_1}\leqslant\mathbf{Y}_j)-\mathbf{Y}_j)\cdots(\mathbf{1}(\mathbf{Y}_{k_m}\leqslant\mathbf{Y}_j)]\}\omega(\mathbf{u})\,d\mathbf{u},\end{aligned}$$

for $m=2,3$. All the summands are centered, except when there are some equalities involving all the indices $k_1,\ldots,k_m$ and i (at least). By splitting $T_{\gamma,m}$, we get several terms. If all the previous indices $i,j,k_1,\ldots,k_m$ are different from each other, the expectation is zero and the variance is

$$\begin{aligned}V_{\gamma,m}^{(1)}=&\frac{4}{n^{4+2m}h^{2m}}\sum_{i_1<j_1}\sum_{i_2<j_2}\sum_{k_1,\ldots,k_m}\sum_{k'_1,\ldots,k'_m}\\ &E\int\{K_h(\mathbf{u}_1-\mathbf{Y}_{i_1})-EK_h(\mathbf{u}_1-\mathbf{Y}_{i_1})\}\cdot\{K_h(\mathbf{u}_2-\mathbf{Y}_{i_2}-EK_h(\mathbf{u}_2-\mathbf{Y}_{i_2})\}\\ &\cdot\{(d^mK)_h(\mathbf{u}_1-\mathbf{Y}_{j_1})\cdot(\mathbf{1}(\mathbf{Y}_{k_1}\leqslant\mathbf{Y}_{j_1})-\mathbf{Y}_{j_1})\ldots(\mathbf{1}(\mathbf{Y}_{k_m}\leqslant\mathbf{Y}_{j_1})-\mathbf{Y}_{j_1})\}\\ &\cdot\{(d^mK)_h(\mathbf{u}_2-\mathbf{Y}_{j_2})\cdot(\mathbf{1}(\mathbf{Y}_{k'_1}\leqslant\mathbf{Y}_{j_2})-\mathbf{Y}_{j_2})\ldots(\mathbf{1}(\mathbf{Y}_{k'_m}\leqslant\mathbf{Y}_{j_2})-\mathbf{Y}_{j_2})\}\\ &\times\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\,d\mathbf{u}_1\,d\mathbf{u}_2.\end{aligned}$$

The corresponding terms are zero except when there are some equalities involving all the indices $k_1,\ldots,k_m,k'_1,\ldots,k'_m$ and i_1,i_2. There are at least $m+1$ equalities. Moreover, there are always three "free" random variables at least, viz three integrations with respect to some $\mathbf{y}$ are available. It is possible to gain another factor h^d by the change of variables $h\tilde{\mathbf{u}}_2=\mathbf{u}_2-\mathbf{u}_1$. Thus, in every case,

$$V_{\gamma,m}^{(1)}=O\left(\frac{1}{n^{4+2m}h^{2m}}\cdot n^{4+2m-(m+1)}\right)=O\left(\frac{1}{n^{1+m}h^{2m}}\right).$$

This quantity is $o(n^{-2}h^{-d})$ when $m=2,3$ since nh^2 tends to the infinity when $n\to\infty$.

Imagine now there are some identities between the indices $i,j,k_1,\ldots,k_m$. The expectation of the corresponding term is zero, except if these equalities involve all $i,k_1,\ldots,k_m$. When $m=2$ (resp. $m=3$), two equalities at least are necessary. This implies the expectation is $O(n^{-2-m}h^{-m}n^m)=O(n^{-2}h^{-m})=o(n^{-1}h^{-d/2})$. Moreover, its variance can be dealt exactly like $V^{(1)}_{\gamma,m}$. Thus, we have proved that, when $m = 2, 3$,

$$T_{\gamma,m} = o_P\left(\frac{1}{nh^{d/2}}\right).$$

B.1.4. Study of $2n^{-2}\sum_{i<j} c_{n,i}c_{n,j}\omega$, $2n^{-2}\sum_{i<j} c_{n,i}b_{n,j}\omega$ *and the other terms of the same type.*

To deal with these terms simultaneously, denote

$$\begin{aligned} &T_{\delta,m,p} \\ &= \frac{2}{n^{2+m+p}h^{m+p}} \sum_{i<j} \sum_{k_1,\dots,k_m} \sum_{l_1,\dots,l_p} \int \{(d^m K)_h(\mathbf{u}-\mathbf{Y}_i)\cdot(\mathbf{1}(\mathbf{Y}_{k_1}\leqslant \mathbf{Y}_i)-\mathbf{Y}_i) \\ &\cdots(\mathbf{1}(\mathbf{Y}_{k_m}\leqslant \mathbf{Y}_i)-\mathbf{Y}_i) - E[(d^m K)_h(\mathbf{u}-\mathbf{Y}_i) \\ &\cdot(\mathbf{1}(\mathbf{Y}_{k_1}\leqslant \mathbf{Y}_i)-\mathbf{Y}_i)\dots(\mathbf{1}(\mathbf{Y}_{k_m}\leqslant \mathbf{Y}_j)]\} \\ &\cdot\{(d^p K)_h(\mathbf{u}-\mathbf{Y}_j)\cdot(\mathbf{1}(\mathbf{Y}_{l_1}\leqslant \mathbf{Y}_j)-\mathbf{Y}_j)\dots(\mathbf{1}(\mathbf{Y}_{l_p}\leqslant \mathbf{Y}_j)-\mathbf{Y}_j) \\ &- E[(d^p K)_h(\mathbf{u}-\mathbf{Y}_j)\cdot(\mathbf{1}(\mathbf{Y}_{l_1}\leqslant \mathbf{Y}_j)-\mathbf{Y}_j)\dots(\mathbf{1}(\mathbf{Y}_{l_p}\leqslant \mathbf{Y}_j)]\}\omega(\mathbf{u})\,d\mathbf{u}, \end{aligned}$$

for m and $l = 1, 2, 3$, $m + p \geqslant 3$. All the summands are centered, except when there are some equalities involving all the indices $k_1, \dots, k_m$ and $l_1, \dots, l_p$ (at least) .

