

# SENSITIVITY ANALYSIS OF VAR AND EXPECTED SHORTFALL FOR PORTFOLIOS UNDER NETTING AGREEMENTS

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## Abstract

In this paper, we characterize explicitly the first derivative of the Value at Risk and the Expected Shortfall with respect to portfolio allocation when netting between positions exists. As a particular case, we examine a simple Gaussian example in order to illustrate the impact of netting agreements in credit risk management. Collateral issues are also dealt with. For practical purposes we further provide nonparametric estimators for sensitivities and derive their asymptotic distributions. An empirical application on a typical banking portfolio is finally provided.

**Key words:** Value at Risk, Expected Shortfall, Sensitivity, Risk Management, Credit Risk, Netting, Collateral.

**JEL Classification:** C14, D81, G10, G21, G22.

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# 1 Introduction

For risk management purposes, the evaluation of marginal impacts of current or new positions on risk measures and regulatory capital has been recognized as an important point (Garman (1996), Jorion (1997)). In practice, this evaluation can be made through explicit estimators of the first order derivatives of some standard risk measures, such as the Value at Risk (VaR) and the Expected Shortfall (ES), with respect to portfolio allocations (Gourieroux, Laurent and Scaillet (2000), hereafter GLS, and Scaillet (2004)). Knowledge of the sensitivity is helpful in reducing the amount of computational time needed to process large portfolios since it avoids the need to recompute risk measures each time the portfolio composition is slightly modified (Kurth and Tasche (2003), Martin, Thompson and Brown (2001), Martin and Wilde (2002)). Besides it allows decomposing global portfolio risk component by component, and identifying the largest risk contributions (Denaault (2001), Garman (1997), Hallerbach (2003), Tasche (1999)). These derivatives are also of particular relevance in portfolio selection problem (see Markowitz (1952) for portfolio selection in a mean-variance framework). They help to characterize and evaluate efficient portfolio allocations<sup>2</sup> when VaR and ES are substituted for variance as measure of risk (GLS (2000), Rockafellar and Uryasev (2000), Yamai and Yoshida (2002b)). In fact, numerical constrained optimization algorithms for computations of optimal allocations usually require consistent estimates of first order derivatives in order to converge properly.

Unfortunately, the results available up to now have fallen short of tackling the problem of netting. Clearly, this is an important omission since most financial positions with respect to one or several counterparties are netted in practice. Neglecting the impact of netting will bias the evaluation of marginal impacts of current or new positions on risk measures and regulatory capital, and will lead to inefficient allocations in portfolio selection problems.

Generally speaking, when trading partners agree to offset their positions or obligations, we say that they are netting. By doing so, they reduce a large number of positions or

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<sup>2</sup>A related topic is dynamic trading strategies under risk limits (Basak and Shapiro (2001), Cuoco, He, and Issaenko (2001), Leippold, Trojani, and Vanini (2002)).

obligations to a smaller number of positions or obligations, and it is on this netted position that the two trading partners settle their outstanding obligations.

In the financial community, positions are most of the time netted inside standardized juridical contracts. Streamlining of documentation has taken place as a result of joint efforts by regulators and financial industry organizations. In 1990, the Bank of International Settlements (BIS) issued minimum standards for the design and operation of netting schemes<sup>3</sup>, while in 1991, the Federal Deposit Insurance Corporation Improvement Act (FDICIA) provided support for netting contracts among banks and other financial institutions. In 1992, the International Swaps and Derivatives Dealers (ISDA) issued its first version of the well-known “ISDA Master Agreement” for over-the-counter (OTC) derivatives markets. Its amended versions are still in force between most market participants around the globe today.

To figure out the relative importance of OTC derivative markets, we recall that the total estimated notional amount of outstanding contracts stood at \$141.7 trillion at the end-of-December 2002 (BIS, 2003), an 11% increase from end-of-June 2002. This compares with a 15% increase in the first half of 2002. At the same time, gross market values grew sharply, rising by 43% to \$6.4 trillion. OTC business continued to accelerate significantly relative to that on exchanges<sup>4</sup>.

As mentioned previously, the term netting is used to describe the process of offsetting mutual positions or obligations e.g. to offset an obligation owed by bank A to bank B with an obligation owed by bank B to bank A. There are three main techniques for netting:

- The payment or settlement netting is the process of settling all deals between two counterparties on a net cash basis, in the same currency. It can be informal or based on a formal agreement. The credit risk between counterparties remains unchanged and they stay legally obligated to settle the gross amounts of their positions. Pay-

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<sup>3</sup>They are known as the Lamfalussy standards after the chairman of the Committee that wrote the report.

<sup>4</sup>Open positions on exchange-traded contracts were stagnant, falling by 1% in the second half of the year 2002. It should be noted, however, that activity in the two types of markets cannot be directly and simply compared owing to inherent differences in the characteristics and uses of the products.

ment netting allows for reducing the need for intra-day liquidity or credit in bridging the timing gaps between gross payments and gross receipts.

- The netting by novation means that a single net amount is contractually substituted for previous individual gross sums owed between two counterparties, i.e. existing obligations are discharged by replacing them with a new obligation. Thus, netting by novation is a formal agreement that aims to reduce liquidity and counterparty credit risk. This novation process may take place automatically within the trading day, on the exchange of confirmations between the two banks. The claims need to be in the same currency. This process can be repeated an infinite number of times until the cut-off time for a particular settlement date. Although netting by novation is, in essence, a bilateral mechanism, it can be operated on multilateral basis within a larger group of banks.
- The close-out netting is an arrangement to settle all contracted but not due liabilities to and claims on an institution by one single payment, immediately after the occurrence of a default or termination event <sup>5</sup>. Since netting by close-out only operates upon the occurrence of a designated event, it cannot have any impact upon the number of payment messages passing between the counterparties in their normal trading relationship. Equally, it has no impact on liquidity or credit risk. If a termination event occurs, each trade is settled individually on due date unless the counterparties also agree on a supplementary netting by novation.

