WEAK CONVERGENCE OF HEDGING STRATEGIES OF CONTINGENT CLAIMS

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Abstract

This paper presents results on the convergence for hedging strategies in the setting of incomplete financial markets. We examine the convergence of the so-called locally risk-minimizing strategy. It is proved that such a choice for the trading strategy, when perfect hedging of contingent claims is infeasible, is robust under weak convergence. Several fundamental examples extracted from the financial modelling literature illustrate this property for both deterministic and random time intervals shrinking to zero.

Keywords: Weak convergence; incomplete financial markets; locally risk-minimizing strategy; hedging strategy; minimal martingale measure.

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1. Introduction

Weak convergence of financial markets is well understood in the setting of complete markets. Indeed, convergence of option prices and gain processes is extensively studied by several authors (see e.g. Cox, Ross and Rubinstein (1979), He (1990), Duffie and Protter (1992), Cutland, Kopp and Willinger (1993), Delbaen and Schachermayer (1994), ...). The most famous example of such a convergence is given by the convergence of the binomial model to the Black-Scholes model, when the time interval between trading dates shrinks to zero. Both models are complete, and contingent claim payoffs can be perfectly replicated by a self-financing strategy. For these models, Pedersen (1999) has recently established the convergence of the trading strategy for European and path dependent payoffs. In a more general framework, Jacod, Méléard and Protter (1999) provide results which can be directly applied to the analysis of the convergence of the trading strategies for European contingent claims when asset prices are martingales under the historical probability. Their results rely fundamentally on the Kunita Watanabe decomposition of martingales.

In this paper our purpose is to extend the results of Pedersen (1999) and Jacod, Méléard and Protter (1999) to the case of incomplete financial markets, when prices are no more martingales. In such an incomplete framework, we adopt the minimal martingale measure (Föllmer and Schweizer (1991), Schweizer (1991)). This choice has the advantage to guarantee both the convergence of the option price processes and the existence of an explicit form of the trading strategy, the so-called locally risk-minimizing strategy. In the following we prove the robustness of this locally risk-minimizing strategy under weak convergence. This complements ideally the results already known on the convergence of option prices in incomplete markets (Runggaldier and Schweizer (1995), Hubalek and Schachermayer (1998), Prigent (1999), and Lesne, Prigent and Scaillet (2000)).

The paper is organized as follows. Section 2 recalls the notion of a locally risk-minimizing strategy, and how this type of strategy can be characterized explicitly in discrete time and continuous time. In Section 3 we provide the main result of the paper, namely the weak convergence of hedging strategies. The regularity of the semigroup of the stock price when it follows a Markov process will play a major role in its derivation. First we examine the case of deterministic time intervals between trading dates when they shrink to zero. Several fundamental examples extracted from the financial modelling literature are developed in order to illustrate the main result. Then, we give convergence results for random time intervals shrinking to zero. This covers the example of discrete portfolio rebalancing. Section 4 gathers some concluding remarks.

2. A review of the locally risk-minimizing strategy

In this section, we first recall the main results concerning the locally risk-minimizing strategy in discrete time (Schweizer (1988, 1990, 1993) and Schäl (1994)) before proceeding further with the continuous time case (Schweizer (1991)). These results constitute the building blocks for the next section in which we develop convergence properties of this type of hedging strategy.

2.1. The locally risk-minimizing strategy in discrete time

We consider a financial market with two traded assets: a riskless money market account and a risky asset, referred to as the bond and the stock, respectively. The value of the money market account at time t is given by $B_t = B_0(1+r)^t$, where r is the riskfree interest rate. Withous loss of generality, we take $r = 0$ and $B_0 = 1$ in the following. The evolution of the stock price is described by a stochastic process $(S_t)_{t\in\{0,1,\ldots,T\}}$ on a probability space $(\Omega, \mathcal{F}, I\!\!P)$, with some filtration $(\mathcal{F}_t)_{t\in\{0,1,\ldots,T\}}$ on $(\Omega, \mathcal{F})$ ($I\!\!P$ is usually called the historical probability). The price S_t is taken $\mathcal{F}_t$-measurable and square-integrable.

A trading strategy will be of the form $\phi = (\alpha, \beta)$. The sequence $(\alpha_t)_{t\in\{0,1,\ldots,T\}}$, resp. $(\beta_t)_{t\in\{0,1,\ldots,T\}}$, describes the successive amounts invested into the stock, resp. the bond. The process $(\alpha_t)_t$ is a square-integrable predictable process, while $(\beta_t)_t$ is adapted and square-integrable.

The value of the portfolio at time t corresponding to the strategy ϕ is then given by:

$$V_t(\phi) = \alpha_t S_t + \beta_t.$$

The cumulated cost up to time t is defined by:

$$C_t = V_t - \sum_{l\leq t} \alpha_{l-1}\Delta S_l,$$

where ΔS_l is equal to $S_l - S_{l-1}$. In particular, $C_0 = V_0$. Both $(V_t)_{t\in\{0,1,\ldots,T\}}$ and $(C_t)_{t\in\{0,1,\ldots,T\}}$ are adapted processes.

Consider now a financial asset, for example an option, written at time $t = 0$ with payoff H_T at maturity T. The payoff H is an $\mathcal{F}_T$-measurable random variable. A European call option with strike K would correspond to $H_T = (S_T - K)^+$. In practice, financial markets are incomplete, and the financial asset exhibits a positive intrinsic risk based on the cumulative cost C_t up to time t. Hence a fair price for hedging the payoff H must take into account optimal hedging strategies ϕ against H.

A strategy $\phi = (\alpha, \beta)$ is called mean-self-financing if the corresponding cost process $(C_t)_t$ is a martingale:

$$I\!\!E[\Delta C_t|\mathcal{F}_t] = 0.$$

In particular, one has:

$$V_0 = I\!\!E[C_T].$$

The locally risk-minimizing strategy minimizes locally the conditional risk:

$$I\!\!E[(C_{t+1} - C_t)^2|\mathcal{F}_t].$$

As in Schweizer (1988) or Schäl (1994) we introduce the nondegeneracy condition (ND):

S is said to satisfy the nondegeneracy condition (ND) if there exists a constant $\delta \in]0, 1[$ such that :

$$(I\!\!E[\Delta S_k|\mathcal{F}_{k-1}])^2 \leq \delta I\!\!E[(\Delta S_k)^2|\mathcal{F}_{k-1}],\ I\!\!P\ a.s. \qquad \text{(ND)}$$

Proposition 2.1.

Under the condition (ND):

i) Every square-integrable contingent claim payoff H admits a decomposition:

$$H = H_0 + \sum_{k=1}^{T} \xi_k^H \Delta S_k + L_T^H, \qquad I\!P \text{ a.s.}, \tag{1}$$

where H_0 is a constant, ξ^H is predictable and $L^H = (L_k^H)_k$ is a square-integrable martingale with $E[L_0^H] = 0$ which is strongly orthogonal to the martingale part of the stock price S.

ii) H_0 is equal to $I\!E_{\hat{I\!P}}[H]$ and by letting $\hat{V}_k = I\!E_{\hat{I\!P}}[H|\mathcal{F}_k]$, ξ^H is determined by:

$$\xi_k^H = \frac{I\!E[\Delta\hat{V}_k \Delta S_k|\mathcal{F}_{k-1}]}{I\!E[\Delta S_k^2|\mathcal{F}_{k-1}]},$$

where $\hat{I\!P}$ denotes the minimal martingale measure.

iii) $\alpha = \xi^H$ and $\beta = \hat{V} - \xi^H S$ determine the unique locally risk-minimizing strategy for H.