Imagine we are dealing with all the terms of the previous sum corresponding to different indices. Thus the expectation is zero and the variance is a sum over $4 + 2(m + p)$ indices (denoted by $i_1, i_2, j_1, j_2, k_1, k'_1, \dots, k_m, k'_m, l_1, l'_1, \dots, l_p, l'_p$ with obvious notations). Nonzero terms occurs when all the k, k', l and l' indices are matched. At least, this provides $m + p$ equalities. Moreover, there are always three opportunities to make some usual changes of variables and to remove the factors h^d. When this factor appears, it means we have an additional equality involving i or j indices. Thus, we win an extra factor n. Therefore, the variance is

$$O\left(\frac{1}{n^{4+2m+2p}h^{m+p}}\cdot(n^{4+m+p}+n^{3+m+p}h^{-d})\right).$$

In every case, this is $o(n^{-2}h^{-d})$.

Now, imagine there are some equalities between $i, j, k_1, \dots, k_m, l_1, \dots, l_p$. The variance of such a term can be dealt as previously. It is sufficient to verify that its expectation is negligible. This expectation is a sum of terms that are nonzero only if there are some equalities involving $k_1, \dots, k_m, l_1, \dots, l_p$. If $m + p$ is even, there are at least $(m + p)/2$ equalities. If $m + p$ is odd, there are at least $[(m + p)/2] + 1$ equalities. In every case, the factors h^d disappear by some changes of variables with respect to $\mathbf{y}_i$ and $\mathbf{y}_j$. To summarize, this expectation is $O(n^{-(m+p)/2}h^{-m-p})$ (resp. $O(n^{-[(m+p)/2]-1}h^{-m-p})$) if $m + p$ is even (resp. odd). These terms are $o(n^{-1}h^{-d/2})$ if $nh^3 \to \infty$.

Thus

$$T_{\delta,m,p} = o_P\left(\frac{1}{nh^{d/2}}\right).$$

B.1.5. Study of the remainder terms

These terms are like $2n^{-2}\sum_{i<j} a_i e_{n,j}\omega$. Actually, every term that involves $e_{n,j}$ is negligible. For instance,

$$\left|2n^{-2}\sum_{i<j} a_i e_{n,j}\omega\right| \leqslant \frac{Cst}{n^2h^4}\cdot\frac{n^2}{h^d}\cdot\sup_j \|\mathbf{Y}_{n,j}-\mathbf{Y}_j\|_\infty^4 = O_P\left(\frac{\ln_2^2 n}{n^2h^{4+d}}\right).$$

This term is $o_P(n^{-1}h^{-d/2})$ under (B). Thus, we have got

$$\begin{aligned} J^*_{n,2} &= \frac{\sqrt{2}}{nh^{d/2}}\mathcal{N}_n + 2T_\alpha + T_\beta + o_P\left(\frac{1}{nh^{d/2}}\right) \\ &= \frac{\sqrt{2}}{nh^{d/2}}\mathcal{N}_n + \frac{(-1)}{nh}\int \tau^2\omega.\sum_{r=1}^d \int K_r^2 + o_P\left(\frac{1}{nh^{d/2}}\right), \end{aligned} \tag{B.7}$$

where $\mathcal{N}_n$ tends in law towards a gaussian r.v. $\mathcal{N}(0,\sigma^2)$.

B.2. Study of $J^*_{n,1}$

With the previous notations

$$\begin{aligned} J^*_{n,1} &= \frac{1}{n^2}\sum_i \int a_{n,i}^2\omega = \frac{1}{n^2}\sum_i\int [a_i + b_{n,i} + c^*_{n,i}]^2\omega \\ &= \frac{1}{n^2}\sum_i\int [a_i^2 + b_{n,i}^2 + (c^*_{n,i})^2 + 2a_ib_{n,i} + 2b_{n,i}c^*_{n,i} + 2a_ic^*_{n,i}]\omega, \end{aligned}$$

where the expansion of K has been stopped at the second order. We denote

$$\begin{aligned} c^*_{n,i}(\mathbf{u}) &= \frac{1}{2h^2}\{(d^2K)_h(\mathbf{u}-\mathbf{Y}^*_{n,i}).(\mathbf{Y}_{n,i}-\mathbf{Y}_i)^{(2)} - E[(d^2K)_h(\mathbf{u}-\mathbf{Y}^*_{n,i}) \\ &\quad \cdot(\mathbf{Y}_{n,i}-\mathbf{Y}_i)^{(2)}]\} \\ &= O_P\left(\frac{1}{h^{d+2}}\sup_i \|\mathbf{Y}_{n,i}-\mathbf{Y}_i\|^2\right) = O_P\left(\frac{\ln_2 n}{nh^{d+2}}\right). \end{aligned}$$