Payment and close-out nettings are part of the ISDA Master agreement. The use of netting techniques can bring significant benefits for balance-sheet purposes, capital usage, credit risk and operational efficiencies <sup>6</sup>. Indeed, it reduces the number of payment messages that have to be exchanged between counterparties. This lowers transaction costs and communication expenses, as well as the chance of mistakes. Moreover, netting is important because it reduces credit and liquidity risks, and ultimately systemic risk. Taking an

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<sup>5</sup>An early termination date is designated as the result of the occurrence of an event of default or a decline in the credit quality of a party following the occurrence of merger-related events.

<sup>6</sup>See also DUFFIE and SINGLETON (2003) for impact of netting in credit derivative pricing.

offsetting position subject to a netting agreement is thus related to credit risk mitigation. The tendency of the regulator is to allow wider range of credit risk mitigants in order to avoid the so-called “domino effect” in the financial sector. In the new Basel Capital Accord (see the 2001 consultative document), on-balance sheet netting agreements of loans and deposits of banks to or from a counterparty will be permitted under some conditions. Note that, in 1995, the 1988 Basel Accord was modified to allow banks to reduce the credit exposures (“credit equivalent” in the Basel terminology) of their derivative positions by bilateral netting procedures (see Crouhy et al. (1998)).

The paper is organized as follows. In Sections 2 and 3 our aim is to extend the sensitivity analysis of VaR and ES to a setting in which netting is allowed, and to propose suitable estimators of the first order derivatives of VaR and ES in that context. We analyze both risk measures since pros and cons exist for each of them (see e.g. the papers of Yamai and Yoshida (2002a,b,c,d)). However ES seems to benefit from clear advantages in a credit environment (Frey and McNeil (2002)). In Section 2, we outline the framework and explain the differences arising from netting agreements in case of default of a given counterparty. The loss function associated with netted positions is no more a simple sum of exposures or mark-to-market valuations, but rather involves some nonlinearities. More precisely, it involves some terms like  $(Y_1 + \dots + Y_I)^+ = \max(Y_1 + \dots + Y_I, 0)$ , when the positions  $Y_i$ ,  $i = 1, \dots, I$ , belong to the same netting agreement at default. Section 2 contains the main result of the paper, namely the explicit characterization of the first order derivatives of VaR and ES for portfolios under netting agreements. The Gaussian case is briefly developed for illustrative purposes. In particular we compare VaR and its sensitivity in the netted and unnetted cases, and show on a simple stylized example that netting is a valuable credit risk mitigation device. In Section 3 we outline an extension of the approach, which allows for the presence of collateral. In Section 4, we derive estimators of sensitivities so that they can be used in practical risk management and portfolio selection procedures. These estimators are of a nonparametric nature and easy to implement. In Section 5 we provide an empirical illustration for a typical portfolio of a large bank. Section 6 contains some concluding remarks. Technical appendices gather proofs.

## 2 Main results

Let us consider a portfolio made of three components:  $a'Y$ ,  $a'_U Y_U$  and  $a'_N Y_N$ . The first component  $a'Y$  is the part which is not sensitive to default risk on a given counterparty, while the two other parts can be affected by its default <sup>7</sup>. The second component  $a'_U Y_U$  gathers unnetted positions, i.e. positions which are outside the netting agreement signed with the counterparty and will not be pooled into a single position in case of counterparty default. The third component  $a'_N Y_N$  is the netted part governed by the master agreement with the given counterparty. At default, unnetted and netted positions should receive a different treatment when they exhibit positive value <sup>8</sup>. The credit loss on unnetted positions corresponds to the sum on the positive part of each position, i.e.  $\sum_j (a_{j,U} Y_{j,U})^+$ , while the credit loss on netted positions corresponds to the positive part of the sum on each position, i.e.  $(a'_N Y_N)^+$ .

Let us now consider an additional single position  $a_Z Z$ . This leads to a portfolio value at the initial date  $t_0$  equal to:

$$V(t_0) = a'Y(t_0) + a'_U Y_U(t_0) + a'_N Y_N(t_0) + a_Z Z(t_0).$$

Hereafter the random variable  $D$  indicates default of the given counterparty at date  $t_1$ , and takes the value 1 (resp. 0) in case of default (resp. no default). More generally,  $D$  corresponds to a credit event that causes some losses, and is the starter of a netting agreement.

In the derivation of the portfolio value at date  $t_1$  we need to distinguish three cases: i)

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<sup>7</sup>Note that the identification of default to a given single counterparty is not a limitation of our analysis. The setting is sufficiently large to accommodate the practical situation of portfolios subject to multiple defaults.

<sup>8</sup>Recall that only contracts showing a positive value at default need to be included in credit loss computations. In a mark-to-market approach, the credit losses are defined as the replacement cost of active positions at the default time. When the market values are negative, there are no losses because the financial institution owes a positive amount to the defaulting counterparty and/or its creditors. This approach agrees with current practice in the financial community. It has been advocated by the Basel Committee for the measurement of credit risk exposures for off-balance-sheet items (see the annex in Basel Committee (1995)).

the position is not subject to default risk on the counterparty; ii) the position is subject to default risk and cannot be netted in case of counterparty default; iii) the position is subject to default risk and falls under the umbrella of the master agreement signed with the counterparty. Then the values of the global portfolio at  $t_1$  are obtained from the contributions of each component under default and no default after taking into account potential netting <sup>9</sup>.