Proof. See Schweizer (1988) or Schäl (1994).

2.2. The locally risk-minimizing strategy in continuous time

We start with the general case before analyzing the particular case of Markov processes.

2.2.1. General case

Let $(\Omega, \mathcal{F}, I\!P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions of right-continuity and completeness. Assume that $\mathcal{F}_0$ is trivial and that $\mathcal{F}_T = \mathcal{F}$. The price S_t is again supposed to be $\mathcal{F}_t$-measurable and square-integrable. It is a semimartingale with a decomposition:

$$S_t = S_0 + A_t + M_t,$$

such that:

1) M is a square-integrable martingale with M_0=0.

2) A is a predictable process of finite variation $|A|$ with A_0=0.

We denote the predictable quadratic variation of the martingale M by $< M, M >$. A trading strategy will be of the form $\phi = (\alpha, \beta)$ satisfying the following conditions:

1) $(\alpha_t)_{t\in[0,T]}$ is a predictable process.

2) The process $\int_0^t \alpha_u dS_u$ is a semimartingale of class $\mathcal{S}^2$, i.e.:

$$I\!E[\int_0^T \alpha_u^2 d < M, M >_u + \int_0^T |\alpha_u| d|A|_u)^2] < \infty.$$

3) $(\beta_t)_{t\in[0,T]}$ is adapted.

The value of the portfolio at time t corresponding to ϕ is then given by the process $V(\phi)$ defined by $V_t(\phi) = \alpha_t S_t + \beta_t$. It is right-continuous and satisfies: $V_t(\phi) \in \mathcal{L}^2(\mathbb{P})$. The cumulated cost up to time t is defined by:

$$C_t = V_t(\phi) - \int_0^t \alpha_u dS_u.$$

As measure of risk, Schweizer (1991) introduces the conditional mean square error process:

$$R_t(\phi) = \mathbb{E}[(C_T(\phi) - C_t(\phi))^2 | \mathcal{F}_t].$$

A strategy ϕ is mean-self-financing if $C(\phi)$ is a martingale. A contingent claim is attainable if and only if there exists an admissible strategy ϕ such that $V_T(\phi) = H_T$ and $R_t(\phi) = 0$, $\mathbb{P}$ *a.s.*

As mentioned in Schweizer (1991), it may happen that no strategy ϕ can minimize $R_t(\phi)$ for all t. Therefore, it is necessary to weaken this condition. This gives birth to the notion of locally risk-minimizing strategy, which is defined as follows:

A trading strategy $\Delta = (\delta, \epsilon)$ is called a small perturbation if it satisfies the following conditions:
1) δ is bounded.
2) $\int_0^t |\delta_u| d|A_u|$ is bounded.
3) $\delta_T = \epsilon_T = 0$.

The underlying idea is to introduce the concept of local variation of a trading strategy. To this end, consider partitions $\tau = (t_i)_i$ of the interval $[0, T]$. Such partitions will always satisfy:

$$0 = t_0 < t_1 < ... < t_N = T,$$

with a mesh size define by: $|\tau| = \max_i (t_i - t_{i-1})$. A sequence $(\tau_n)_{n\in\mathbb{N}}$ of partitions will be increasing if $\tau_n \subseteq \tau_{n+1}$ for all n. It will be called 0-convergent if it satisfies:

$$\lim_{n\to+\infty} |\tau_n| = 0.$$

If Δ is a small perturbation and $(s, t]$ a subinterval of $[0, T]$, the small perturbation $\Delta|(s,t] = (\delta|(s,t], \epsilon|(s,t])$ is defined by setting:

$$\delta|(s,t](\omega, u) = \delta_u(\omega) 1_{(s,t]}(u) \text{ and } \epsilon|(s,t](\omega, u) = \epsilon_u(\omega) 1_{[s,t)}(u).$$

The asymetry corresponds to the fact that δ is predictable and ϵ adapted.

Let ϕ be a trading strategy, Δ a small perturbation and τ a partition of $[0, T]$. Then, the R-quotient is defined by

$$r^\tau[\phi, \Delta](\omega, t) = \sum_{t_i \in \tau} \frac{R_{t_i}(\phi + \Delta|(t_i, t_{i+1}]) - R_{t_i}(\phi)}{\mathbb{E}[< M, M >_{t_{i+1}} - < M, M >_{t_i} | \mathcal{F}_{t_i}]}(\omega) \quad 1_{(t_i, t_{i+1}]}(t).$$

The strategy ϕ is called locally risk-minimizing if

$$\lim \inf_{n \to +\infty} r^{\tau_n}[\phi, \Delta] \geq 0, \qquad I\!P_M \ a.s.$$

for every small perturbation Δ and every increasing 0-convergent sequence (τ_n) of partitions of $[0, T]$. The ratio $r^{\tau}[\phi, \Delta]$ can be interpreted as a measure for the total change of riskiness if ϕ is locally perturbated by Δ along the partition τ.

Suppose that:

H1) For $I\!P$-almost all ω, the measure on $[0, T]$ induced by $< M, M > (\omega)$ has the whole interval $[0, T]$ as support (the martingale M is not locally constant: for example, a diffusion process with a strictly positive diffusion coefficient or a point process with a strictly positive intensity). Then, Schweizer (1991) shows that if a trading strategy is locally risk-minimizing then it is mean-self-financing.

Recall a main result about the computation of the locally risk-minimizing strategy in continuous time. As in the discrete time setting, we denote by $\hat{I\!P}$ the minimal martingale measure, and by $\hat{V}$ the conditional expectation of the contingent claim payoff H under $\hat{I\!P}$:

$$\hat{V}_t = I\!E_{\hat{I\!P}}[H|\mathcal{F}_t].$$

Suppose furthermore that:

H2) A is continuous.

H3) A is absolutely continuous with respect to $< M, M >$ with a density α satisfying $E_M[|\alpha|Log^+|\alpha|] < \infty$.

H4) S is continuous at time T (no fixed time of discontinuity at T).

Assume that:

H5) There exists a square-integrable $I\!P$-martingale N such that M and N form a $I\!P$-basis of $\mathcal{L}^2(I\!P)$, and S et N form a $\hat{I\!P}$-basis of $\mathcal{L}^2(\hat{I\!P})$ (see Schweizer (1991) p. 355).

Proposition 2.2.

Under conditions H1-5:

i) Every square-integrable contingent claim payoff H admits a decomposition:

$$H = I\!E[H]_{\hat{I\!P}} + \int_0^T \xi_u^{H,\hat{I\!P}} dS_u + \int_0^T \zeta_u^{H,\hat{I\!P}} dN_u, \qquad \hat{I\!P} \ a.s., \tag{2}$$

where $\xi^{H,\hat{I\!P}}$ and $\zeta^{H,\hat{I\!P}}$ are predictable.

ii) If $(S_t)_{t\in[0,T]}$ has continuous trajectories, then the locally risk-minimizing strategy is given by:

$$\alpha_t = \frac{d < \hat{V}, S >_t^{\hat{I\!P}}}{d < S, S >_t^{\hat{I\!P}}},$$

where $< ., . >^{\hat{I\!P}}$ is the predictable quadratic covariation with respect to $\hat{I\!P}$.

iii) The unique locally risk-minimizing strategy for the contingent claim H is given by $\alpha = \xi^{H,\hat{I\!P}}$ and $\beta = \hat{V} - \xi^{H,\hat{I\!P}}$.

Proof. See Theorem 3.2 of Schweizer (1991).