Therefore, it is easy to bound $\int a_ic^*_{n,i}\omega$, $\int b_{n,i}c^*_{n,i}\omega$, and $\int (c^*_{n,i})^2\omega$. All the corresponding terms in $J^*_{n,1}$ are negligible if

$$\frac{\ln_2 n}{n^2h^{d+2}} + \frac{\ln_2^{3/2} n}{n^{5/2}h^{d+3}} + \frac{\ln_2^2 n}{n^3h^{d+4}} \ll \frac{1}{nh^{d/2}}.$$

This is satisfied under condition (B). The main term of $J^*_{n,1}$ is provided by $\int a_i^2\omega$. Note that

$$\begin{aligned} E\frac{1}{n^2}\sum_i\int a_i^2\omega &= \frac{1}{nh^d}\int K^2(\mathbf{t})(\tau\omega)(\mathbf{u}-h\mathbf{t})\,d\mathbf{t} + O(n^{-1}) \\ &= \frac{1}{nh^d}\int K^2\int \tau\omega + O\left(\frac{h^2}{nh^d}\right), \end{aligned}$$

since K is even. Moreover, the variance is

$$
\begin{aligned}
V \equiv \frac{1}{n^4} E \sum_{i,j} \int & \left[\left\{ K_h(\mathbf{u}_1 - \mathbf{Y}_i) - \int K(\mathbf{t})\tau(\mathbf{u}_1 - h\mathbf{t})\, d\mathbf{t} \right\}^2 \right. \\
& \left. - E\{K_h(\mathbf{u}_1 - \mathbf{Y}_i) - \int K(\mathbf{t})\tau(\mathbf{u}_1 - h\mathbf{t})\, d\mathbf{t}\}^2 \right] \\
& \cdot \left[\left\{ K_h(\mathbf{u}_2 - \mathbf{Y}_j) - \int K(\mathbf{t})\tau(\mathbf{u}_2 - h\mathbf{t})\, d\mathbf{t} \right\}^2 \right. \\
& \left. - E \left\{ K_h(\mathbf{u}_2 - \mathbf{Y}_j) - \int K(\mathbf{t})\tau(\mathbf{u}_2 - h\mathbf{t})\, d\mathbf{t} \right\}^2 \right] \omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\, d\mathbf{u}_1\, d\mathbf{u}_2.
\end{aligned}
$$

The nonzero terms are obtained when $i = j$. By the change of variables $h\tilde{\mathbf{y}}_i = \mathbf{u}_1 - \mathbf{y}_i$ and $h\tilde{\mathbf{u}}_2 = \mathbf{u}_2 - \mathbf{u}_1$, it is easy to verify that $V = O(n^{-3} \cdot h^{-2d})$. Thus, since $nh^d \to \infty$, we get

$$
\frac{1}{n^2} \sum_i \int a_i^2 \omega = \frac{1}{nh^d} \int K^2(\mathbf{t})(\tau\omega)(\mathbf{u} - h\mathbf{t})\, d\mathbf{t} + o_P(n^{-1}h^{-d/2}).
$$

Let us consider now $T \equiv n^{-2} \sum_i \int a_i b_{n,i} \omega$. Its expectation is

$$
\begin{aligned}
E \left[n^{-2} \sum_i \int a_i b_{n,i} \omega \right] &= n^{-1} \int E[a_1 b_{n,1}]\omega \\
&= \frac{(-1)}{n^2 h} \int \left\{ K_h(\mathbf{u} - \mathbf{y}) - \int K(\mathbf{t})\tau(\mathbf{u} - h\mathbf{t})\, d\mathbf{t} \right\} \\
&\quad \cdot \{(dK)_h(\mathbf{u} - \mathbf{y}) \cdot (1 - \mathbf{y}) - E[(dK)_h(\mathbf{u} - \mathbf{Y}) \cdot (1 - \mathbf{Y})]\}\tau(\mathbf{y})\omega(\mathbf{u})\, d\mathbf{u}\, d\mathbf{y} \\
&= \frac{(-1)}{n^2 h^{1+d}} \int \{K(\mathbf{v}) - \int K(\mathbf{t})\tau(\mathbf{u} - h\mathbf{t})\, d\mathbf{t}\} \cdot (dK)(\mathbf{v}) \cdot (1 - \mathbf{u} - h\mathbf{v}) \\
&\quad \cdot \tau(\mathbf{u} - h\mathbf{v})\omega(\mathbf{u})\, d\mathbf{v}\, d\mathbf{u} + O(n^{-2}h^{-1}).
\end{aligned}
$$