i) No default risk on  $a_Z Z$

$$V(t_1) = a'Y(t_1) + a_Z Z(t_1) + (a'_U Y_U(t_1) + a'_N Y_N(t_1)) - \left( \sum_j (a_{j,U} Y_{j,U}(t_1))^+ + (a'_N Y_N(t_1))^+ \right) D.$$

ii) Default risk on unnetted  $a_Z Z$

$$V(t_1) = a'Y(t_1) + (a'_U Y_U(t_1) + a_Z Z(t_1) + a'_N Y_N(t_1)) - \left( \sum_j (a_{j,U} Y_{j,U}(t_1))^+ + (a_Z Z(t_1))^+ + (a'_N Y_N(t_1))^+ \right) D.$$

iii) Default risk on netted  $a_Z Z$

$$V(t_1) = a'Y(t_1) + (a'_U Y_U(t_1) + a'_N Y_N(t_1) + a_Z Z(t_1)) - \left( \sum_j (a_{j,U} Y_{j,U}(t_1))^+ + (a'_N Y_N(t_1) + a_Z Z(t_1))^+ \right) D.$$

In the following, we are interested in risk measures computed on the portfolio value at date  $t_1$ , namely VaR and ES (see Szegö (2002) for a discussion about conditions for proper use of risk measures) defined by:

$$\alpha = P[V(t_1) < -VaR], \quad (1)$$

$$ES = -E[V(t_1) | V(t_1) < -VaR], \quad (2)$$

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<sup>9</sup>In case of default it may happen that the asset  $Z$  urgently needs to be replaced because of hedging demand. It is then sufficient to subtract the replacement value in case of default at date  $t_1$ , i.e. add the short position  $-a_Z Z(t_1)D$ . Stated results may be easily adapted to handle that case.

where  $\alpha$  is a small loss probability level, say 1%<sup>10</sup>. The dependence of VaR and ES with respect to the loss probability level and portfolio allocations will be assumed implicit in the notations. Equations (1) and (2) may also be expressed in terms of losses using  $L = -V$ :

$$\alpha = P[L(t_1) > VaR], \quad (3)$$

$$ES = E[L(t_1)|L(t_1) > VaR]. \quad (4)$$

Note that our definition of “Expected Shortfall” does not correspond to the most general definition of this coherent measure of risk (see e.g. Acerbi and Tasche (2002), Rockafellar and Uryasev (2002)):

$$\tilde{ES} = E[L(t_1)|L(t_1) > VaR] - (\alpha - P(L(t_1) > VaR)) \frac{1}{\alpha} VaR.$$

Nonetheless, under our assumptions, the law of the loss functions will be absolutely continuous with respect to the Lebesgue measure. In this case, we know that  $ES$  and  $\tilde{ES}$  coincide.

As already mentioned, it is interesting to compute the contribution of each position to the risk of the whole portfolio and monitor the most risky ones for precautionary reasons. Since VaR and ES are homogeneous of degree one, we get by Euler Theorem (see Tasche (1999), for instance) :

$$a' \frac{\partial VaR}{\partial a'} + a_Z \frac{\partial VaR}{\partial a'_Z} + a'_U \frac{\partial VaR}{\partial a'_U} + a'_N \frac{\partial VaR}{\partial a'_N} = VaR,$$

and the same expression holds with  $ES$  substituted for  $VaR$ . The contribution associated with  $a_Z Z$  to the global risk measured by  $VaR$  (resp.  $ES$ ) is simply the sensitivity of  $VaR$  (resp.  $ES$ ) with respect to  $a_Z$  multiplied by the allocation  $a_Z$ , i.e.  $a_Z \partial VaR / \partial a_Z$ ,

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<sup>10</sup>These risk measures may also be computed on portfolio value *changes*:

$$\begin{aligned} \alpha &= P[V(t_1) - V(t_0) < -\bar{VaR}], \\ \bar{ES} &= -E[V(t_1) - V(t_0) | V(t_1) - V(t_0) < -\bar{VaR}]. \end{aligned}$$

Results obtained for  $VaR$  and  $ES$  can be immediately translated to  $\bar{VaR}$  and  $\bar{ES}$  using the two changes of variable  $\bar{VaR} = VaR + V(t_0)$  and  $\bar{ES} = ES + V(t_0)$ . In particular, this will lead to  $\frac{\partial \bar{VaR}}{\partial a_Z} = \frac{\partial VaR}{\partial a_Z} + \frac{\partial V(t_0)}{\partial a_Z}$  and  $\frac{\partial \bar{ES}}{\partial a_Z} = \frac{\partial ES}{\partial a_Z} + \frac{\partial V(t_0)}{\partial a_Z}$  with  $\frac{\partial V(t_0)}{\partial a_Z} = Z(t_0)$ .



resp.  $a_Z \partial ES / \partial a_Z$ . Such a quantity is known in the literature (see e.g. Garman (1997)) as incremental or component VaR.

The first order derivative of  $VaR$  and  $ES$  with respect to  $a_Z$  can be easily computed by applying the following proposition.

**Proposition 1.** *Consider the loss functions*

$$\begin{aligned} L_1 &= X - \varepsilon Z, \\ L_2 &= X - \varepsilon Z + (\varepsilon Z)^+ D, \\ L_3 &= X - \varepsilon Z + (Y + \varepsilon Z)^+ D, \end{aligned}$$

where  $\varepsilon$  is a positive real number<sup>11</sup>, and  $(X, Y, Z)$  is a random vector admitting a continuous conditional density with respect to the Lebesgue measure, conditionally to the event  $D = 1$  and  $D = 0$ . Then

**A. First derivative of Value at Risk.**

$$\partial_\varepsilon VaR_1 = -E[Z | L_1 = VaR_1], \quad (5)$$

$$\begin{aligned} \partial_\varepsilon VaR_2 &= -E[Z \mathbf{1}\{Z < 0\} | L_2 = VaR_2, D = 1] \times P[D = 1 | L_2 = VaR_2] \\ &\quad - E[Z | L_2 = VaR_2, D = 0] \times P[D = 0 | L_2 = VaR_2], \end{aligned} \quad (6)$$

$$\begin{aligned} \partial_\varepsilon VaR_3 &= -E[Z \mathbf{1}\{Y + \varepsilon Z < 0\} | L_3 = VaR_3, D = 1] \times P[D = 1 | L_3 = VaR_3] \\ &\quad - E[Z | L_3 = VaR_3, D = 0] \times P[D = 0 | L_3 = VaR_3]. \end{aligned} \quad (7)$$

