A KOLMOGOROV-SMIRNOV TYPE TEST FOR POSITIVE QUADRANT DEPENDENCE

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This version: January 2005 (first version : January 2004)

Abstract

We consider a consistent test, that is similar to a Kolmogorov-Smirnov test, of the complete set of restrictions that relate to the copula representation of positive quadrant dependence. For such a test we propose and justify inference relying on a simulation based multiplier method and a bootstrap method. We also explore the finite sample behavior of both methods with Monte Carlo experiments. A first empirical illustration is given for US insurance claim data. A second one examines the presence of positive quadrant dependence in life expectancies at birth of males and females among countries.

Keywords and phrases: Nonparametric, Positive Quadrant Dependence, Copula, Risk Management, Loss Severity Distribution, Bootstrap, Multiplier Method, Empirical Process.

JEL Classification: C12, D81, G10, G21, G22.
AMS 2000 Subject Classification: 60E15, 62G10, 62G30, 62P05, 91B28, 91B30.

1. INTRODUCTION

The concept of positive quadrant dependence (PQD) was introduced by Lehmann (1966). Two random variables are said to be PQD when the probability that they are simultaneously large (or small) is at least as great as it would be were they independent. Recent work in finance, insurance and risk management has emphasized the importance of PQD; e.g., Dhaene & Goovaerts (1996), Denuit, Dhaene & Ribas (2001), Embrechts, McNeil & Straumann (2000). For example, one interest of this dependence structure is that it allows the risk manager to compare directly the sum of PQD random variables with the corresponding sum under the independence assumption. The comparison is in the sense of different stochastic orderings expressing the common preferences of rational decision-makers. Inferring that two claims are PQD, no matter what is the strength of this dependence, immediately allows one to conclude to the underestimation of most insurance premiums involving a portfolio of these two claims if the independence assumption is made instead. In a financial setting the same holds true but for risk measures and derivative prices related to a portfolio of two PQD financial assets. We refer the reader to Denuit & Scaillet (2004) for an extensive discussion of examples of application of PQD in finance and actuarial sciences. Implications of PQD are also found in reliability theory (Lai & Xie (2003)), and several other fields (see e.g. Levy (1992), Shaked & Shanthikumar (1994), Mari & Kotz (2001)).

Formally, two random variables X and Y are said to be PQD if, for all $(x, y) \in \mathbb{R}^2$,

$$P[X \leq x, Y \leq y] \geq P[X \leq x]P[Y \leq y]. \tag{0.1}$$

Of course, (0.1) is equivalent to

$$P[X > x, Y > y] \geq P[X > x]P[Y > y] \tag{0.2}$$

which enjoys a similar interpretation (with “small" replaced with “large"). Note further that negative quadrant dependence (NQD) is defined analogously if we subsitute $\leq$ for $\geq$ in (0.1).

Considering (0.1)-(0.2), PQD appears as a comparison of the joint distribution of (X, Y) to that of $(X, Y)^{\perp}$, where $(X, Y)^{\perp}$ denotes an independent version of the random vector (X, Y), that is, (X, Y) and $(X, Y)^{\perp}$ have identical univariate marginals, and $(X, Y)^{\perp}$ has independent components. It can thus be considered as a special case of comparisons of pairs of bivariate distributions with identical marginals in terms of stochastic dominance.

Clearly, X and Y are PQD if, and only if, $a(X)$ and $b(Y)$ are PQD for any strictly increasing functions a and b. This indicates that PQD is a property of the underlying copula, and is not influenced by the marginals. In fact Inequality (0.1) can be written in terms of the copula C of the two random variables, since (0.1) is equivalent to the condition that, for all $(u, v) \in [0, 1]^2$,

$$C(u, v) \geq C^{\perp}(u, v) := uv. \tag{0.3}$$

Recall that the copula C is such that $P[X \leq x, Y \leq y] = C(P[X \leq x], P[Y \leq y])$ (Sklar (1959)); see, e.g., Joe (1997) & Nelsen (1999) for detailed explanations on copulas, their properties and their use. Note that parametric copulas may or may not exhibit PQD per se. For example families that only allow PQD are the Cook-Johnson family and the Gumbel family. On the contrary a member of the Farlie-Gumbel-Morgenstern family, the Frank family, or the Gaussian family, induces PQD when the parameter is positive, and NQD when the parameter is negative.