Thus, this expectation is $o(n^{-1}h^{-d/2})$. Moreover, its variance is

$$
\begin{aligned}
Var(T) &= \frac{1}{n^4} E \sum_{i,j} \int a_i(\mathbf{u}_1) a_j(\mathbf{u}_2) b_{n,i}(\mathbf{u}_1) b_{n,j}(\mathbf{u}_2) \omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\, d\mathbf{u}_1\, d\mathbf{u}_2 - E[T]^2 \\
&= \frac{1}{n^3} E \int a_1(\mathbf{u}_1) a_1(\mathbf{u}_2) b_{n,1}(\mathbf{u}_1) b_{n,1}(\mathbf{u}_2) \omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\, d\mathbf{u}_1\, d\mathbf{u}_2 - E[T]^2 \\
&= \frac{1}{n^3 h^2} E \int a_1(\mathbf{u}_1) a_1(\mathbf{u}_2)(dK)_h(\mathbf{u}_1 - \mathbf{Y}_1) \cdot (\mathbf{Y}_{n,1} - \mathbf{Y}_1) \\
&\quad \cdot (dK)_h(\mathbf{u}_2 - \mathbf{Y}_1) \cdot (\mathbf{Y}_{n,1} - \mathbf{Y}_1)\omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\, d\mathbf{u}_1\, d\mathbf{u}_2 + O(n^{-4}h^{-2-2d}).
\end{aligned}
$$

Invoking an a.e. upper bound for the empirical process, we get

$$Var(T) \leqslant \frac{1}{n^3h^2} E \int |a_1(\mathbf{u}_1)a_1(\mathbf{u}_2)| \cdot \|(dK)_h\|^2_\infty(\mathbf{u}_1 - \mathbf{Y}_1) \cdot \|(dK)_h\|_\infty(\mathbf{u}_2 - \mathbf{Y}_1)$$
$$\cdot \|\mathbf{Y}_{n,1} - \mathbf{Y}_1\|_\infty \omega(\mathbf{u}_1)\omega(\mathbf{u}_2)\, d\mathbf{u}_1\, d\mathbf{u}_2 + O\left(\frac{1}{n^4h^{2+2d}}\right) \leqslant \frac{Cst}{n^3h^2} \cdot \frac{1}{h^d} \cdot \frac{\ln_2 n}{n}.$$

The latter upper bound is $o(n^{-2}h^{-d})$. Thus, we have proved $T = o_P(n^{-1}h^{-d/2})$.

It remains to deal with $n^{-2}\sum_i \int b^2_{n,i}\omega$. By a change of variable with respect to $\mathbf{u}$, we get directly the upper bound

$$n^{-2}\sum_i \int b^2_{n,i}\omega = O_P\left(\frac{1}{nh^2} \cdot \frac{1}{h^d} \cdot \frac{\ln_2 n}{n}\right) = o_P(n^{-1}h^{-d/2}),$$

if $nh^{2+d/2}/\ln_2 n \to \infty$. The latter condition could be relaxed by a more cautious analysis of the latter term, as done previously. It is useless, facing the set of technical assumptions we have already done. To conclude,

$$J^*_{n,1} = \frac{1}{nh^d} \int K^2(\mathbf{t})(\tau\omega)(\mathbf{u} - h\mathbf{t})\, d\mathbf{t} + o_P\left(\frac{1}{nh^{d/2}}\right). \tag{B.8}$$

B.3. Study of J_{I}

Recall that

$$J_{\mathrm{I}} = \int (\tau_n - E\tau_n) \cdot (E\tau_n - K_h * \hat{\tau})\omega, \text{ and}$$

$$\tau_n(\mathbf{u}) - E\tau_n(\mathbf{u}) = n^{-1}\sum_i \int [a_i(\mathbf{u}) + b^*_{n,i}(\mathbf{u})]\omega(\mathbf{u})\, d\mathbf{u}, \text{ with}$$

$$b^*_{n,i}(\mathbf{u}) = \frac{(-1)}{nh}\sum_{i=1}^n \{(dK)_h(\mathbf{u} - \mathbf{Y}^*_{n,i}) \cdot (\mathbf{Y}_{n,i} - \mathbf{Y}_i)$$
$$- E[(dK)_h(\mathbf{u} - \mathbf{Y}^*_{n,i}) \cdot (\mathbf{Y}_{n,i} - \mathbf{Y}_i)]\},$$

for some random variable $\mathbf{Y}^*_{n,i}$, $\|\mathbf{Y}^*_{n,i} - \mathbf{Y}_i\| \leqslant \|\mathbf{Y}_{n,i} - \mathbf{Y}_i\|$ a.e. Thus,

$$J_{\mathrm{I}} = \int \left\{\frac{1}{n}\sum_i [a_i(\mathbf{u}) + b^*_{n,i}(\mathbf{u})]\right\} \cdot \left\{E\beta^*_{n,i}(\mathbf{u}) - K_h * (\hat{\tau} - \tau)(\mathbf{u})\right\} \omega(\mathbf{u})\, d\mathbf{u}$$
$$= \frac{1}{n}\sum_i \int a_i(\mathbf{u})K_h * (\tau - \hat{\tau})(\mathbf{u})\omega(\mathbf{u})\, d\mathbf{u} + \frac{1}{n}\sum_i \int a_i(\mathbf{u})E\beta^*_{n,i}(\mathbf{u})\omega(\mathbf{u})\, d\mathbf{u}$$
$$+ \frac{1}{n}\sum_i \int b^*_{n,i}(\mathbf{u})K_h * (\tau - \hat{\tau})(\mathbf{u})\omega(\mathbf{u})\, d\mathbf{u} + \frac{1}{n}\sum_i \int b^*_{n,i}(\mathbf{u})E\beta^*_{n,i}(\mathbf{u})\omega(\mathbf{u})\, d\mathbf{u}$$
$$\equiv J^{(0)}_{\mathrm{I}} + J^{(1)}_{\mathrm{I}} + J^{(2)}_{\mathrm{I}} + J^{(3)}_{\mathrm{I}},$$

by denoting

$$\beta^*_{n,i}(\mathbf{u}) = \frac{(-1)}{h}(dK)_h(\mathbf{u} - \mathbf{Y}^*_{n,i}) \cdot (\mathbf{Y}_{n,i} - \mathbf{Y}_i).$$