**B. First derivative of Expected Shortfall.**

$$\partial_\varepsilon ES_1 = -E[Z | L_1 > VaR_1], \quad (8)$$

$$\begin{aligned} \partial_\varepsilon ES_2 &= -E[Z \mathbf{1}\{Z < 0\} | L_2 > VaR_2, D = 1] \times P[D = 1 | L_2 > VaR_2] \\ &\quad - E[Z | L_2 > VaR_2, D = 0] \times P[D = 0 | L_2 > VaR_2], \end{aligned} \quad (9)$$

$$\begin{aligned} \partial_\varepsilon ES_3 &= -E[Z \mathbf{1}\{Y + \varepsilon Z < 0\} | L_3 > VaR_3, D = 1] \times P[D = 1 | L_3 > VaR_3] \\ &\quad - E[Z | L_3 > VaR_3, D = 0] \times P[D = 0 | L_3 > VaR_3]. \end{aligned} \quad (10)$$

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<sup>11</sup>if  $\varepsilon$  is negative, the results can be applied by simply changing  $Z$  into  $-Z$ .

**Remark 1.** *Note that we can rewrite simply*

$$\partial_\varepsilon VaR_2 = -E[Z\mathbf{1}\{Z < 0, D = 1\}|L_2 = VaR_2] - E[Z\mathbf{1}\{D = 0\}|L_2 = VaR_2],$$

$$\partial_\varepsilon VaR_3 = -E[Z\mathbf{1}\{Y + \varepsilon Z < 0, D = 1\}|L_3 = VaR_3] - E[Z\mathbf{1}\{D = 0\}|L_3 = VaR_3],$$

and

$$\partial_\varepsilon ES_2 = -E[Z\mathbf{1}\{Z < 0, D = 1\}|L_2 > VaR_2] - E[Z\mathbf{1}\{D = 0\}|L_2 > VaR_2],$$

$$\partial_\varepsilon ES_3 = -E[Z\mathbf{1}\{Y + \varepsilon Z < 0, D = 1\}|L_3 > VaR_3] - E[Z\mathbf{1}\{D = 0\}|L_3 > VaR_3].$$

In the aforementioned loss functions we have implicitly assumed zero recovery rates. This point of view, not necessarily unrealistic, could appear as too conservative in some cases. Similar formulas to the ones presented in Proposition 1 may be obtained with arbitrary fixed or random recovery rates. Unfortunately, these formulas are a lot more complicated. Indeed, with a non zero recovery rate  $R$ , the loss function, say  $L_3$ , becomes  $X - \varepsilon Z + (1 - R)(Y + \varepsilon Z)^+$ . When differentiating with respect to  $\varepsilon$ , the simplification between the terms that are coming from the market value  $X - \varepsilon Z$  and those that are coming from the credit exposure  $(1 - R)(Y + \varepsilon Z)^+$  disappears. Fortunately this problem is avoided when dealing with credit exposures only, as we will see below.

Previous computations have been made in an “integrated” credit and market risk environment. Indeed, Var and ES computations have included impact of both market conditions and credit events on the portfolio value. The previous random variables  $X, Y, Z$  are usually functions of some market factors (equity indices, interest rates, exchange rates, credit spreads, ...). Let us remark that we do not rely on the assumption of independence between credit events and changes in these underlying market factors when deriving our results. Actually, some standard credit risk models are based on a dependence between credit events and the variations of some underlying market factors such as equity prices (“structural approaches”) or spreads.

For credit analysis purposes, it is common to focus on credit risk solely. Practitioners and regulators are then concerned with so-called exposures to default. The exposure is a random variable related to positions with respect to a single counterparty, say  $i$ , and

corresponds to a specific amount potentially lost due to the default of that counterparty in the future. Lost amounts are often calculated by some calibrated pricing models. Formally, the exposure at  $t_1$  to counterparty  $i$  can be written as

$$Exp_{U,i}(t_1) = \left( \sum_j (a_{j,U} Y_{j,U}(t_1))^+ + (a_Z Z(t_1))^+ + (a'_N Y_N(t_1))^+ \right), \quad (11)$$

in the unnetted case, and

$$Exp_{N,i}(t_1) = \left( \sum_j (a_{j,U} Y_{j,U}(t_1))^+ + (a'_N Y_N(t_1) + a_Z Z(t_1))^+ \right), \quad (12)$$

in the netted case.

Let us stress that positions are related to the same single counterparty  $i$ . In the first case,  $Z$  is outside a netting agreement while, in the second case, it is not. Note further that exposures have nothing to do with default probabilities of the counterparty. The VaR associated with a single exposure is commonly called a potential exposure, and is a pure credit risk measure linked to positions with a single counterparty.

Moreover, to compute provisions to cover multiple credit losses, it is necessary to aggregate exposures and take into account the probability of default of several counterparties. Losses can be aggregated, for example, inside some portfolios, sections, or perimeters. The total credit loss of a portfolio  $\mathcal{P}$  at  $t_1$  is then equal to

$$L(t_1) = \sum_{i \in \mathcal{P}} Exp_i(t_1) D_i,$$

denoting by  $D_i$  the default of the  $i$ -th counterparty at date  $t_1$ . To study the marginal impact of a particular position  $Z$  related to a given counterparty  $i$  on the VaR and ES of the portfolio  $\mathcal{P}$  is approximately equivalent, depending on the status of  $Z$  in terms of netting, to calculate the sensitivity to  $a_Z$  of the VaR and ES of

$$L_U(t_1) = X + \left( \sum_j (a_{j,U} Y_{j,U}(t_1))^+ + (a_Z Z(t_1))^+ + (a'_N Y_N(t_1))^+ \right) D_i,$$

and

$$L_N(t_1) = X + \left( \sum_j (a_{j,U} Y_{j,U}(t_1))^+ + (a'_N Y_N(t_1) + a_Z Z(t_1))^+ \right) D_i,$$

for some random variable  $X$ .