Let us remark that the decompositions (1) and (2) leading to the characterization of the locally risk-minimizing strategy in discrete time and continous time, respectively, look similar. They hold under two different probability measures, namely under the historical probability for the first one, and under the minimal martingale measure for the second one. Nevertheless, as mentioned in Heath, Platen and Schweizer (1999), under the assumptions H1-5, a strategy is locally risk-minimizing if and only if it is pseudo-locally risk-minimizing (i.e. its cost process C is a square integrable P-martingale, strongly P-orthogonal to the P-martingale part M of S). In that case, finding a locally risk-minimizing strategy is equivalent to finding a decomposition of H:

$$H = H_0 + \int_0^T \xi_u^H dS_u + L_T^H,$$

so that H_0 is in $L^2(\mathcal{F}_0, P)$ and L^H is a square integrable P-martingale null at 0 and strongly P-orthogonal to M. In that case, $\alpha = \xi^H$ and $C_t = H_0 + L_t^H$.

2.2.2. Particular case of Markov processes

Consider now a risky asset whose price S follows a Markov diffusion process with jumps:

$$dS_t = a(t, S_{t-})dt + \sigma(t, S_{t-})dW_t + \int_{I\!R} \gamma(t, S_{t-}, x)(\mu(dt, dx) - \nu(dt, dx)),$$

where $(W_t)_t$ is a standard Brownian motion, $\mu(dt, dx)$ is the counting measure of a compound Poisson process, independent of $(W_t)_t$, and $\nu(dt, dx)$ is its compensator measure. The coefficient $a(.,.)$, $\sigma(.,.)$, $\gamma(.,.)$ are supposed to satisfy usual conditions to guarantee existence and unicity of the above equation and $\sigma(.,.) > 0$.

The compensator takes the form $\nu(dt, dx) = ldtK(dx)$ where K is a deterministic probability kernel and l a nonnegative constant(see Section 3.2 for further elaboration). We assume that both $(W_t)_t$ and $\mu(dt, dx)$ are defined on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ where $\mathcal{F}_t$ is generated by S. In that case, according to Colwell and Elliot (1993), the locally risk-minimizing strategy is determined as follows.

Consider the value $\hat{V}$ of the contingent claim under the minimal martingale measure $\hat{I\!P}$. Applying the Kunita Watanabe decomposition to $\hat{V}$, we get :

$$\hat{V}_t = \hat{V}_0 + \int_0^t \phi_s^c d\hat{W}_s + \int_0^t \int_{I\!R} \phi_s^d(\mu(ds, dx) - \hat{\nu}(ds, dx)),$$

where $(\hat{W}_t)_t$ is a standard Brownian motion under $\hat{I\!P}$ and $\hat{\nu}$ denotes the compensator of the measure of jumps under $\hat{I\!P}$.

Then, from Theorem 3.1 of Colwell and Elliot (1993), the locally risk-minimizing trading strategy exists and is given by:

$$\alpha(t, S_{t-}) = \frac{\sigma(t, S_{t-})\phi^c(t, S_{t-}) + \int_{I\!R} \gamma(t, S_{t-}, x)\phi^d(t, S_{t-}, x)lK(dx)}{\sigma^2(t, S_{t-}) + \int_{I\!R} \gamma^2(t, S_{t-}, x)lK(dx)}$$

and

$$\beta(t, S_{t-}) = \hat{V}_t - \alpha(t, S_{t-})S_t.$$

Note that $\alpha(t, S_{t-})$ is still equal to $d < \hat{V}, S >_t^{\hat{I\!P}} / d < S, S >_t^{\hat{I\!P}}$.
Moreover, $\phi^c = \sigma \partial \hat{V} / \partial x$ and $\phi^d(t, S_{t-}, x) = \hat{V}(t, S_{t-} + \gamma(t, S_{t-}, x)) - \hat{V}(t, S_{t-})$.

In order to analyze $\alpha(t, S_{t-})$ and in particular ϕ^c and ϕ^d, let us examine another way to compute the previous locally risk-minimizing trading strategy. Since under the minimal martingale measure $\hat{I\!P}$, S is a martingale, it is possible to use the result of Jacod, Méléard and Protter (1999). They give an alternative form to the Clark-Haussman formula for Markov processes.

Under $\hat{I\!P}$, S is the solution of the equation:

$$S_t^x = x + \int_0^t \sigma(s, S_{s-}^x) d\hat{W}_s + \int_0^t \int_{I\!R} \gamma(s, S_{s-}^x, x) \left(\mu(ds, dx) - \hat{\nu}(ds, dx)\right).$$

The functions σ and γ are supposed to be continuously differentiable functions with bounded derivatives σ' and γ'. Consider the solution S' of the following linear equation :

$$S_t'^x = 1 + \int_0^t \sigma'(s, S_{s-}^x) S_s'^x d\hat{W}_s + \int_0^t \int_{I\!R} \gamma'(s, S_{s-}^x, x) S_s'^x \left(\mu(ds, dx) - \hat{\nu}(ds, dx)\right).$$

Assume that the contingent claim has the form $H = f(S_T)$ where f is a measurable function with at most linear growth. Introduce :

$$\hat{P}_t f(x) = I\!E_{\hat{I\!P}}[f(S_t^x)],$$

and

$$\hat{Q}_t f(x) = I\!E_{\hat{I\!P}}[f(S_t^x) S_t'^x].$$

Note that $(\hat{P}_t)$ is the semi-group of the Markov process S^x.

Then we can determinate ϕ^c and ϕ^d by using a modified version of Theorem 2.6 of Jacod, Méléard and Protter (1999). In particular, the following formulas involve the quantity $\hat{Q}_t f'$ which plays a key role in establishing convergence results.

Proposition 2.3.

If f is differentiable with bounded derivative f' then the unique locally risk-minimizing strategy for the contingent claim H is characterized by:

$$\begin{array}{rcl} \phi^c(t, S_{t-}) & = & \hat{Q}_{T-t} f'(S_{t-}^x), \\ \phi^d(t, S_{t-}, z) & = & \gamma(t, S_{t-}^x, z) \int_0^1 \hat{Q}_{T-t} f'\left(S_{t-}^x + \gamma(t, S_{t-}^x, z)u\right) du, \end{array}$$

which gives:

$$\alpha(t, f(S_{t-}^x)) = \hat{Q}_{T-t} f'(S_{t-}^x)$$
$$+ \tfrac{1}{\tilde{c}(t, S_{t-}^x)} \int_{I\!R} l\gamma^2(t, S_{t-}^x, z) K(dz) \int_0^1 \left(\hat{Q}_{T-t} f'(S_{t-}^x + \gamma(S_{t-}^x, z)u) - \hat{Q}_{T-t} f'(S_{t-}^x)\right) du,$$

where $\tilde{c}(t, S^x_{t-}) = \sigma^2(t, S^x_{t-}) + \int_{I\!R} \gamma^2(t, S^x_{t-}, z) lK(dz)$.

Proof. It follows basically the lines of Jacod, Méléard and Protter (1999), Theorem 2.6. The key point is that $\frac{\partial}{\partial x}\hat{P}_t f(x) = \hat{Q}_t f'(x)$ when the jumps are bounded, σ, γ are infinitely differentiable with bounded derivatives of all orders and f is twice continuously differentiable with f, f' and f" bounded. For the general case, the result can be established by using suitable approximations of both the Poisson process and the functions σ, γ and f.

3. Convergence of the locally risk-minimizing strategy

The standard weak convergence denoted by $X_n \overset{\mathcal{L}(I\!D)}{\Rightarrow} X$ is considered on the set $I\!D[0, T]$ of right-continuous with left-hand limits (r.c.l.l.) functions on $[0, T]$ with values in $I\!R$ endowed with the Skorokhod topology (see Jacod and Shiryaev (1987)). All processes indexed by n are defined on the filtered probability spaces $(\Omega_n, \mathcal{F}_n, (\mathcal{F}_{n,t}), I\!P_n)$ for $[0, T]$ and satisfy the usual conditions.