In this paper we propose a consistent test of PQD, that is similar to a Kolmogorov-Smirnov test, of the complete set of restrictions that relate to the copula representation (0.3) of PQD. Observe that Denuit & Scaillet (2004) has already suggested some nonparametric ways to test for PQD. These are inspired by traditional stochastic dominance tests as in Anderson (1996), Dardanoni & Forcina (1999), Davidson & Duclos (2000), and are either based on distance tests or intersection-union tests for inequality constraints. However these tests rely on pairwise comparisons made at a fixed number of arbitrary chosen points. This is not a desirable feature since it introduces the possibility of test inconsistency.

The paper is organised as follows. In Section 2 we describe the test statistic, and analyse the asymptotic properties of the test for PQD. We follow closely Barrett & Donald (BD) (2003), who extend and justify the procedure of McFadden (1989) (see also Abadie (2002), Linton, Maasoumi & Whang (2001)) leading to consistent tests of stochastic dominance. From a technical point of view, we differ from their work by the multivariate aspect of our distributional setting as well as the use of empirical copula processes instead of univariate empirical processes. In Section 3 we discuss two practical ways to compute the p-values for testing PQD. The first one relies on a simulation-based multiplier method while the second relies on a bootstrap method. In Section 4 we explore the finite-sample behavior of both methods with Monte Carlo experiments. A first empirical illustration is given for US insurance claim data in Section 5. A second one examines the presence of PQD in life expectancies at birth of males and females across countries. We give some concluding remarks and discuss some potential extensions for dimensions higher than two in Section 6. Proofs are gathered in an appendix.

2. TEST STATISTIC AND ASYMPTOTIC PROPERTIES

We consider a setting made of pairs of i.i.d. observations $\{(X_i, Y_i);\ i = 1, ..., n\}$ of a random vector taking values in $\mathbb{R}^2$. These data may correspond to either observed individual losses on insurance contracts, the amounts of claims reported by a given policy holder on different guarantees in a multiline product, or observed returns of financial assets. The margins are denoted by F and G, respectively.

Let us define the empirical copula function by

$$C_n(u,v) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{F_n(X_i) \le u, G_n(Y_i) \le v\}, \qquad (u,v) \in [0,1]^2,$$

where F_n and G_n are the empirical cdf computed from $\{X_i;\ i = 1, ..., n\}$ and $\{Y_i;\ i = 1, ..., n\}$, respectively.

Observe that C_n is actually a function of the ranks of the observations since $nF_n(X_i)$, resp. $nG_n(Y_i)$, gives the rank of X_i, resp. Y_i.

Let $D_n(u,v) := uv - C_n(u,v)$, $D(u,v) := uv - C(u,v)$ and $l^\infty([0,1]^2)$ be the set of all locally bounded real functions on $[0,1]^2$.

Lemma 0.1. *Let the copula function $C(u,v)$ have continuous partial derivatives. Then $\sqrt{n}\{D_n(u,v)-D(u,v)\}$ converges weakly to a tight mean zero Gaussian process $\{\mathbb{G}_C(u,v), 0 \leq u,v \leq 1\}$ in $l^\infty([0,1]^2)$, whose covariance function is*

$$\begin{aligned}\Omega_{\mathbb{G}}(u,v,u',v') &:= E[\mathbb{G}_C(u,v)\mathbb{G}_C(u',v')] \\ &= \Omega(u,v,u',v') - \partial_1 C(u',v')\Omega(u,v,u',1) - \partial_2 C(u',v')\Omega(u,v,1,v') \\ &\quad -\partial_1 C(u,v)\left(\Omega(u,1,u',v') - \partial_1 C(u',v')\Omega(u,1,u',1) - \partial_2 C(u',v')\Omega(u,1,1,v')\right) \\ &\quad -\partial_2 C(u,v)\left(\Omega(1,v,u',v') - \partial_1 C(u',v')\Omega(1,v,u',1) - \partial_2 C(u',v')\Omega(1,v,1,v')\right),\end{aligned}$$

for each $0 \leq u,u',v,v' \leq 1$, where

$$\Omega(u,v,u',v') := C(u \wedge u', v \wedge v') - C(u,v)C(u',v'),$$

and $x \wedge y := \min(x,y)$.

This lemma is deduced from the weak convergence properties of the empirical copula process. We rely here on Theorem 4 in Fermanian, Radulovic & Wegkamp (FRW) (2004), which shows that the result obtained by van der Vaart & Wellner (VW) (1996 p. 389) (see also Stute (1984), Gänssler & Stute (1987) for weak convergence in the Skorokhod space $D([0,1]^2)$) holds true in a larger space under weaker assumptions. Note that uniform almost sure convergence is a by-product of this type of weak convergence. The weak convergence of $\sqrt{n}\{C_n(u,v) - C(u,v)\}$ towards a completely-tucked Brownian sheet when the two margins are independent has already been given, for example, in Deheuvels (1981) in the context of a Kolmogorov-Smirnov test for independence.