These sensitivities can be calculated using the following result.

**Proposition 2.** *Consider the loss functions*

$$\begin{aligned} L_4 &= X + (\varepsilon Z)^+ D, \\ L_5 &= X + (Y + \varepsilon Z)^+ D, \end{aligned}$$

where  $\varepsilon$  is a positive real number, and  $(X, Y, Z)$  is a random vector admitting a continuous conditional density with respect to the Lebesgue measure, conditionally to the event  $D = 1$  and  $D = 0$ . Then

**A. First derivative of Value at Risk.**

$$\partial_\varepsilon VaR_4 = E[Z\mathbf{1}\{Z \geq 0\}|L_4 = VaR_4, D = 1] \times P[D = 1|L_4 = VaR_4], \quad (13)$$

$$\partial_\varepsilon VaR_5 = E[Z\mathbf{1}\{Y + \varepsilon Z \geq 0\}|L_5 = VaR_5, D = 1] \times P[D = 1|L_5 = VaR_5]. \quad (14)$$

**B. First derivative of Expected Shortfall.**

$$\partial_\varepsilon ES_4 = E[Z\mathbf{1}\{Z \geq 0\}|L_4 > VaR_4, D = 1]P[D = 1|L_4 > VaR_4], \quad (15)$$

$$\partial_\varepsilon ES_5 = E[Z\mathbf{1}\{Y + \varepsilon Z > 0\}|L_5 > VaR_5, D = 1]P[D = 1|L_5 > VaR_5]. \quad (16)$$

**Remark 2.** *Note that we can rewrite*

$$\begin{aligned} \partial_\varepsilon VaR_4 &= E[Z\mathbf{1}\{Z \geq 0, D = 1\}|L_4 = VaR_4], \\ \partial_\varepsilon VaR_5 &= E[Z\mathbf{1}\{Y + \varepsilon Z \geq 0, D = 1\}|L_5 = VaR_5], \end{aligned}$$

and

$$\begin{aligned} \partial_\varepsilon ES_4 &= E[Z\mathbf{1}\{Z > 0, D = 1\}|L_4 > VaR_4], \\ \partial_\varepsilon ES_5 &= E[Z\mathbf{1}\{Y + \varepsilon Z > 0, D = 1\}|L_5 > VaR_5]. \end{aligned}$$

Clearly sensitivities of potential exposures as defined in Equations (11) and (12) can be deduced from the previous formulas by forcing  $D = 1$ . The above formulas can also be easily modified to incorporate the presence of a non zero (constant or random) recovery rate.

All aforementioned expressions are explicit. In principle they can be computed, at least numerically, if the joint distribution of  $(X, Y, Z, D)$  is known. The exposures are clearly functions of some underlying market factors. Thus, as we have said previously,  $(X, Y, Z)$  may depend on  $D$ . Nonetheless, once the distribution of the underlying random variables  $(X, Y, Z)$  is known and independence with respect to the default event is assumed, estimation is much more simple. This will be the case for reduced form credit risk models where default is not predictable and whose law does not depend on the current or past values of market factors. Implicitly, we will do this assumption in our numerical illustration.

For instance, assume the joint law of  $(X, Y, Z)$  is Gaussian  $\mathcal{N}(m, \Omega)$ . Then it is possible to get a closed form expression for  $\partial_\varepsilon VaR_j$  and  $\partial_\varepsilon ES_j$ ,  $j = 1, \dots, 5$ . Nevertheless, the formulas are tedious to obtain since the positive parts in the losses functions  $L_j$  require to evaluate some truncated distributions.

The assumption of normality may seem odd to some readers: the exposures are usually sums of more or less nonlinear instruments with respect to market factors, and these factors themselves often exhibit non Gaussian features (fat tails, volatility clustering). Normality can be understood under a standard market risk approach, where exposures are sums of hundreds of net values. Alternatively, when the time horizon is growing, the distributions of exposures tend to be Gaussian. Thus, we could consider market exposures in the long run, i.e. for a time horizon of several years, to justify this type of assumption.

In the remaining of this section, we only study a simple portfolio example  $E_1 + \epsilon E_2$  where the vector of exposures  $E = (E_1, E_2)'$  is Gaussian with mean  $m = (m_1, m_2)'$  and covariance matrix

$$\Omega = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

This stylized example is used to illustrate the impact of netting on VaR and their sensitivities.

Let us consider the credit losses when there is no netting agreement between  $E_1$  and  $E_2$ :

$$L_U = (E_1^+ + (\epsilon E_2)^+)D,$$

and when all positions are netted:

$$L_N = (E_1 + \epsilon E_2)^+ D.$$

The credit VaR themselves do not admit explicit forms (even in this simple case) and need to be computed numerically. However, once they are known, they can be plugged in the explicit form of the sensitivities derived in appendix D.

Figure 1 shows the impact of netting as a function of  $\rho$  for  $m_1 = 10$ ,  $m_2 = -1$ ,  $\epsilon = 1$  and  $\sigma_1 = \sigma_2 = 1$ . This portfolio can be viewed as made of a long aggregated position in one category of assets and a short position in another one with the same counterparty. In the first case the two positions cannot be netted while they can in the second case. VaR are estimated by computing empirical quantiles on 100,000 simulations delivered by an antithetic variate Monte Carlo procedure. The loss probability level  $\alpha$  is equal to 1% (99% VaR) and the default probability  $p$  is equal to 20%.