Clearly,

$$\hat{\tau}(\mathbf{u}) - \tau(\mathbf{u}) = \partial_\theta \tau(\mathbf{u}, \theta_0) \cdot (\hat{\theta} - \theta_0) + 2^{-1}\partial^2_\theta \tau(\mathbf{u}, \tilde{\theta}) \cdot (\hat{\theta} - \theta_0)^{(2)}, \tag{B.9}$$

for some $\tilde{\theta}$, $\|\tilde{\theta} - \theta_0\| \leqslant \|\hat{\theta} - \theta_0\|$ a.e. Implicitly, $\tilde{\theta}$ depends on $\mathbf{u}$.

B.3.1. Study of $J_{\mathrm{I}}^{(0)}$

Note that

$$\begin{aligned} J_{\mathrm{I}}^{(0)} &= n^{-1}\sum_{i=1}^n \int a_i(\mathbf{u})K(\mathbf{v})(\tau - \hat{\tau})(\mathbf{u} - h\mathbf{v})\omega(\mathbf{u})\, d\mathbf{u}\, d\mathbf{v} \\ &= n^{-1}\sum_{i=1}^n \int a_i(\mathbf{u})K(\mathbf{v})\partial_\theta \tau(\mathbf{u} - h\mathbf{v}, \theta_0) \cdot (\hat{\theta} - \theta_0)\omega(\mathbf{u})\, d\mathbf{u}\, d\mathbf{v} \\ &\quad +(2n)^{-1}\sum_{i=1}^n \int a_i(\mathbf{u})K(\mathbf{v})\partial^2_\theta \tau(\mathbf{u} - h\mathbf{v}, \tilde{\theta}) \cdot (\hat{\theta} - \theta_0)^{(2)}\omega(\mathbf{u})\, d\mathbf{u}\, d\mathbf{v} \equiv J_{\mathrm{I},1}^{(0)} + J_{\mathrm{I},2}^{(0)}. \end{aligned}$$

Actually, the latter random quantity $\tilde{\theta}$ depends on $\mathbf{u} - h\mathbf{v}$. The first previous term $J_{\mathrm{I},1}^{(0)}$ can be dealt exactly as in Fan [13]. This author has assumed $\hat{\theta}$ is the maximum likelihood estimator of θ, which implies $B(\theta_0, \mathbf{Y}_i)$ is a score function. Actually, by reading carefully her proof, we notice we need only $B(\theta_0, \mathbf{Y}_i)$ is centered and belongs in L^2, viz our assumption (E). Thus, $J_{\mathrm{I},1}^{(0)} = O_P(n^{-1})$. Moreover, by some change of variables,

$$\begin{aligned} \|J_{\mathrm{I},2}^{(0)}\| \leqslant \frac{Cst}{n}\sum_{i=1}^n \int |K|(\tilde{\mathbf{u}})|K|(\mathbf{v})\|\partial^2_\theta \tau(\mathbf{Y}_i - h\tilde{\mathbf{u}} - h\mathbf{v}, \tilde{\theta})\| \\ \omega(\mathbf{Y}_i - h\tilde{\mathbf{u}})\, d\tilde{\mathbf{u}}\, d\mathbf{v} \cdot \|\hat{\theta} - \theta_0\|^2. \end{aligned}$$

To bound the previous right hand side, we could assume

$$E\left[\sup_{\{(\mathbf{u},\mathbf{v},\theta)\,|\,\|\mathbf{u}\|+\|\mathbf{v}\| \leqslant 2h,\, \|\theta-\theta_0\| \leqslant \varepsilon\}} \|\partial^2_\theta \tau(\mathbf{Y}_i - \mathbf{u}, \theta)\| \cdot |\omega|(\mathbf{Y}_i - \mathbf{v})\right] < \infty. \tag{B.10}$$

This assumption is satisfied under the stronger condition (T), for n sufficiently large.

Thus, under (B.10), we get $J_{\mathrm{I},2}^{(0)} = O_P(\|\hat{\theta} - \theta_0\|^2) = O_P(n^{-1})$.