Since we wish to test for PQD, namely

$$\begin{aligned} H_0 :&\quad uv \leq C(u,v) \quad \text{for all}\,(u,v) \in [0,1]^2, \\ H_1 :&\quad uv > C(u,v) \quad \text{for some}\,(u,v) \in [0,1]^2,\end{aligned}$$

we consider the test statistic

$$S_n := \sqrt{n}\sup_{u,v} D_n(u,v),$$

and a test based on the decision rule:

$$\text{“ reject} \quad H_0 \quad \text{if} \quad S_n > c\text{”},$$

where c is some critical value that will be discussed later.

The following result characterizes the properties of the test, where $\bar{S} := \sup_{u,v}\mathbb{G}_C(u,v)$.

Proposition 0.2. *Let c be a positive finite constant, then:*

i) if H_0 is true,

$$\lim_{n\to\infty} P[\textit{reject}\, H_0] \leq P[\bar{S} > c] := \alpha(c),$$

with equality when $C(u,v) = uv$ for all $(u,v) \in [0,1]^2$;

ii) if H_0 is false,

$$\lim_{n\to\infty} P[\textit{reject}\, H_0] = 1.$$

The first part of the result provides a random variable that dominates the limiting random variable corresponding to the test statistic under the null hypothesis. The inequality tells us that the test will never reject more often than $\alpha(c)$ when the null hypothesis is satisfied. Furthermore the probability of rejection will asymptotically be exactly $\alpha(c)$ when the copula corresponds to the independent copula. The first part also implies that if one could find a c to set the $\alpha(c)$ to some desired probability level (say the conventional 0.05 or 0.01) then this would be the significance level for composite null hypotheses in the sense described by Lehmann (1986). The second part of the result indicates that the test is capable of detecting any violation of the full set of restrictions of the null hypothesis.

Of course, in order to make the result operational, we need to find an appropriate critical value c. Since the distribution of the test statistic depends on the underlying copula, this is not an easy task. Indeed, recall that the null hypothesis is not independence. Therefore, we cannot directly simulate under the independence hypothesis; i.e., draw from the independent copula. Such a procedure would not reflect the dependence of the distribution of the test statistic on the underlying copula. Hereafter we rely on two different methods to simulate p-values.

3. SIMULATING p-VALUES

3.1. MULTIPLIER METHOD

In this section we use a similation-based method that exploits the multiplier central limit theory discussed in VW (1996) Section 2.9 (see BD for use in stochastic dominance test and Hansen (1996), Glidden (1999), Guay & Scaillet (2003) for other uses). The idea is to rely on artificial pseudo-random numbers to simulate a process that is identical but (asymptotically) independent of $\mathbb{G}_C$. To do this let $\{U_i;\ i = 1, ..., n\}$ denote a sequence of i.i.d. $N(0, 1)$ random variables that are independent of the data sample. Then since $\mathbb{G}_C(u, v) = \mathbb{B}_C(u, v) - \partial_1 C(u, v)\mathbb{B}_C(u, 1) - \partial_2 C(u, v)\mathbb{B}_C(1, v)$, where $\mathbb{B}_C$ is a tight Brownian bridge on $[0, 1]^2$ (see the proof of Lemma 0.1), the process is easily generated from:

$$\begin{aligned}\mathbb{G}_{C_n}(u,v) \;&:=\; \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\mathbb{I}\{F_n(X_i)\le u, G_n(Y_i)\le v\} - C_n(u,v)\right]U_i \\ &\quad -c_{1,n}(u,v)\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\mathbb{I}\{F_n(X_i)\le u\} - u\right]U_i \\ &\quad -c_{2,n}(u,v)\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\mathbb{I}\{G_n(Y_i)\le v\} - v\right]U_i,\end{aligned}$$

where $c_{i,n}(u, v)$ is a consistent estimate of $\partial_i C(u, v)$ $i = 1, 2$. Consistent estimates are easily obtained from smoothed versions of the empirical copula process. For example, one can use the nonparametric estimators $c_{1,n}(u, v) = \partial_1 H_n\{F_n^{-1}(u), G_n^{-1}(v)\}/f_n\{F_n^{-1}(u)\}$ and $c_{2,n}(u, v) = \partial_2 H_n\{F_n^{-1}(u), G_n^{-1}(v)\}/g_n\{G_n^{-1}(v)\}$, where estimates of H, f and g are obtained from kernel based estimators H_n, f_n and g_n, respectively; see Fermanian & Scaillet (2003) for details and proofs of asymptotic properties of such estimators. The p-value can be estimated from:

$$p_n := P_U[\sup_{u,v} \mathbb{G}_{C_n}(u, v) > S_n],$$

where P_U is the probability function associated with the normal random variable U and is conditional on the realized sample. The following result provides the decision rule in this environment.