- Please insert Figure 1 approximately here -

It can be observed that netting is valuable from a credit risk management point of view. Indeed, VaR are always lower in the netted case <sup>12</sup>. These results show that netting can be thought as sound credit risk mitigation practice and should be encouraged. Moreover, the range of VaR sensitivities is larger in the netted case. Hence this example also shows that the sensitivities can differ substantially between the two situations, and that netting agreements should be taken into account when computing VaR sensitivities. In the netted case we may observe that the VaR sensitivity becomes negative for highly negative correlations. The explanation is simple. If  $E_1$  and  $E_2$  are strongly negatively correlated,  $E_2$  is usually very negative when  $E_1$  is very positive. Hence increasing slightly the position in  $E_2$  has a favorable effect on the VaR of the global netted position. On the contrary it is not possible to benefit from the offsetting effect in the unnetted case, and the VaR sensitivity stays positive always.

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<sup>12</sup>Note that if we increase the mean of  $E_2$  in absolute terms, the benefits of netting increase since the probability of taking positive values diminishes.

### 3 Extension to collateral agreements

Netting agreements are only one part of the credit mitigation techniques. The possibility to exchange collateral is often used by lots of dealers. It is recognized as a way to reduce significantly the amount of credit exposures, even if it may entail other risks: liquidity risk, legal risk, custody risk and operational risk.

The ISDA credit forms - each referred to as a Credit Support Annex (CSA) - that go with international master agreements are now part of the standard documentation. Most national master agreements also have a specific collateral annex. The CSA includes a schedule of elections to be negotiated between the counterparties, including a counterparty threshold, that is the amount of uncollateralized credit exposure the other is willing to accept. In general, the threshold reflects the counterparty credit rating: the higher the rating, the larger the threshold.

The collateral reduces the current exposure of the collateral taker to the collateral giver by the amount of collateral held. Even if the net value of the collateral is varying, we will assume for the sake of computational convenience that the collateral process bounds the net exposure from above by some constant threshold  $M$ . Thus, the loss functions  $L_j$ ,  $j = 1, \dots, 5$  stay unchanged except for the positivity condition being replaced by the belonging to the interval  $[0, M]$ . The new sensitivities can be calculated as previously. With obvious notations we provide the results for the most general loss functions only.

**Proposition 3.** *Consider the loss functions*

$$\begin{aligned}L'_3 &= X - \varepsilon Z + (Y + \varepsilon Z) \cdot \mathbf{1}\{Y + \varepsilon Z \in [0, M]\}D + M \cdot \mathbf{1}\{Y + \varepsilon Z > M\}D, \\L'_5 &= X + (Y + \varepsilon Z) \cdot \mathbf{1}\{Y + \varepsilon Z \in [0, M]\}D + M \cdot \mathbf{1}\{Y + \varepsilon Z > M\}D,\end{aligned}$$

where  $\varepsilon$  is a positive real number, and  $(X, Y, Z)$  is a random vector admitting a continuous conditional density with respect to the Lebesgue measure, conditionally to the event  $D = 1$  and  $D = 0$ . Then

**A. First derivative of Value at Risk.**

$$\begin{aligned} \partial_\varepsilon VaR'_3 &= -E[Z\mathbf{1}\{Y + \varepsilon Z \notin [0, M], D = 1\}|L'_3 = VaR'_3] \\ &\quad -E[Z\mathbf{1}\{D = 0\}|L'_3 = VaR'_3], \end{aligned} \tag{17}$$

$$\partial_\varepsilon VaR'_5 = E[Z\mathbf{1}\{Y + \varepsilon Z \in [0, M], D = 1\}|L'_5 = VaR'_5]. \tag{18}$$

**B. First derivative of Expected Shortfall.**

$$\begin{aligned} \partial_\varepsilon ES'_3 &= -E[Z\mathbf{1}\{Y + \varepsilon Z \notin [0, M], D = 1\}|L'_3 > VaR'_3] \\ &\quad -E[Z\mathbf{1}\{D = 0\}|L'_3 > VaR'_3], \end{aligned} \tag{19}$$

$$\partial_\varepsilon ES'_5 = E[Z\mathbf{1}\{Y + \varepsilon Z \in [0, M], D = 1\}|L'_5 > VaR'_5]. \tag{20}$$

Note that the formulas are remarkably similar to previous ones. This was not obvious from the start, in particular for  $L'_3$ .

## 4 Estimation

Since all sensitivities can be evaluated through conditional expectations, they are amenable to estimation by nonparametric techniques such as the well-known kernel method. Note further that the simple example of Section 2 shows that even a bivariate setting is difficult to handle analytically. Other parametric multivariate distributions would probably not be tractable. Hence we do not see how one could do without a nonparametric method when netting is introduced. An additional advantage is avoiding any misspecification problems.

To be specific, assume the joint distribution of  $(X, Y, Z, D)$  is known explicitly, or that we can get at least  $n$  random draws  $(x_i, y_i, z_i, d_i)_{i=1, \dots, n}$  from the joint distribution of  $(X, Y, Z, D)$ . This is the usual situation because most of the risk models are parametric and/or simulation-based. This setting is large enough to host some dependence between “the market factors”  $(X, Y, Z)$  and the default event <sup>13</sup>.