To ensure the weak convergence of stochastic integrals, the property of uniform tightness (UT) (see Stricker (1985)) is used. Recall that this condition is the following:

For all t, the family of distributions $\{I\!P^n_{H_n.X_{n,t}}, H_n \in \mathcal{H}_{n,t}\}$ is tight, where $\mathcal{H}_{n,t}$ is the set of all elementary predictable processes H_n such that $H_{n,s} = Y_{n,0} + \sum_{i=0}^{k} Y_{n,t_i} I_{]t_i,t_{i+1}]}(s)$ with Y_{n,t_i} any $\mathcal{F}_{n,t_i}$-measurable variable satisfying $|Y_{n,t_i}| \leq 1$ and $\{t_0, ..., t_k\}$ any partition of $[0, t]$.

This property will essentially guarantee the existence of the limit integrals $H.X$. Details on this condition can be found for instance in Jakubowski, Mémin and Pages (1989) and also in Mémin and Slominski (1991).

As it can be seen in Propositions 2.2 and 2.3, the locally risk-minimizing strategy is based on expectations under the minimal martingale measure. So, to get convergence of this strategy, we have first to ensure the convergence of options prices under the minimal martingale measure.
This problem has been previously solved in Runggaldier and Schweizer (1995) and in Prigent (1999).

According to Ansel and Stricker (1992, 1993), the stock prices S_n has the canonical decomposition:

$$S_n = M_n + \lambda_n. < M_n, M_n >,$$

where λ_n is a predictable process that satisfies :

$$\lambda_n^2. < M_n, M_n > < \infty,$$

and the Radon-Nikodym derivative of the minimal martingale measure is given by $\mathcal{E}(-\lambda_n.M_n)$.

Recall the result in Prigent (1999) on the joint convergence of the price process and the Radon-Nikodym derivative of the minimal martingale measure:

Proposition 3.1.

If M_n has the property UT , and if M is a locally square-integrable martingale with $\lambda^2. < M, M > < \infty$, while:

$$(M_n, < M_n, M_n >, \lambda_n) \stackrel{\mathcal{L}(\mathbb{D}^3)}{\Longrightarrow} (M, < M, M >, \lambda),$$

then

$$(S_n, \hat{\eta}_n) \stackrel{\mathcal{L}(\mathbb{D}^2)}{\Longrightarrow} (S, \hat{\eta}).$$

This proposition shows that the minimal martingale measure is a suitable choice to guarantee the robustness of option pricing under weak convergence.

Now that we are fully equipped, we may examine the topic of our paper, namely convergence of locally risk-minimizing strategies. In the following both deterministic and random time intervals between trading dates will be examined. We start with the analysis of the former.

3.1. Convergence for deterministic time intervals

For the sake of simplicity, we take $T = 1$. Let f be a differentiable function with a bounded and Lipschitz derivative. The contingent claim H is given by $H = f(S_1)$.

The discrete time model consists of a sequence of i.i.d. variables $(Y_{n,k})_n$ defined on a probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ such that:

$$\mathbb{E}_n[Y_{n,k}] = 0, \quad \mathbb{E}_n[Y_{n,k}^2] = \frac{1}{n}, \quad \mathbb{E}_n[Y_{n,k}^4] \leq \frac{\epsilon_n}{n},$$

with $\epsilon_n \to 0$.

Define:

$$Z_n(t) = \sum_{k=1}^{[nt]} Y_{n,k}.$$

We then know that $(Z_n)_n$ converges weakly to the process $Z_t = W_t$, where $(W_t)_t$ is a standard Brownian motion.

Let us introduce the filtration $\mathcal{F}_{n,k} = \sigma(Y_{n,l}, l \leq k)$, and let S_n be solution of:

$$S^x_{n,k} = S^x_{n,k-1} + \frac{a(S^x_{n,k-1})}{n} + \sigma(S^x_{n,k-1}) Y_{n,k}, \qquad S_{n,0} = x,$$

where a and σ are continuously differentiable with bounded and Lipschitz derivatives a' and σ'.

Let also S'_n be solution of:

$$S'^x_{n,k} = S'^x_{n,k-1} + \frac{a'(S^x_{n,k-1})}{n} S'^x_{n,k-1} + \sigma'(S^x_{n,k-1}) S'^x_{n,k-1} Y_{n,k}, \qquad S'_{n,0} = 1.$$

We further denote by P^n_Y the common distribution of the $Y_{n,k}$ under $I\!P_n$, and

$$\begin{aligned}
\hat{P}^n_{\frac{k}{n}} f(x) &= I\!E_{\hat{I\!P}_n}[f(S^x_{n,k})], \\
\hat{Q}^n_{\frac{k}{n}} f(x) &= I\!E_{\hat{I\!P}_n}[f(S^x_{n,k}) S'^x_{n,k}].
\end{aligned}$$

First we need to compute the form of the locally risk-minimizing strategy. Note that this strategy differs from the one in Jacod, Méléard and Protter (1999) since we are not in the martingale case.

Lemma 3.1.

The explicit form of $\xi^H_{n,k}$ is given by

$$\xi^H_{n,k} = \frac{n}{\sigma(S^x_{n,k-1})} \int \left[\hat{P}^n_{1-\frac{k}{n}} f\left(S^x_{n,k-1} + \frac{a(S^x_{n,k-1})}{n} + y\sigma(S^x_{n,k-1}) \right) - \hat{P}^n_{1-\frac{k-1}{n}} f(S^x_{n,k-1}) \right] y P^n_Y(dy).$$

Proof. It is a direct consequence of Proposition 2.1 (ii) and previous assumptions.

Second from standard arguments (see Mémin and Slominsky (1991) for example), we get the weak convergence of the sequence of price processes $(S^x_{n,t})_n$ defined by $S^x_{n,t} = S^x_{n,k}$ if $t \in [\frac{k}{n}, \frac{k+1}{n}[$ to:

$$S^x_t = x + \int_0^t a(S^x_{s-}) ds + \int_0^t \sigma(S^x_{s-}) dW_s.$$

Furthermore we also have the weak convergence of the sequence $(S'^x_{n,t})_n$ to:

$$S'^x_t = 1 + \int_0^t a'(S'^x_{s-}) S'^x_{s-} ds + \int_0^t \sigma'(S'^x_{s-}) S'^x_{s-} dW_s.$$

Thus we get the main result of our paper, namely:

Proposition 3.2.

Under previous assumptions, the locally risk-minimizing strategy in discrete-time converges to the locally risk-minimizing strategy in continuous time.

Proof. We parallel computations given in Jacod, Méléard and Protter (1999), but under the non martingale case. From the properties of a and σ, it can be deduced that the mapping from x to $S^x_{n,t}$ is differentiable and its derivative is given by $S'^x_{n,t}$. Therefrom we have that $\frac{\partial}{\partial x} P_{n,t} f(x) = Q_{n,t} f'(x)$ and, by a Taylor expansion, that:

$$\begin{aligned}
&\int \left[\hat{P}^n_{1-\frac{k}{n}} f\left(S^x_{n,k-1} + \frac{a(S^x_{n,k-1})}{n} + y\sigma(S^x_{n,k-1}) \right) - \hat{P}^n_{1-\frac{k-1}{n}} f(S^x_{n,k-1}) \right] y P^n_Y(dy)) \\
&= \int \left[\left(\frac{a(S^x_{n,k-1})}{n} + y\sigma(S^x_{n,k-1}) \right) \left(\hat{Q}^n_{1-\frac{k}{n}} f'(S^x_{n,k-1}) + \epsilon_{n,k} \right) \right] y P^n_Y(dy),
\end{aligned}$$

with $\sup_k \ \epsilon_{n,k} \to 0$.