Proposition 0.3. *Assuming that $\alpha < 1/2$, a test for PQD based on the rule:*

$$\text{“ } \textit{reject}\, H_0 \quad \textit{if} \quad p_n < \alpha \text{ ”},$$

satisfies the following

$$\begin{aligned} \lim P[\textit{reject}\, H_0] &\leq \alpha \quad \textit{if} \quad H_0 \textit{ is true,} \\ \lim P[\textit{reject}\, H_0] &= 1 \quad \textit{if} \quad H_0 \textit{ is false.} \end{aligned}$$

The multiplier method can be justified by showing that the simulated process converges weakly to an identical independent copy of the Gaussian process $\mathbb{G}_C$. Then an application of the continuous mapping theorem shows that we get a simulated copy of the bounding random variable that appears in Proposition 0.2. In practice, we use Monte-Carlo methods to approximate the probability and a grid to approximate the supremum. The p-value is simply approximated by

$$p_n \approx \frac{1}{R} \sum_{r=1}^{R} \mathbb{I}\{S_{n,r} > S_n\},$$

where the averaging is made on R replications and $S_{n,r}$ is computed from a fine grid on $[0,1]^2$. Note that the replication number and the grid mesh can be chosen to make the approximations as accurate as one desires given time and computer constraints.

3.2. BOOTSTRAP METHOD

The second method relies on the standard bootstrap (see BD and Abadie (2002) for use in stochastic dominance tests). An alternative resampling technique could be subsampling, for which similar results can be shown to hold as well (see Linton, Maasoumi & Whang (2001) for use in stochastic dominance tests).

Let us denote $\{(X_i^*, Y_i^*);\ i = 1, ..., n\}$ a random sample drawn from the observed pairs of data, and $C_n^*(u, v)$ the empirical copula function built from this bootstrap sample. Let us further take

$$S_n^* := \sqrt{n} \sup_{u,v} \{C_n^*(u,v) - C_n(u,v)\},$$

and define

$$p_n^* := P[S_n^* > S_n].$$

Then the bootstrap method is justified by the next statement.

Proposition 0.4. *Assuming that $\alpha < 1/2$, a test for PQD based on the rule:*

$$\text{“ } \textit{reject}\, H_0 \quad \textit{if} \quad p_n^* < \alpha \text{ ”},$$

satisfies the following

$$\begin{aligned} \lim P[\textit{reject}\, H_0] &\leq \alpha \quad \textit{if} \quad H_0 \textit{ is true,} \\ \lim P[\textit{reject}\, H_0] &= 1 \quad \textit{if} \quad H_0 \textit{ is false.} \end{aligned}$$

Again we need to rely on Monte-Carlo methods to approximate the probability and a grid to approximate the supremum in a manner analogous to the one of the previous subsection.

4. MONTE CARLO RESULTS

In this section we examine the performance of the Kolmogorov-Smirnov type test in small samples. The grid is made of the values (u, v) evenly spaced inside $\{0.05, 0.10, ..., 0.95\} \times \{0.05, 0.10, ..., 0.95\}$, while the nonparametric estimator of the derivatives of the copula function rely on a Gaussian product kernel and the quick standard rule of thumb (Silverman (1986)) to select the two individual bandwidths. The replication number R to approximate the p-value is set equal to 1000. For each case 1000 Monte Carlo simulations are performed, and the rejection rates are computed for the multiplier method and the bootstrap method w.r.t. the two conventional significance levels of $\alpha = 0.05$ and $\alpha = 0.01$. Samples are generated with both margins corresponding to an exponential distribution with a unit parameter. This can be seen as mimicking the behaviour of claim or duration data. Note that the numerical results below remain exactly the same if we use other strictly monotonic continuously differentiable cdfs (such as Gaussian or Student margins to mimick financial returns) and keep the same seeds in the pseudo-random generators. The reason is that both procedures rely intrinsically on ranks.