B.3.2. Study of $J_{\mathrm{I}}^{(1)}$

$$J_{\mathrm{I}}^{(1)} = \frac{(-1)}{nh}\sum_i \int a_i(\mathbf{u})E[(dK)_h(\mathbf{u} - \mathbf{Y}^*_i) \cdot (\mathbf{Y}_{n,i} - \mathbf{Y}_i)]\omega(\mathbf{u})\, d\mathbf{u}.$$

Clearly, this term is centered. By a limited expansion of K up to the pth order, we prove that

$$E[(dK)_h(\mathbf{u}-\mathbf{Y}_i^*)\cdot(\mathbf{Y}_{n,i}-\mathbf{Y}_i)] = O\left(\frac{1}{n}+\frac{1}{h^{p+d}}\cdot\left(\frac{\ln_2 n}{n}\right)^{(p+1)/2}\right). \tag{B.11}$$

The latter upper bound is uniform with respect to $\mathbf{u}$. Therefore, the variance of $J_{\mathrm{I}}^{(1)}$ is

$$\begin{aligned} E[(J_{\mathrm{I}}^{(1)})^2] = \; & \frac{1}{n^2h^2}\sum_i E\int a_i(\mathbf{u}_1)a_i(\mathbf{u}_2)E[(dK)_h(\mathbf{u}_1-\mathbf{Y}_i^*)\cdot(\mathbf{Y}_{n,i}-\mathbf{Y}_i)] \\ & \cdot E[(dK)_h(\mathbf{u}_2-\mathbf{Y}_i^*)\cdot(\mathbf{Y}_{n,i}-\mathbf{Y}_i)]\omega(\mathbf{u}_1) \\ & \times\omega(\mathbf{u}_2)\,d\mathbf{u}_2\,d\mathbf{u}_1 \\ = \; & O\left(\frac{1}{nh^2}\cdot\frac{1}{n^2}+\frac{1}{nh^2}\cdot\frac{1}{h^{2p+2d}}\cdot\left(\frac{\ln_2 n}{n}\right)^{p+1}\right) = o\left(\frac{1}{n^2h^d}\right), \end{aligned}$$

by a change of variables with respect to $\mathbf{y}$ and $\mathbf{u}_2$, and if $n^p h^{2+2p+d}/(\ln_2 n)^{p+1}\to\infty$. The latter condition is satisfied under our assumptions with $p=2$.

B.3.3. Study of $J_{\mathrm{I}}^{(2)}$

With obvious notations,

$$\begin{aligned} J_{\mathrm{I}}^{(2)} &= n^{-1}\sum_{i=1}^n\int b_{n,i}^*(\mathbf{u})K(\mathbf{v})(\tau-\hat\tau)(\mathbf{u}-h\mathbf{v})\omega(\mathbf{u})\,d\mathbf{u}\,d\mathbf{v} \\ &= n^{-1}\sum_{i=1}^n\int [b_{n,i}+c_{n,i}^*](\mathbf{u})K(\mathbf{v})[\partial_\theta\tau(\mathbf{u}-h\mathbf{v},\theta_0)\cdot(\hat\theta-\theta_0) \\ &\quad +2^{-1}\partial_\theta^2\tau(\mathbf{u}-h\mathbf{v},\tilde\theta)\cdot(\hat\theta-\theta_0)^{(2)}]\omega(\mathbf{u})\,d\mathbf{u}\,d\mathbf{v} \\ &= n^{-1}\sum_{i=1}^n\int b_{n,i}(\mathbf{u})K(\mathbf{v})[\partial_\theta\tau(\mathbf{u}-h\mathbf{v},\theta_0)\cdot(\hat\theta-\theta_0)]\omega(\mathbf{u})\,d\mathbf{u}\,d\mathbf{v} \\ &\quad +O_P\left(\frac{\ln_2 n}{n}\cdot\frac{1}{h^2n^{1/2}}+\frac{1}{h}\cdot\left(\frac{\ln_2 n}{n}\right)^{1/2}\cdot\frac{1}{n}\right), \end{aligned}$$

under the condition (B.10). The main term of the latter expansion is

$$T \equiv \frac{1}{n^2}\sum_{i,j}\int b_{n,i}(\mathbf{u})K(\mathbf{v})\partial_\theta\tau(\mathbf{u}-h\mathbf{v},\theta_0)A(\theta_0)^{-1}B(\theta_0,\mathbf{Y}_j)\omega(\mathbf{u})\,d\mathbf{u}.$$

Thus, when $i\neq j$, the expectation of the summand is $O(n^{-1})$, and

$$\begin{aligned} E[T] = \; & \frac{1}{n^2}\sum_i E[b_{n,i}(\mathbf{u})K(\mathbf{v})\partial_\theta\tau(\mathbf{u}-h\mathbf{v},\theta_0)A(\theta_0)^{-1}B(\theta_0,\mathbf{Y}_i)]\omega(\mathbf{u})\,d\mathbf{u}+O(n^{-1}) \\ = \; & O\left(\frac{1}{nh}\cdot\left(\frac{\ln_2 n}{n}\right)^{1/2}+\frac{1}{n}\right) = o\left(\frac{1}{nh^{d/2}}\right). \end{aligned}$$

Moreover, by the same reasoning, its variance is

$$Var(T) = O\left(\frac{1}{n^2h^2}\cdot\left(\frac{\ln_2 n}{n}\right)\right) = o(\frac{1}{n^2h^d}).$$

Note that one remainder term is

$$\frac{1}{n}\sum_i \int b_{n,i}(\mathbf{u})K(\mathbf{v})\partial_\theta\tau(\mathbf{u}-h\mathbf{v},\theta_0)\omega(\mathbf{u})\,d\mathbf{u}\cdot o_P(r_n).$$