Using the assumptions on $Y_{n,k}$, it is equal to:

$$\frac{\sigma(S^x_{n,k-1})}{n}\left(\hat{Q}^n_{1-\frac{k}{n}} f'(S^x_{n,k-1}) + \epsilon_{n,k}\right).$$

Thus, the locally risk-minimizing strategy satisfies:

$$\xi^H_{n,k} = \hat{Q}^n_{1-\frac{k}{n}} f'(S^x_{n,k-1}) + \epsilon_{n,k}.$$

Recall that $\hat{Q}^n_{1-\frac{k}{n}} f'(S^x_{n,k-1}) = \mathbb{E}_{\hat{\mathbb{P}}_n}[f'(S^x_{n,k})S'^x_{n,k}]$. Now, applying Proposition 3.1, and standard arguments, we get the convergence

$$(S^x_n, S'^x_n, \hat{\eta}_n) \overset{\mathcal{L}(\mathbb{D}^3)}{\Longrightarrow} (S, S'^x, \hat{\eta})$$

Then, from the assumptions on f, we can deduce that $(\mathbb{E}_{\hat{\mathbb{P}}_n}[f'(S^x_{n,\frac{[nt]}{n}})S'^x_{n,\frac{[nt]}{n}}])_t$ converges weakly to $(\mathbb{E}_{\hat{\mathbb{P}}}[f'(S^x_t)S'^x_t])_t$, which is equal to $\hat{Q}_{1-t}f'(S^x_t)$.

Finally, since the amount invested on the bond is given by $\beta_{n,k} = \mathbb{E}_{\hat{\mathbb{P}}_n}[f(S^x_{n,k})] - \xi^H_{n,k}S^x_{n,k}$, we obtain:

$$\left(\xi^H_{n,\frac{[nt]}{n}}; \beta_{n,\frac{[nt]}{n}}\right)_t \overset{\mathcal{L}(\mathbb{D}^2)}{\Longrightarrow} \left(\hat{Q}_{1-t}f'(S^x_t); \hat{V}_t - \left(\hat{Q}_{1-t}f'(S^x_t)\right) S_t\right)_t.$$

Now take the continuous time model. From Propositions 2.2 and 2.3, since processes are continuous, the locally risk-minimizing strategy ξ^H_t is also equal to $\hat{Q}_{1-t}f'(S^x_{t-})$, and the stated result follows.

Now that the general result has been presented, we may turn our attention to the analysis of several leading examples borrowed from the financial literature.

Example 3.1. Trinomial trees

The so-called trinomial tree is a discrete time model defined on the probability space $\Omega_n = \{\omega_1, \omega_2, \omega_3\}^n$ with:

$$S_{n,t} = S_{n,[nt]} = S_{n,[nt]-1}(1 + \tilde{Y}_{n,[nt]}),$$

$$\tilde{Y}_{n,[nt]} = \frac{\mu}{n} + Y_{n,[nt]},$$

where $(Y_{n,k})_k$ is a sequence of i.i.d. trinomial trials such that

$$\mathbb{P}_{n,k}[Y_{n,k} = \frac{\alpha}{\sqrt{n}}] = p_1, \qquad \mathbb{P}_{n,k}[Y_{n,k} = \frac{\beta}{\sqrt{n}}] = p_2, \qquad \mathbb{P}_{n,k}[Y_{n,k} = \frac{-\gamma}{\sqrt{n}}] = p_3.$$

We assume

$$\mu > 0, \quad \alpha > \beta \geq 0, \quad 1 + \mu > \gamma > \mu \qquad \text{and} \qquad p_1 > 0, \quad p_2 > 0, \quad p_3 > 0,$$

and $p_1\alpha + p_2\beta - p_3\gamma = 0$. The last condition entails $\mathbb{E}[Y_{n,k}] = 0$.

From usual arguments (see Jacod and Shiryaev (1987) p. 404-405 for the Lindeberg-Feller theorem for rowwise triangular arrays), we get convergence to the Black-Scholes model, in which

$$S_t = S_0 \exp[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t),$$

where $(W_t)_t$ is a standard Brownian motion and $\sigma^2 = p_1\alpha^2 + p_2\beta^2 + p_3\gamma^2$.

Corollary 3.1.

Under previous assumptions, the prices $\hat{V}_n(t)$ of European options H computed under the minimal martingale measure, converge to the option prices V_t of the Black-Scholes model and the discrete time locally risk-minimizing strategies converge to their continuous time analogues:

$$\xi_t^H = \frac{\partial V_t}{\partial S_t}, \qquad \beta_t^H = V_t - \xi_t^H S_t.$$

Proof. By straightforward application of Propositions 3.1 and 3.2.

Note that $\hat{\mathbb{P}}$ is here the standard unique martingale measure, since the limiting market is complete. Furthermore the trading strategy ξ_t^H corresponds to the well-known delta $\frac{\partial V_t}{\partial S_t}$ of the option. In fact Corollary 3.1 extends Pedersen's result from the (complete) binomial setting to the (incomplete) trinomial setting. General multinomial models may be examined along the same lines.

Example 3.2. GARCH models

GARCH models are very popular in the empirical financial literature, and have been shown by Nelson (1990) to be suitable tools in approximating stochastic volatility models (Hull and White (1987)). Nelson's approximation built on:

$$\begin{cases} X_{n,k} &= X_{n,k-1} + g_n(\sigma_{n,k}^2) + \sigma_{n,k} Z_{n,k}, \\ \sigma_{n,k}^2 &= \omega_n + \sigma_{n,k-1}^2[\beta_n + \delta_n Z_{n,k-1}], \end{cases}$$

where $\Delta X_{n,k} = \ln(S_{n,k}) - \ln(S_{n,k-1})$, $(Z_{n,k})$ is an i.i.d. triangular array, and coefficients are choosen to ensure $(\sigma_{n,[nt]+1}^2, X_{n,[nt]})_t \overset{\mathcal{L}(\mathbb{D}^2)}{\Longrightarrow} (\sigma_t^2, X_t)_t$.

For example, let us take: ω_n , β_n , δ_n non-negative, $\lim_n(n\omega_n) = \omega$, $\lim_n(n[\beta_n - 1 + \frac{\delta_n}{n}]) = -\theta$, $\lim_n(2\frac{\delta_n^2}{n}) = \delta^2$, $g_n(.) = \frac{1}{n}g(.)$, where g is bounded and continuous, and the variance $\rho_{n,k}^2$ of $(Z_{n,k})_k$ is equal to $\frac{1}{n}$.

Then we get convergence to the Hull-White model:

$$\begin{cases} dX_t = g(\sigma_t^2)dt + \sigma_t dW_{1,t}, \\ d\sigma_t^2 = (\omega - \theta\sigma_t^2)dt + \delta\sigma_t^2 dW_{2,t}, \end{cases}$$

where $(W_{1,t})$ and $(W_{2,t})$ are independent standard Brownian motions.
Note that $S_t = S_0 e^{X_t}$ and so by Ito's formula: $dS_t = S_t[(g(\sigma_t^2) + \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_{1,t}]$.

Corollary 3.2.