In Table I the true copula is the independent copula. Then Proposition 0.2 suggests that the test should reject the null hypothesis with a frequency close to the chosen nominal significance level. This experiment should give us some idea about the validity of the asymptotic theory and the two methods used to simulate the p-values in small samples. The values shown in Table I indicate that the test tends to overreject with the multiplier method and underreject with the bootstrap method, but with a rejection rate converging in both cases to the chosen nominal significance level as n increases.

TABLE I: Independent copula

$\alpha = 0.05$	$n = 50$	$n = 100$	$n = 200$	$n = 400$
KSm	.103	.097	.066	.064
KSb	.024	.020	.027	.032
$\alpha = 0.01$	$n = 50$	$n = 100$	$n = 200$	$n = 400$
KSm	.024	.023	.021	.014
KSb	.003	.002	.004	.007

Table II gathers results concerning the power of the testing procedure when the true copula is a Frank copula, a Gaussian copula, or a Farlie-Gumbel-Morgenstern (FGM) copula inducing NQD. These parametric families are often used in actuarial and financial applications, and permit quick simulations (Genest (1987), Nelsen (1999)). The chosen values of the parameter θ are $\theta \in \{-1, -2, -3\}$ for the Frank copula, $\theta \in \{-.17, -.32, -.46\}$ for the Gaussian copula, and $\theta \in \{-.495, -.945, -1.395\}$ for the FGM copula. They match low and moderate negative dependences as exhibited by the corresponding true values of the Kendall tau, $\tau \in \{-.11, -.21, -.31\}$. The sample size is fixed at $n = 200$. The reported numbers show that both testing procedures have nice power properties under different negative dependence structures.

TABLE II: Frank, Gaussian and FGM copulas

	$\tau = -0.11$			$\tau = -0.21$			$\tau = -0.31$		
$\alpha = 0.05$	F	G	FGM	F	G	FGM	F	G	FGM
KSm	.686	628.	.680	.993	.973	.991	1.000	1.000	1.000
KSb	.495	.421	.493	.979	.924	.974	1.000	1.000	1.000
$\alpha = 0.01$	F	G	FGM	F	G	FGM	F	G	FGM
KSm	.430	.345	.431	.963	.896	.953	1.000	.999	1.000
KSb	.208	.177	.202	.863	.753	.851	.998	.994	.998

5. EMPIRICAL ILLUSTRATIONS

5.1. US INSURANCE CLAIMS

Various processes in casualty insurance involve correlated pairs of variables. A prominent example is the loss and allocated loss adjustment expenses (ALAE, for short) on a single claim. Here ALAE are type of insurance company expenses that are specifically attributable to the settlement of individual claims such as lawyers' fees and claims investigation expenses. The joint modelling in parametric settings of those two variables has been examined by Frees & Valdez (1998), and Klugman & Parsa (1999). The data used in these empirical studies were collected by the US Insurance Services Office, and comprise general liability claims randomly choosen from late settlement lags. Frees & Valdez (1998) choose the Pareto distribution to model the margins, and select Gumbel and Frank copulas (on the basis of a graphical procedure suitable for Archimedean copulas). Both models express PQD by their estimated parameter values. Klugman & Parsa (1999) opt for the Inverse Paralogistic for the losses and for the Inverse Burr for ALAE's. They use the Frank copula. Again, the estimated value of the dependence parameter entails PQD for losses and ALAE's. In the following we rely on a nonparametric approach to assess PQD. This assessment has many implications in insurance, for example, for the computation of reinsurance premiums (where the sharing of expenses between the ceding company and the reinsurer has to be decided on) and for the determination of the expense level for a given loss level (for reserving an appropriate amount to cover future settlement expenses). We refer to Denuit & Scaillet (2004) for further discussion and practical implications on the design of reinsurance treaties when PQD is present.

The data consist in $n = 1,466$ uncensored observed values of the pair (LOSS,ALAE). The grid, the number of replications, the bivariate kernel and the bandwidthes are chosen as in the previous Monte Carlo experiments. We have found $p_n = 1.000$ (multiplier method) and $p_n^* = 1.000$ (bootstrap method) for $S_n = -0.0356$, which means that we cannot reject PQD.

5.2. LIFE EXPECTANCIES AT BIRTH

This second empirical illlustration aims to detect a PQD behavior in life expectancies at birth of males and females across 225 different countries. These data are available at `http://www.odci.gov/cia/publications/factbook/`. A slightly different type of data (life expectancy on total population versus difference between life expectancy of males and females) has been examined in Amblard & Girard (2003) in the context of semiparametric estimation of bivariate copulas under a PQD assumption. The grid, the number of replications, the bivariate kernel and the bandwidthes are again chosen as in the previous Monte Carlo experiments. We have found $p_n = 1.000$ (multiplier method) and $p_n^* = 1.000$ (bootstrap method) for $S_n = -0.0208$, which means that we cannot reject PQD.