The latter term is negligible if $\left(\frac{\ln_2 n}{n}\right)^{1/2}\cdot\frac{r_n}{h} << \frac{1}{nh^{d/2}}$, viz if

$$r_n = o\left(\frac{1}{\sqrt{n}\ln_2^{1/2} n}\cdot\frac{1}{h^{d/2-1}}\right).$$

B.3.4. Study of $J_{\mathrm{I}}^{(3)}$

Clearly, under the previous assumptions,

$$J_{\mathrm{I}}^{(3)} = O_P\left(\frac{1}{h}\cdot\left(\frac{\ln_2 n}{n}\right)^{1/2}\cdot\frac{1}{nh}\right) = o_P\left(\frac{1}{nh^{d/2}}\right),$$

since $nh^2/\ln_2 n \to \infty$. To conclude,

$$J_{\mathrm{I}} = o_P\left(\frac{1}{nh^{d/2}}\right). \tag{B.12}$$

B.4. Study of J_{II}

With the previous notations,

$$\begin{aligned} J_{\mathrm{II}} &= \int (E\tau_n - K_h*\hat\tau)^2\omega \\ &= \int [K_h*(\hat\tau-\tau)]^2\omega + \int [E\beta^*_{ni}]^2\omega - 2\int K_h*(\hat\tau-\tau)E\beta^*_{ni}\omega. \end{aligned}$$

Applying Eq. (B.11) with $p=2$, we get

$$E\beta^*_{ni}(\mathbf{u}) = O\left(\frac{1}{n}+\frac{1}{h^4}\cdot\left(\frac{\ln_2 n}{n}\right)^{3/2}\right),$$

uniformly with respect to $\mathbf{u}$. Thus, it is straightforward that

$$\int [E\beta^*_{ni}]^2\omega = o\left(\frac{1}{nh^{d/2}}\right).$$

Moreover, under assumption (T) and by a limited expansion with respect to $\mathbf{u}$,

$$\int [K_h*(\hat\tau-\tau)]^2\omega = O_P\left(\frac{1}{n}\right).$$

By applying Schwartz's inequality, we obtain

$$J_{\mathrm{II}} = o_P\left(\frac{1}{nh^{d/2}}\right). \tag{B.13}$$

Theorem 3 results from Eqs. (B.1), (B.2), (B.7), (B.8), (B.12) and (B.13). □

AppendixC.. Proof of Corollary 4

It is sufficient to prove that

$$\frac{1}{nh^d}\int K^2(\mathbf{t})((\hat{\tau}-\tau)\omega)(\mathbf{u}-h\mathbf{t})\,d\mathbf{t}\,d\mathbf{u} = o_P\left(\frac{1}{nh^{d/2}}\right), \text{ and} \tag{C.1}$$

$$\frac{1}{nh}\int K^2(\mathbf{t})(\hat{\tau}^2-\tau^2)\omega = o_P\left(\frac{1}{nh^{d/2}}\right). \tag{C.2}$$

Note that, under (T) and by a limited expansion with respect to θ, we have

$$\sup_{\mathbf{u}\in[\varepsilon,1-\varepsilon]^d} \|\hat{\tau}(\mathbf{u},\hat{\theta}) - \tau(\mathbf{u},\theta_0)\| = O_P(\|\hat{\theta}-\theta_0\|) = O_P(n^{-1/2}).$$

Thus, Eqs. (C.1) and (C.2) are clearly satisfied because nh^d tends to the infinity when $n\to\infty$, proving the result. □

AppendixD.. The semiparametric estimator

Consider the parametric family $\mathcal{C} = \{\tau(\cdot,\theta), \theta\in\Theta\}$. The semiparametric estimator of θ satisfies, by definition, $\hat{\theta} = \arg\max_{\theta\in\Theta} Q_n(\theta)$, where

$$Q_n(\theta) = n^{-1}\sum_{i=1}^n \ln\tau(\mathbf{Y}_{ni},\theta).$$

We prove that $\hat{\theta}$ satisfies condition (3.1). By a limited expansion, there exists some random vector θ^* such that $\partial^2_{\theta\theta}Q_n(\theta^*)\cdot(\hat{\theta}-\theta_0) = -\partial_\theta Q_n(\theta_0)$, with $\|\theta^*-\theta_0\| \leqslant \|\hat{\theta}-\theta_0\|$ a.e. First, with obvious notations,

$$\begin{aligned}\partial_\theta Q_n(\theta_0) &= n^{-1}\sum_{i=1}^n \partial_\theta \ln\,\tau(\mathbf{Y}_i,\theta_0) + n^{-1}\sum_{i=1}^n \partial^2_{\mathbf{y},\theta}\ln\,\tau(\mathbf{Y}_i,\theta_0)\cdot(\mathbf{Y}_{n,i}-\mathbf{Y}_i)\\ &\quad + \frac{1}{2n}\sum_{i=1}^n \partial^3_{\mathbf{yy}\theta}\ln\,\tau(\mathbf{Y}^*_{ni},\theta_0)\cdot(\mathbf{Y}_{n,i}-\mathbf{Y}_i)^{(2)} \equiv S_0+S_1+S_2.\end{aligned}$$

We assume that

$$E[\|\partial_\theta\ln\tau(\mathbf{Y},\theta_0)\| + \|\partial^2_{\theta,\mathbf{y}}\ln\tau(\mathbf{Y},\theta_0)\| + \|\partial^3_{\theta,\mathbf{y},\mathbf{y}}\ln\tau(\mathbf{Y},\theta_0)\|] < \infty. \tag{D.1}$$