Under previous assumptions, the prices $\hat{V}_n(t)$ of European options computed under the minimal martingale measure converge to the option prices $\hat{V}_t$ given by the Hull-White model and the discrete time locally risk-minimizing strategies converge to their continuous time analogues:

$$\xi_t^H = \frac{\partial \hat{V}_t}{\partial S_t}, \qquad \beta_t^H = \hat{V}_t - \xi_t^H S_t.$$

Proof. Let $H = f(S_{n,n}^x)$. Recall that $\xi_{n,k}^H = \frac{\mathbb{E}[\Delta\hat{V}_{n,k}\Delta S_{n,k}^x|\mathcal{F}_{n,k-1}]}{\mathbb{E}[\Delta(S_{n,k}^x)^2|\mathcal{F}_{n,k-1}]}$, which implies:

$$\xi_{n,k}^H = \frac{1}{S_{n,k-1}^x}\frac{\mathbb{E}[\Delta\hat{V}_{n,k}\left(e^{\Delta X_{n,k}} - 1\right)|\mathcal{F}_{n,k-1}]}{\mathbb{E}[\left(e^{\Delta X_{n,k}} - 1\right)^2|\mathcal{F}_{n,k-1}]}.$$

Denote P_Z^n the common distribution of the $Z_{n,k}$. Let us remark that $\sigma_{n,k}$ is predictable. Then, we deduce:

$$\begin{aligned}
\xi_{n,k}^H \quad = \quad & \frac{1}{S_{n,k-1}^x}\frac{1}{\int\left(e^{g_n(\sigma_{n,k}^2)+\sigma_{n,k}z} - 1\right)^2 P_Z^n(dz)} \\
& \times \int \Big[\hat{P}_{1-\frac{k}{n}}^n f\left(S_{n,k-1}^x + S_{n,k-1}^x\left(e^{g_n(\sigma_{n,k}^2)+\sigma_{n,k}z} - 1\right)\right) \\
& \qquad -\hat{P}_{1-\frac{k-1}{n}}^n f(S_{n,k-1}^x)\left(e^{g_n(\sigma_{n,k}^2)+\sigma_{n,k}z} - 1\right)\Big] P_Z^n(dz).
\end{aligned}$$

Hence the results follows for similar argument as before.[3]

Note that the relation $\xi_t^H = \partial\hat{V}_t/\partial S_t$ is established in Pham, Rheinländer and Schweizer (1998).

Example 3.3. "Direct discretizations"

In some cases, investors can only trade at fixed dates although price processes evolve in continuous time. For such traders markets are incomplete (see e.g. Kind, Liptser, and Runggaldier (1991), Eberlein (1992)).

The continuous time process is given by $S_t = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right)$ where $(W_t)_t$ is a standard Brownian motion under the filtration $(\mathcal{F}_t)_t$.

We consider the following direct approximation:

$$S_{n,t} = S_{n,k} \;=\; S_0\exp[(\mu - \frac{1}{2}\sigma^2)t_{n,k} + \sigma W_{t_{n,k}}], \qquad t \in [t_{n,k}, t_{n,k+1}, [,$$

[3] Details on request

where $(t_{n,k})_{\{k\in\{0,\ldots,\, m(n)\}}$ is a partition of the interval [0,1] (i.e. $0 = t_{n,0} < \ldots < t_{n,m(n)} = 1$) whose mesh size $\max_{t_{n,k}} |t_{n,k+1} - t_{n,k}|$ tends to 0. We put $(\mathcal{F}_{n,k})_{n,k} = (\mathcal{F}_{t_{n,k}})_{n,k}$. and $X_{n,k} = \ln(\frac{S_{n,k}}{S_0})$ so that:

$$\Delta X_{n,k} = (\mu - \frac{1}{2}\sigma^2)(t_{n,k} - t_{n,k-1}) + \sigma(W_{t_{n,k}} - W_{t_{n,k-1}}).$$

Corollary 3.3.

Under previous assumptions, the prices $\hat{V}_n(t)$ of European options computed under the minimal martingale measure converge to the option prices V_t of the Black-Scholes model and the discrete time locally risk-minimizing strategies converge to their continuous time analogues:

$$\xi_t^H = \frac{\partial V_t}{\partial S_t}, \qquad \beta_t^H = V_t - \xi_t^H S_t.$$

Proof. By straightforward application of Propositions 3.1 and 3.2.[4]

3.2. Convergence for random time intervals

We present here some results on the convergence of the locally risk-minimizing strategy when time intervals between trading dates are random instead of being deterministic. Such randomness can be induced for example by traders who decide to readjust their positions only after significant moves in the underlying asset price. Traders may decide to discretely rebalance their hedging portfolios only if the price increases (or decreases) by a given percentage a, i.e. according to relative price movements.

This type of portfolio rebalancing is analyzed in Prigent, Renault and Scaillet (1999, 2000). In such a framework price movements are best described by a process whose logarithmic variations have fixed sizes (a or $-a$) and occur at random times. Such a process is a marked point process on the real line with one dimensional mark space. The real line depicts time while the marks describe the random jumps taking place at random times of occurrence of events.

Let S be given by

$$S_t = S_0 e^{X_t},$$

with

$$X_t = \sum_{j:T_j \leq t} Z_j,$$

built from the marked points (T_j, Z_j).

Consider the filtration $(\mathcal{F}_t)_t$ generated by the marked point process. The process X is a purely discontinuous process with jumps $Z_j = \Delta X_{T_j}$ at random times T_j. It takes the

[4]Details on request

form of an integral process $x * \mu$ defined by the sum of the jumps :

$$X_t = \sum_{j:T_j \leq t} \Delta X_{T_j} = x * \mu_t.$$

The integer-valued random measure $\mu(dt, dx)$ on $I\!R_+ \times E$ is the counting measure associated to the marked point process with marks in set $E = \{a, -a\}$.

If $E[\sum_{j:T_j \leq t} \mid Z_j \mid]$ is finite for all t (Jacod and Shiryaev (1987) p. 72), such a process can be decomposed as :

$$X = x * (\mu - \nu) + x * \nu.$$

The measure ν is a predictable measure, called the compensator, with the property that $\mu - \nu$ is a local martingale measure. This measure can be disintegrated as (Jacod and Shiryaev (1987) p. 67) :

$$\nu(dt, dx) = d\Lambda_t K(t, dx),$$

where Λ is a predictable integrable increasing process and K is a transition kernel.

Proposition 3.3.

Under previous assumptions, the locally risk-minimizing strategy is given by: for $t \in]]T_j, T_{j+1}]]$,

$$\alpha_t = \xi_t^H = \frac{\int_{I\!R} \hat{\phi}_{T_j}(x)\delta(x)K(T_{j+1}, dx)}{S_{T_j} \int_E \delta(x)^2 K(T_{n,j+1}, dx)},$$

with $\hat{\phi}_{T_j}(x) = \hat{V}(T_j, S_{T_j}e^x) - \hat{V}(T_j, S_{T_j})$, *and* $\delta(x) = e^x - 1$.

Proof. see Colwell and Elliott (1993).

Let us now examine the convergence to the Black-Scholes model in which the stock price evolves according to a geometric Brownian motion:

$$S_t = S_0 \exp\left((m - \frac{s^2}{2})t + sW_t\right),$$

and when the triggering price increment shrinks to zero. We look at two particular cases: a marked Poisson process and a jump process driven by a latent geometric Brownian motion (see Prigent, Renault, and Scaillet (1999, 2000) for further details on option pricing in this framework).