6. CONCLUDING REMARKS AND EXTENSIONS

In this paper we have considered a Kolmogorov-Smirnov type test for PQD. This test is consistent since it is based on an examination of the complete set of restrictions that result from the copula representation of PQD. Two empirical examples have illustrated its practical use in detecting PQD in US insurance claim data and life expectancy data.

The test has been designed in the spirit of a Kolmogorov-Smirnov functional, but other possibilities are available. We may for example opt for a weighted supremum test statistic $\sqrt{n}\sup_{u,v}\{D_n(u,v)w(u,v)\}$ for some non-negative weighting function $w(u,v)$. The results of this paper carry over in that case. We may also design tests based on Cramer-von Mises type functionals, such as $\int\int[\max\{0, D_n(u,v)\}]^r w(u,v)dudv$ for some positive r. However the bootstrap procedure needs then to be modified to make it consistent. Note also that there is no obvious ranking across these sorts of tests as which functional yields asymptotic efficiency depends on the alternative being tested (see Nikitin (1995)).

Let us further remark that the procedure is rather straighforward to extend to accommodate dimensions higher than two. This will lead to tests for positive orthant dependences (see e.g. Newman (1984)) as described in the next lines.

A d-dimensional random vector Y is said to be positively lower orthant dependent (PLOD, in short) if

$$C(u_1,...,u_d) \geq C^{\perp}(u_1,...,u_d) := \prod_{i=1}^{d} u_i, \qquad \forall (u_1,...,u_d) \in [0,1]^d, \tag{0.4}$$

while it is said to be positively upper orthant dependent (PUOD, in short) if

$$\bar{C}(u_1,...,u_d) \geq \bar{C}^{\perp}(u_1,...,u_d) := \prod_{i=1}^{d}(1-u_i), \qquad \forall (u_1,...,u_d) \in [0,1]^d, \tag{0.5}$$

where $\bar{C}$ denotes the survival copula associated with C (see Nelsen (1999)). Of course, (0.4) and (0.5) are no more equivalent when $d \geq 3$. When (0.4) and (0.5) simultaneously hold, then Y is said to be positively orthant dependent (POD, in short).

The extension of the testing procedure in the PLOD case is immediate. Indeed we have that the empirical copula process converges weakly to

$$\begin{aligned}\mathbb{G}_C(u_1,...,u_d) &= \mathbb{B}_C(u_1,...,u_d) - \partial_1 C(u_1,...,u_d)\mathbb{B}_C(u_1,1,..,1) \\ &\qquad ... - \partial_d C(u_1,...,u_d)\mathbb{B}_C(1,...,1,u_d),\end{aligned}$$

where $\mathbb{B}_C$ is a tight Brownian bridge on $[0,1]^2$ with covariance function

$$E[\mathbb{B}_C(u_1,...,u_d)\mathbb{B}_C(u_1',...,u_d')] = C(u_1 \wedge u_1',...,u_d \wedge u_d') - C(u_1,...,u_d)C(u_1',...,u_d'),$$

for each $0 \leq u_1, ..., u'_d \leq 1$. Hence the aforementioned results remain valid with $S_n := \sqrt{n} \sup_{u_1,...,u_d} \{C^{\perp}(u_1, ..., u_d) - C_n(u_1, ..., u_d)\}$.

The PUOD case is more delicate to handle. One could rely on the link between C and $\bar{C}$. When $d = 3$, we have $\bar{C}(u_1, u_2, u_3) = u_1 + u_2 + u_3 - 2 + C(1 - u_1, 1 - u_2) + C(1 - u_1, 1 - u_3) + C(1 - u_2, 1 - u_3) - C(1 - u_1, 1 - u_2, 1 - u_3)$ (see Georges, Lamy, Nicolas, Quibel & Roncalli (2001) for a translation formula in the general case), and build the estimator $\bar{C}_n$ obtained from substituting C_n for C. The weak convergence of the empirical copula process can then be again invoked to conclude that $\sqrt{n}(\bar{C}_n - \bar{C})$ has a Gaussian limit. This means that the properties of a testing procedure based on $\bar{S}_n := \sqrt{n} \sup_{u_1,...,u_d} \{\bar{C}^{\perp}(u_1, ..., u_d) - \bar{C}_n(u_1, ..., u_d)\}$ should be similar.