Obviously, S_0 is asymptotically normal. The expectation of S_1 is $O(n^{-1})$ and its variance is $O(n^{-2})$. Thus, S_1 is $O_P(n^{-1})$. Moreover,

$$\|S_2\| \leqslant Cte \cdot \frac{1}{n}\sum_{i=1}^{n} \|\partial^3_{\mathbf{yy}\theta} \ln\, \tau(\mathbf{Y}^*_{ni}, \theta_0)\| \cdot \|\mathbf{Y}_{ni} - \mathbf{Y}_i\|^2. \tag{D.2}$$

Assume the following conditions of regularity:

1. There exist some constants α et β such that, a.e.,

$$\|\partial^3_{\mathbf{yy}\theta} \ln \tau(\mathbf{Y}^*_{ni}, \theta_0)\| \leqslant \alpha \|\partial^3_{\mathbf{yy}\theta} \ln \tau(\mathbf{Y}_i, \theta_0)\| + \beta \|\partial^3_{\mathbf{yy}\theta} \ln \tau(\mathbf{Y}_{ni}, \theta_0)\|, \text{ and}$$

2. For every $\mathbf{u} \in (0,1)^d$, $\|\partial^3_{\mathbf{yy}\theta} \ln\, \tau(\mathbf{u}, \theta_0)\| \leqslant Cst.r(u_1)^{a_1} \ldots r(u_d)^{a_d}$, where $a_k = (-1 + \delta)/p_k$, $1/p_1 + \cdots + 1/p_k = 1$, $\delta > 0$, and $r(t) = t(1-t)$.

The latter condition ensures the consistency of the empirical mean of $\|\partial^3_{\mathbf{yy}\theta} \ln \tau(\mathbf{Y}_{ni}, \theta_0)\|$ (see Genest et al. [20], Proposition A.1). Thus, we get $\|S_2\| = O_P(n^{-1} \ln_2 n)$. We have obtained

$$\partial_\theta Q_n(\theta_0) = n^{-1} \sum_{i=1}^{n} \partial_\theta \ln\, \tau(\mathbf{Y}_i, \theta) + O_P(\ln_2 n/n).$$

Moreover, with obvious notations,

$$\begin{aligned}\partial^2_\theta Q_n(\theta^*) &= \partial^2_\theta Q_n(\theta_0) + n^{-1} \sum_{i=1}^{n} \partial^3_\theta \ln\, \tau(\mathbf{Y}_{ni}, \tilde{\theta}) \cdot (\theta^* - \theta_0) \\ &= \lim_{n\infty} E[\partial^2_\theta Q_n(\theta_0)] + O_P(n^{-1/2}),\end{aligned}$$

if $\partial^2_\theta Q_n(\theta)$ is asymptotically normal, and if

$$n^{-1} \sum_{i=1}^{n} \sup_{\theta \in \mathcal{V}(\theta_0)} \|\partial^3_\theta \ln \tau(\mathbf{Y}_{ni}, \theta)\| < \infty \text{ a.e.} \tag{D.3}$$

Here, $\mathcal{V}(\theta_0)$ denotes a neighborhood of θ_0. Applying Proposition A.1 of Genest et al. [21], these two conditions can be are ensured if:

1. For every $\mathbf{u} \in (0,1)^d$, $M(\mathbf{u}) \equiv \|\partial^2_\theta \ln \tau(\mathbf{u}, \theta_0)\| \leqslant Cst.r(u_1)^{b_1} \cdots r(u_d)^{b_d}$, where $b_k = (-0.5 + v)/q_k$, $1/q_1 + \ldots + 1/q_k = 1$, $v > 0$. Moreover, $M(\mathbf{u})$ has continuous partial derivatives $M_k(\mathbf{u}) = \partial M(\mathbf{u})/\partial u_k$, such that $M_k(\mathbf{u}) \leqslant Cst.r(u_1)^{d_1^{(k)}} \ldots r(u_d)^{d_d^{(k)}}$, $d_k^{(k)} = b_k$, $d_j^{(k)} = b_j - 1$ if $j \neq k$.
2. For every $\mathbf{u} \in (0,1)^d$, $\sup_{\theta \in \mathcal{V}(\theta_0)} \|\partial^3_\theta \ln \tau(\mathbf{u}, \theta)\| \leqslant Cst.r(u_1)^{c_1} \ldots r(u_d)^{c_d}$, where $c_k = (-1 + \eta)/p'_k$, $1/p'_1 + \ldots + 1/p'_k = 1$, $\eta > 0$.

Condition (1) ensures the asymptotic normality of the empirical mean of $M(\mathbf{Y}_{ni})$. Condition (2) ensures condition (D.3).

It can be checked that the previous conditions are satisfied by a large number of commonly used copula families. Particularly, it is the case for the gaussian copula.

Thus, under the previous conditions, we get

$$\sqrt{n}(\hat{\theta} - \theta_0) = \frac{1}{\sqrt{n}} A(\theta_0)^{-1} \cdot \sum_{i=1}^{n} \partial_\theta \ln\, \tau(\mathbf{Y}_i, \theta_0) + O_P\left(\frac{\ln_2\, n}{n}\right),$$

$$A(\theta_0) = -\lim_{n\infty} E\left[\partial_\theta^2 Q_n(\theta)\right]$$

and (3.1) is satisfied. □

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