Let us consider a consistent estimator  $\widehat{VaR}_4$  of  $VaR_4$ , whose rate of convergence is the parametric one ( $n^{-1/2}$ ). Such an estimator can be obtained for example by computing an

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<sup>13</sup>For instance this dependence exists when equity prices and default probabilities are linked.



empirical quantile on the simulated losses  $l_{4,i}$ ,  $i = 1, \dots, n$ . Then, an estimate of  $\partial_\varepsilon VaR_4$  can be defined by the usual nonparametric kernel regression estimator:

$$\widehat{\partial_\varepsilon VaR_4} = \frac{\sum_{i=1}^n z_i \mathbf{1}\{z_i \geq 0, d_i = 1\} K_h(l_{4,i} - \widehat{VaR_4})}{\sum_{i=1}^n K_h(l_{4,i} - \widehat{VaR_4})}, \quad (21)$$

where  $K : \mathbb{R} \rightarrow \mathbb{R}$  is a kernel (an integrable function whose integral is one),  $h = h(n)$  is a smoothing parameter, called the bandwidth (a sequence of positive real numbers which tends to 0 when  $n$  tends to the infinity) and  $K_h(u) = K(u/h)/h$ <sup>14</sup>. Similarly,

$$\widehat{\partial_\varepsilon VaR_5} = \frac{\sum_{i=1}^n z_i \mathbf{1}\{y_i + \varepsilon z_i \geq 0, d_i = 1\} K_h(l_{5,i} - \widehat{VaR_5})}{\sum_{i=1}^n K_h(l_{5,i} - \widehat{VaR_5})}. \quad (22)$$

Both estimators are easy to implement since they correspond to ratios of straight averages of known functions of the observed data points. They can be even computed in a standard spreadsheet.

Denote the density of  $L_k$  by  $f_k$ ,  $k = 4, 5$ , and set

$$\begin{aligned} \mu_4(v) &= E[Z^2 \mathbf{1}\{Z > 0, D = 1\} | L_4 = v] - (\partial_\varepsilon VaR_4)^2, \\ \mu_5(v) &= E[Z^2 \mathbf{1}\{Y + \varepsilon Z > 0, D = 1\} | L_5 = v] - (\partial_\varepsilon VaR_5)^2. \end{aligned}$$

Then we prove the following asymptotic normality result in Appendix D.

**Proposition 4.** *Assume for  $k = 4, 5$ ,*

- $\sqrt{n}(\widehat{VaR}_k - VaR_k) = O_P(1)$ ,
- $K$  is an even kernel function,  $\int |t|^3 |K|(t) dt < \infty$ ,  $\lim_{|t| \rightarrow \infty} |t|^3 K(t) = 0$ . It is three times continuously differentiable,  $K'$  and  $K''$  are integrable and  $K'''$  is bounded.
- $E[|Z|^p] < \infty$  for every integer  $p$ ,
- $t \mapsto E[Z^4 | L_k = t]$  is bounded in a neighborhood of  $VaR_k$ ,
- $nh^5 \rightarrow 0$  and  $nh^{7/2} \rightarrow \infty$ ,

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<sup>14</sup>Obviously, direct differentiation of  $\widehat{VaR}_4$  through computation of finite difference has no meaning here since the empirical quantile is *not* differentiable w.r.t.  $\varepsilon$ .

- $f_k$  and the function  $VaR_k \mapsto \partial_\varepsilon VaR_k$  are two times continuously differentiable,
- $f_k$  and  $\mu_k$  are continuous and strictly positive in a neighborhood of  $VaR_k$ .

Then

$$\sqrt{nh} \left\{ \widehat{\partial_\varepsilon VaR_k} - \partial_\varepsilon VaR_k \right\} \xrightarrow[n \rightarrow \infty]{law} \mathcal{N}(0, \sigma_k^2).$$

$$\sigma_k^2 = \frac{\int K^2 \cdot \mu_k(VaR_k)}{f_k(VaR_k)}.$$

Similar kernel estimators of  $\partial_\varepsilon VaR_j$ ,  $j = 1, 2, 3$  can be easily inferred. They are asymptotically normal under similar regularity conditions.

As usually these asymptotic normality results can be used to build confidence intervals around estimates <sup>15</sup>. The previous asymptotic variances  $\sigma_k^2$  can be easily estimated. For instance, to get an estimate of  $\sigma_5^2$ , we may replace  $\mu_5(VaR_5)$  by

$$\hat{\mu}_5 = \frac{\sum_{i=1}^n z_i^2 \mathbf{1}\{y_i + \varepsilon z_i \geq 0, d_i = 1\} K_h(l_{5,i} - \widehat{VaR_5})}{\sum_{i=1}^n K_h(l_{5,i} - \widehat{VaR_5})} - \left( \widehat{\partial_\varepsilon VaR_5} \right)^2,$$

and  $f_5(VaR_5)$  by

$$\hat{f}_5 = \frac{1}{n} \sum_{i=1}^n K_h(l_{5,i} - \widehat{VaR_5}).$$

We can also estimate the sensitivity of the expected shortfall by

$$\widehat{\partial_\varepsilon ES_4} = \frac{\sum_{i=1}^n z_i \mathbf{1}\{z_i \geq 0, d_i = 1, l_{4,i} > \widehat{VaR_4}\}}{\sum_{i=1}^n \mathbf{1}\{l_{4,i} > \widehat{VaR_4}\}}, \quad (23)$$

and

$$\widehat{\partial_\varepsilon ES_5} = \frac{\sum_{i=1}^n z_i \mathbf{1}\{y_i + \varepsilon z_i \geq 0, d_i = 1, l_{5,i} > \widehat{VaR_5}\}}{\sum_{i=1}^n \mathbf{1}\{l_{5,i} > \widehat{VaR_5}\}}. \quad (24)$$

The strong consistency of the two latter estimators can be deduced easily from the strong uniform consistency of the empirical process (see Van der Vaart (1994)). Furthermore, their asymptotic normality can be established under some regularity conditions.

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<sup>15</sup>Since we may rely on an arbitrarily large number of drawings, the lack of large samples is typically not an issue in a credit risk environment. In particular a large number  $n$  (say 100,000) of losses can be simulated so that confidence intervals are sufficiently narrow to guarantee statistically precise sensitivity estimates.