Example 3.4. Marked Poisson process

The compensator $\nu^a(dt, dx)$ on $I\!R+ \times \{a, -a\}$ of a marked Poisson process with independent binomial marks satisfies:

$$\nu^a(dt, dx) = l^a dt K^a(dx),$$

with

$$
\begin{aligned}
K^a(dx) &= p^a && \text{if} \quad dx = a, \\
&= 1 - p^a && \text{if} \quad dx = -a.
\end{aligned}
$$

The Radon-Nikodym derivative of the minimal martingale measure is characterized by:

$$
\begin{aligned}
\frac{\hat{\eta}^a_T}{\hat{\eta}^a_t} = & \prod_{j:t<T_j\leq T} \left(1 - \frac{\delta(a)p^a + \delta(-a)(1-p^a)}{\delta(a)^2 p^a + \delta(-a)^2(1-p^a)} \delta(Z_j)\right) \\
& \exp\left(l^a \left((T_{j+1}\wedge T) - T_j\right) \frac{\left(\delta(a)p^a + \delta(-a)(1-p^a)\right)^2}{\delta(a)^2 p^a + \delta(-a)^2(1-p^a)}\right).
\end{aligned}
$$

All relevant quantities are indexed by a, and convergence should be understood when letting a go to zero.

Proposition 3.4.

Under the assumptions:

$$
p^a - 1/2 \sim ma/(2s^2) \quad \textit{and} \quad l^a \sim s^2/a^2,
$$

the following conditions are satisfied:

$$
\begin{aligned}
\delta * \nu^a_t &\longrightarrow mt, \\
\delta^2 * \nu^a_t &\longrightarrow s^2 t,
\end{aligned}
$$

from which we deduce:
i)

$$
(S^a_t, \hat{\eta}^a_t) \overset{\mathcal{L}(I\!D^2)}{\Longrightarrow} \left(S_t, \mathcal{E}\left(-\frac{m}{s}W_t\right)\right).
$$

ii) The locally risk-minimizing strategies converge to the Black-Scholes strategies.

Proof. The above conditions can be rewritten as :

$$
\begin{aligned}
l^a \left[(e^a - 1)\,p^a + (e^{-a} - 1)\,(1-p^a)\right] &\longrightarrow m, \\
l^a \left[(e^a - 1)^2 p^a + (e^{-a} - 1)^2 (1-p^a)\right] &\longrightarrow s^2.
\end{aligned}
$$

From Taylor expansions, it can be verified that both conditions are satisfied if : $p^a - 1/2 \sim ma/(2s^2)$ and $l^a \sim s^2/a^2$.

These conditions and the jump boundedness ensure the convergence of the first Doléans-Dade exponential $\mathcal{E}\left(\delta * \mu^a_t\right)$ to $\mathcal{E}\left(mt + sW_t\right)$ (see Jacod and Shiryaev (1987) 3.11 p. 432 and note that both $\delta * \nu_t^a$ and $\delta^2 * \nu_t^a$ are here determistic). Since the martingale part of the discounted price is uniformly tight (the jumps are bounded and the predictable part is increasing) we conclude that the second Doléans-Dade exponential $\hat{\eta}_t^a = \mathcal{E}\left((-\lambda S \delta) * (\mu^a - \nu^a)_t\right)$ converges to the Radon-Nikodym derivative $\hat{\eta}_t$ of the minimal martingale measure as in Prigent (1999) Proposition 3.3 or Lesne, Prigent and Scaillet (1998) Proposition 1.

The stated result (ii) is deduced from the study of $\hat{\phi}_{T_j}^a(x) = \hat{V}(T_j, S_{T_j}^a e^x) - \hat{V}(T_j, S_{T_j}^a)$ as in Proposition 3.2. In fact, from Proposition 2.3:

$$\alpha^a(t, f(S_{t-}^a)) = \frac{\int_{I\!R} \delta^2(z) S_{T_j}^a \left(\int_0^1 \left(\hat{Q}_{T-T_j} f'(S_{T_j}^a + \delta(z) S_{T_j}^a u) \right) du \right) K^a(T_{j+1}, dz)}{S_{T_j}^a \int_{I\!R} \delta(z)^2 K^a(T_{j+1}, dz)},$$

which is also equal to:

$$\frac{\delta^2(a) \left(\int_0^1 \hat{Q}_{T-T_j} f'(S_{T_j}^a + \delta(a) S_{T_j}^a u) du \right) p^a + \delta^2(-a) \left(\int_0^1 \hat{Q}_{T-T_j} f'(S_{T_j}^a + \delta(-a) S_{T_j}^a u) du \right) (1 - p^a)}{p^a \delta(a)^2 + (1 - p^a) \delta(-a)^2}.$$

Besides $\delta^2(a) \sim a^2$, $\delta^2(-a) \sim a^2$, from Taylor expansions, and $p^a \to 1/2$. Furthermore both $\int_0^1 \hat{Q}_{T-T_j} f'(S_{T_j}^a + \delta(a) S_{T_j}^a u) du$ and $\int_0^1 \hat{Q}_{T-T_j} f'(S_{T_j}^a + \delta(-a) S_{T_j}^a u) du$ converge to $\hat{Q}_{T-t} f'(S_t)$, and the stated result follows. [5]

This proposition gives the conditions on the probability p^a and the directing intensity l^a so that the marked Poisson model coincides with the Black-Scholes model in the limit. It embodies the convergence of the incomplete model based on the marked point process to the complete Black-Scholes model.

Example 3.5. Latent geometric Brownian motion

This last example is akin to Example 3.3, and can be viewed as “direct discretizations” in space in place of time.

Let us constitute the set of barriers $B = \{S_0 \exp ja, j \in \mathbb{Z}\}$, and consider the random-crossings of such barriers by the geometric Brownian motion S. This will define a marked point price process S^a. It is enough to take : $\exp X_t = S_t / S_0$, if $S_t \in B$. The entire path of the geometric Brownian motion is assumed to be not observable and the continuous time process driving the jumps is thus latent (hidden). The distributional features of the geometric Brownian motion can nevertheless be used to identify the distribution of the jump process X.

The conditional distribution of arrival times is characterized by the probability that a Brownian motion with drift escapes the corridor $\{a, -a\}$. This probability can be deduced from the trivariate distribution of the running minimum, running maximum and end value of a Brownian motion (Revuz and Yor (1994) p. 104) after a suitable change of measure to incorporate the presence of a drift (see also Kunitomo and Ikeda (1992), Géman and Yor (1994), He, Keirstead and Rebholz (1998)). The conditional distribution of marks is given by the probability that a Brownian motion with drift hits one barrier before the other (see Karlin and Taylor (1975) p. 361).