Finally to derive a test for POD, one could rely on a test statistic equal to the supremum of the bivariate vector made of $\sqrt{n}\{C^{\perp}(u_1, ..., u_d) - C_n(u_1, ..., u_d)\}$ and $\sqrt{n}\{\bar{C}^{\perp}(u_1, ..., u_d) - \bar{C}_n(u_1, ..., u_d)\}$, and parallel the previous developments.

APPENDIX

All limits are taken as n goes to infinity.

Proof of Lemma 0.1

Under continuous differentiability of the copula function, Theorem 4 of FRW states that the empirical copula process converges weakly in $l^\infty([0, 1]^2)$ towards

$$\mathbb{G}_C(u, v) = \mathbb{B}_C(u, v) - \partial_1 C(u, v)\mathbb{B}_C(u, 1) - \partial_2 C(u, v)\mathbb{B}_C(1, v),$$

where $\mathbb{B}_C$ is a tight Brownian bridge on $[0, 1]^2$ with covariance function

$$\Omega(u, v, u', v') := E[\mathbb{B}_C(u, v)\mathbb{B}_C(u', v')] = C(u \wedge u', v \wedge v') - C(u, v)C(u', v'),$$

for each $0 \leq u, u', v, v' \leq 1$. This yields the result after computing the covariance function.

Proof of Proposition 0.2

1. Proof of Part i):

From the definitions of S_n and the fact that under H_0, $D(u, v) \leq 0$ for all $(u, v) \in [0, 1]^2$, we get that

$$\begin{aligned} S_n &\leq \sup_{u,v} \sqrt{n}\{D_n(u, v) - D(u, v)\} + \sup_{u,v} \sqrt{n} D(u, v) \\ &\leq \sup_{u,v} \sqrt{n}\{D_n(u, v) - D(u, v)\}. \end{aligned}$$

Hence the results follows from the weak convergence of $\sqrt{n}\{D_n(u, v) - D(u, v)\}$ and the definition of $\bar{S}$.

2. Proof of Part ii):

If the alternative is true, then there is some (u,v), say $(\bar{u},\bar{v}) \in [0,1]^2$, for which $D(\bar{u},\bar{v}) = \delta > 0$. Then the result follows using the inequality $S_n \geq \sqrt{n} D_n(\bar{u},\bar{v})$ and Theorem 4 of FRW.

Proof of Proposition 0.3

Let us write

$$\begin{aligned}
\mathbb{G}_{C_n}(u,v) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\mathbb{I}\{F_n(X_i) \leq u, G_n(Y_i) \leq v\} - C(u,v) \right] U_i \\
&\quad - c_{1,n}(u,v) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\mathbb{I}\{F_n(X_i) \leq u\} - u \right] U_i \\
&\quad - c_{2,n}(u,v) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\mathbb{I}\{G_n(Y_i) \leq v\} - v \right] U_i, \\
&\quad - \{C_n(u,v) - C(u,v)\} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i.
\end{aligned}$$

First consider the last term. Note that Theorem 4 of FRW implies that almost every observed sample has the property that

$$\sup_{u,v} |C_n(u,v) - C(u,v)| \longrightarrow 0.$$

Then since the U_i are i.i.d. $N(0,1)$, we have that conditional on the sample:

$$\begin{aligned}
&P_U[\sup_{u,v} |\{C_n(u,v) - C(u,v)\} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i| > \epsilon] \\
&\qquad = P_U[\sup_{u,v} |C_n(u,v) - C(u,v)| |\frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i| > \epsilon] \\
&\qquad \leq \frac{\{\sup_{u,v} |C_n(u,v) - C(u,v)|\}^2 E[\frac{1}{n} \sum_{i=1}^{n} U_i^2]}{\epsilon^2} \longrightarrow 0.
\end{aligned}$$

Consequently for this sample we have that $\{C_n(u,v) - C(u,v)\} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i \stackrel{p}{\longrightarrow} 0$, (where 0 is the zero function, a member of the space $l^{\infty}([0,1]^2)$) which implies $\{C_n(u,v) - C(u,v)\} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i \Longrightarrow 0$. But this holds for almost all samples so that $\{C_n(u,v) - C(u,v)\} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i \stackrel{a.s}{\Longrightarrow} 0$.