[5] details on request

The compensator $\nu^a(dt,dx)$ on $I\!\!R^+ \times \{a,-a\}$ satisfies :

$$\nu^a(dt,dx) = l_t^a dt K^a(dx),$$

where for $t \in]]T_j, T_{j+1}]]$:

$$l_t^a = \left[-\frac{d}{dt} F^a(t-T_j) \right] / F^a(t-T_j),$$

and

$$\begin{aligned} K^a(dx) &= \frac{e^{m'a/s^2}}{e^{-m'a/s^2} + e^{m'a/s^2}} \qquad if \quad dx = a, \\ &= \frac{e^{-m'a/s^2}}{e^{-m'a/s^2} + e^{m'a/s^2}} \qquad if \quad dx = -a, \end{aligned}$$

with $m' = m - s^2/2$

In this setting the survival function $F^a(u)$ is equal to the probability that a Brownian motion lives during a time period u between $-a$ and a, or equivalently to the probability that the running minimum and maximum stay above $-a$ and below a, respectively (see Duffie and Lando (1997) for similar computations of hazard rate processes in default event modelling).

$$\begin{aligned} F^a(u) &= \sum_{k=-\infty}^{+\infty} e^{4km'a/s^2} \left\{ \left[\Phi\left(\frac{a - m'u - 4ka}{s\sqrt{u}} \right) - \Phi\left(\frac{-a - m'u - 4ka}{s\sqrt{u}} \right) \right] \right. \\ &\qquad \left. - e^{-2am'/s^2} \left[\Phi\left(\frac{3a - m'u - 4ka}{s\sqrt{u}} \right) - \Phi\left(\frac{a - m'u - 4ka}{s\sqrt{u}} \right) \right] \right\}. \end{aligned}$$

The Radon-Nikodym derivative of the minimal martingale measure is characterized by:

$$\begin{aligned} \frac{\hat{\eta}_T^a}{\hat{\eta}_t^a} &= \prod_{j:t<T_j\leq T} \left(1 - \frac{\delta(a)K^a(a) + \delta(-a)K^a(-a)}{\delta(a)^2 K^a(a) + \delta(-a)^2 K^a(-a)} \delta(Z_j) \right) \\ &\qquad \exp\left(-\frac{(\delta(a)K^a(a) + \delta(-a)K^a(-a))^2}{\delta(a)^2 K^a(a) + \delta(-a)^2 K^a(-a)} \log F^a((T_{j+1}\wedge T) - T_j) \right). \end{aligned}$$

To close this section it remains to exhibit the composition (α_t, β_t) of the hedging strategy which generates $\hat{V}(T, S_T^a)$ (recall that the processes α and β represent the quantities held in the risky and riskless asset, respectively).

The locally risk-minimizing trading strategy (α_t^a, β_t^a) is given by : for $t \in]]T_j, T_{j+1}]]$

$$\alpha_t = \frac{\int_E \tilde{\phi}_{T_j}^a(x)\delta(x)K^a(T_{j+1},dx)}{S_{T_j}^a \int_E \delta^2(x)K^a(T_{j+1},dx)},$$

and :

$$\beta_t^a = \hat{V}(t, S_t^a) - \alpha_t^a S_t^a,$$

with : $\tilde{\phi}^a_{T_j}(x) = \hat{V}(T_j, S^a_{T_j} + S^a_{T_j}(e^x - 1)) - \hat{V}(T_j, S^a_{T_j})$.

Again all relevant quantities are indexed by a, and convergence should be understood when letting a go to zero.

In the derivation of the convergence results, we will need to rely on the following lemma:

Lemma 3.2.[6]

For small a, if N^a_t denotes the number of jumps on the time interval $[0,t]$, then:

$$\int_0^\infty e^{-lt}\mathbb{E}[N^a_t]dt \sim \frac{1}{l^2}\frac{s^2}{a^2}.$$

Proposition 3.5.

Under previous assumptions, we deduce:
i)

$$(S^a_t, \hat{\eta}^a_t) \overset{\mathcal{L}(\mathbb{D}^2)}{\Longrightarrow} \left(S_t, \mathcal{E}\left(-\frac{m}{s}W_t\right)\right).$$

ii) The locally risk-minimizing strategies converge to the Black-Scholes strategies.

Proof .
- The convergence of $(S^a_t)^a$ is immediately deduced from the following property:

$$\sup_{t\in[0,T]} |X^a_t - (m't + sW_t)| \leq a,$$

which is true by construction of X^a (See Jacod and Shiryaev Remark 3.30 and Lemma 3.31 p. 316 with $t^a = X^a - m't + sW_t$ and $Y^a = m't + sW_t$.

- Now, we establish that X^a has the property UT (uniform tightness). Recall that X^a_t is equal to $\sum_{T_j\leq t} Z_j$ and so $X^a_t = (x * \nu^a)_t + x * (\mu^a - \nu^a)_t$.

From Condition (ii) 2 of Theorem 1-4 of Mémin and Slominski (1991), we have to check that $Var(B^a)$ is $\mathbb{P}^a$-stochastically bounded where B^a is the bounded variation part of X^a and so is equal to $x * \nu^a$.
Here, since $x * \nu^a$ is an increasing process, we have:

$$\sup_{[0,T]} Var(x*\nu^a) = (\int xK^a(dx)) \times \int_0^T l^a_t dt.$$

Moreover, for all $L > 0$:

$$\mathbb{P}[(\int xK^a(dx)) \times \int_0^T l^a_t dt > L] \leq \frac{1}{L}(\int xK^a(dx)) \times \mathbb{E}[\int_0^T l^a_t dt].$$

[6]Proof available on request

Recall that $\mathbb{E}[\int_0^T l_t^a dt] = \mathbb{E}[N_t^a]$.

Note also that $\int x K^a(dx) \sim \frac{m'}{s^2} a^2$ since $K(a) \sim \frac{1}{2} + \frac{1}{2}\frac{m'a}{s^2}$.

From Lemma 3.2 and previous relations, we deduce that :

$$\forall \epsilon > 0, \exists L, \mathbb{P}[\sup_{[0,T]} Var(x * \nu^a) > L] < \epsilon,$$

and so $Var(B^a)$ is $\mathbb{P}^a$-stochastically bounded.

- Since $m't + sW_t$ is continuous, we get from Proposition 2.2 of Mémin and Slominski (1991) that the martingale part $x * (\mu^a - \nu^a)$ of X^a converges to the martingale $(sW_t)_t$ and that the bounded variation part $x * \nu^a$ converges to the bounded variation part $m't$. In fact, for small a and fixed t, $\mathbb{E}[\int_0^T l_t^a dt] \sim \frac{s^2}{a^2} t$.

So we obtain:

$$\begin{aligned} \delta * \nu_t^a &\xrightarrow{P} m't, \\ \delta^2 * \nu_t^a &\xrightarrow{P} s^2 t, \end{aligned}$$

The last condition guarantees the convergence of the predictable compensator of the martingales $x * (\mu^a - \nu^a)$. Moreover, since the jumps of these martingales are uniformly bounded, the sequence $(x * (\mu^a - \nu^a))_a$ satisfies the uniform tightness (UT) condition.

Now, we examine the convergence of the Radon-Nikodym density of the minimal martingale measure (see Proposition 3.1). Standard Taylor expansions of λ_t^a prove that λ^a converges weakly to λ with $\lambda_t = \frac{1}{S_t}\frac{m}{s^2}$, which leads to the result.

All the previous properties ensure the convergence of the first Doléans-Dade exponential $\mathcal{E}(\delta * \mu_t^a)$ to $\mathcal{E}(mt + sW_t)$ (Jacod and Shiryaev (1987) Theorem 3.11 p. 432). We conclude that the second Doléans-Dade exponential $\hat{\eta}_t^a = \mathcal{E}((-\alpha S \delta) * (\mu^a - \nu^a)_t)$ converges to the Radon-Nikodym derivative $\hat{\eta}_t$ of the minimal martingale measure as in Prigent (1999) Proposition 3.3 or in Lesne, Prigent and Scaillet (1998) Proposition 1.

The stated result (ii) is deduced of the study of $\hat{\phi}_{T_j}^a(x) = \hat{V}(T_j, S_{T_j}^a e^x) - \hat{V}(T_j, S_{T_j}^a)$ as in the proof of Proposition 3.4[7].

4. Concluding remarks

We have shown that locally risk-minimizing strategies are robust under weak convergence. Some leading examples borrowed from the financial literature have been analyzed for illustration purposes of our general results. Deterministic and random time intervals between trading dates have been considered in these examples. The robustness results contained in this paper concern European options, and ought to be extended to path dependent options and American options. These extensions still await further research.

[7]details on request

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