For the first term since $\|U\|_1 \equiv E|U| < \infty$, $\|U\|_{2,1} \equiv \int_0^{\infty} \sqrt{P(|U| > x)} dx < \infty$, and $E \max_{1 \leq i \leq n} |U_i| / \sqrt{n} \longrightarrow 0$, we deduce from the multiplier inequalities of Lemma 2.9.1 of VW that the asymptotic equicontinuity conditions for the three empirical and multiplier processes are equivalent, respectively. This means that the sum of first three terms converge weakly to an independent copy $\mathbb{G}'_C$ of $\mathbb{G}_C$. As in Theorem 2.9.7 of VW this leads to the almost sure conditional convergence $\sup_{h \in BL_1} |E_U h(\mathbb{G}_{C_n}) - E h(\mathbb{G}_C)| \stackrel{a.s.}{\longrightarrow} 0$ where BL_1 is the set of bounded Lipschitz functions on $l^{\infty}([0,1]^2)$.

Now to show the result concerning the asymptotic behavior of the p-values, let $P_n(t)$ be the c.d.f. of the process (conditional on the original sample) generated by $\sup_{u,v} \mathbb{G}_{C_n}(u,v)$. The CMT gives that

$$\sup_{u,v} \mathbb{G}_{C_n}(u,v) \overset{a.s.}{\Longrightarrow} \sup_{u,v} \mathbb{G}'_C(u,v), \tag{0.6}$$

where the latter random variable is an independent copy of $\bar{S}$. Note that the median of the distribution $P^0(t)$ of $\sup_{u,v} \mathbb{G}'_C(u,v)$ is strictly positive and finite. Since $\mathbb{G}'_C$ is a Gaussian process indexed by two parameters living in the compact set $[0,1]^2$, P^0 is absolutely continuous (Tsirel'son (1975)), while $c(\alpha)$ defined by $P[\bar{S} > c(\alpha)] = \alpha$ is finite and positive for any $\alpha < 1/2$ (Proposition A.2.7 of VW). The event $\{p_n < \alpha\}$ is equivalent to the event $\{S_n > c_n(\alpha)\}$ where

$$\inf\{t : P_n(t) > 1-\alpha\} = c_n(\alpha) \overset{a.s.}{\longrightarrow} c(\alpha), \tag{0.7}$$

by (0.6) and the aforementioned properties of P^0. Then:

$$\begin{aligned} \lim P[\text{reject } H_0|H_0] &= \lim P[S_n > c_n(\alpha)] \\ &= \lim P[S_n > c(\alpha)] + \lim\{P[S_n > c_n(\alpha)] - P[S_n > c(\alpha)]\} \\ &\le P[\bar{S} > c(\alpha)] := \alpha, \end{aligned}$$

where the last statement comes from (0.7), part *i*) of Proposition 0.2 and $c(\alpha)$ being a continuity point of the distribution of $\bar{S}$. On the other hand part *ii*) of Proposition 0.2 and $c(\alpha) < \infty$ ensure that $\lim P[\text{reject } H_0|H_1] = 1$.

Proof of Proposition 0.4

Let C_n^* be the empirical copula associated to the bootstrap sample. Theorem 6 of FRW states that $\sqrt{n}(C_n^* - C_n)$ converges weakly to an independent copy $\mathbb{G}''_C$ of $\mathbb{G}_C$ in probability conditionnally on the sample in the sense $\sup_{h\in BL_1} |E_{X,Y} h(\mathbb{G}_{C_n}) - Eh(\mathbb{G}_C)| \overset{p}{\longrightarrow} 0$, where $E_{X,Y}$ is the expectation given the original sample. Hence we deduce from the CMT that $S_n^* \overset{p}{\Longrightarrow} \sup_{u,v} \mathbb{G}''_C(u,v)$, where the latter random variable is an independent copy of $\bar{S}$, and we can pursue as in the proof of Proposition 0.3 but using convergence in probability instead of almost sure convergence to get the final result.

ACKNOWLEDGEMENTS

The author would like to thank Christian Genest and two referees for constructive criticism, as well as Professors Frees and Valdez for kindly providing the Loss-ALAE data, which were collected by the US Insurance Services Office (ISO). The author also wishes to express his deep gratitude to Bruno Rémillard for his valuable help in designing the simulation-based multiplier method in an appropriate way. He is further grateful to the participants at seminars at ULB and CERN as well as at the DeMoSTAFI conference for their comments. The author acknowledges financial support by the Swiss National Science Foundation through the National Center of Competence: Financial Valuation and Risk Management (NCCR FINRISK). Part of his research was done when he was visiting THEMA and ECARES.

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