However the forms of their asymptotic normal distributions depend on the choice of the estimators  $\widehat{VaR}_k$ , and are more complicated than in the VaR case. Note also that their rate of convergence will be the parametric one ( $n^{-1/2}$ ) instead of the slower nonparametric one ( $(nh)^{-1/2}$ ), when  $n^{1/2}(\widehat{VaR}_k - VaR_k)$ ,  $k = 4, 5$  are asymptotically normal sequences.

Clearly, the previous estimation procedures are straightforward to modify to account for the presence of collateral. We only have to replace the indicator functions  $\mathbf{1}\{y_i + \varepsilon z_i > 0\}$  by  $\mathbf{1}\{y_i + \varepsilon z_i \in [0, M]\}$  as before.

## 5 Empirical application

In this empirical section, we wish to illustrate how the estimators of the previous section can be used in practice. We study a real life banking example and analyze the sensitivity w.r.t. an additional short position in a bond (borrowing) with a single counterparty. The default probability is 2% at one year. We examine a short position to emphasize importance of netting in credit risk calculations, and aim to quantify what is the impact of this additional position on credit VaR and credit ES at a 1% loss probability level. 35 other positions are already standing with this counterparty. Each of these 35 positions corresponds to an aggregated position inside a netting agreement (some of them are in fact stand alone positions). Positions can be long or short, and include several categories of assets such as forward contracts, swaps and options in different currencies. Table 1 gives the current value of the 35 aggregated positions in Euros. The current value of the additional short position is equal to -1,150,000 Euros.

- Please insert Table 1 approximately here -

We compare the situation where the additional position can be incorporated inside one of the 35 existing netting agreements with the case where it is left outside any netting <sup>16</sup>. The 35 netted cases are numbered from 1 to 35, and the unnetted case is called case 0. The number of simulations of default times is  $n = 100,000$ , and random future position

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<sup>16</sup>Let us remark that this illustrative example may look a bit fictitious since one cannot usually choose the status of an additional position. We do so because this enlarges the number of studied cases.

values are computed from internal pricing models. The estimation procedure is based on a Gaussian kernel and a bandwidth selected according to the usual rule of thumb, namely  $h = \hat{\sigma}n^{-1/5}$  where  $\hat{\sigma}$  is the empirical standard deviation of the credit losses. Table 2 gives the one year credit VaR and one year credit ES together with their sensitivity for each case.

- Please insert Table 2 approximately here -

As already observed on the Gaussian numerical example, credit VaR and credit ES may differ substantially between netted and unnetted cases. Credit ES are larger than credit VaR revealing much of the danger of using the VaR concept in a credit environment (see Frey and McNeil (2002) for further elaboration). Besides if one enjoys the possibility to choose the agreement, which will host the additional position, then cases 17, 26, 27, 30, 35, resp. 28, seem to be more suitable from a credit risk management point of view since they minimize VaR, resp. ES, at one year. It can be further observed that a small VaR sensitivity can coexist with a larger ES sensitivity (see for instance case 15). This can be explained by the fact that VaR only concerns a single point in the loss distribution, while ES is about an average of points.

## 6 Concluding remarks

Risk measures answer the need of quantifying the risk of potential losses on a portfolio of assets. This need may arise due to internal concerns (risk-reward tradeoff) or external constraints (prudential rules imposed by regulators). In this paper we have proposed estimation procedures allowing for a sensitivity analysis w.r.t. changes in the portfolio allocation. The setting explicitly takes into account the possibility of netting and collateral agreements. The estimation procedures are nonparametric, fast and easy to implement. They have also been shown to be of practical relevance in real life banking situations, and should help to achieve better credit risk management in the future.

**Table 1: Current aggregated position values**

1	2	3	4	5
9,100,000	510	752,076	556,260	-2,140,000
6	7	8	9	10
-2.510 10 <sup>9</sup>	-17,100,000	-22,500,000	3,990,000	-1,520,000
11	12	13	14	15
654,270	688,179	6,890,000	893,581	-28,000,000
16	17	18	19	20
-1,190,000	30,100,000	156,448	-1,680,000	0
21	22	23	24	25
0	0	-29,400,000	-2,010,000	3,220,000
26	27	28	29	30
-872,964	24,900,000	3,700,000	388,944	3,850,000
31	32	33	34	35
-346,792	294,316	-77,199	-1,620,000	3,340,000

**Table 2: One year credit Var, one year credit ES and their sensitivity**

case	$\widehat{VaR}$	$\widehat{\partial_\varepsilon VaR}$	$\widehat{ES}$	$\widehat{\partial_\varepsilon ES}$
0	7.033 10 <sup>7</sup>	0	8.380 10 <sup>7</sup>	0
1	6.937 10 <sup>7</sup>	-0.087 10 <sup>7</sup>	8.304 10 <sup>7</sup>	-0.093 10 <sup>7</sup>
2	7.033 10 <sup>7</sup>	0	8.379 10 <sup>7</sup>	0
3	6.954 10 <sup>7</sup>	-0.062 10 <sup>7</sup>	8.318 10 <sup>7</sup>	-0.060 10 <sup>7</sup>
4	7.007 10 <sup>7</sup>	-0.003 10 <sup>7</sup>	8.371 10 <sup>7</sup>	-0.002 10 <sup>7</sup>
5	7.033 10 <sup>7</sup>	0	8.380 10 <sup>7</sup>	0
6	7.033 10 <sup>7</sup>	0	8.380 10 <sup>7</sup>	0
7	7.033 10 <sup>7</sup>	0	8.380 10 <sup>7</sup>	0
8	7.033 10 <sup>7</sup>	0	8.380 10 <sup>7</sup>	0
9	6.981 10 <sup>7</sup>	0	8.329 10 <sup>7</sup>	0
10	7.001 10 <sup>7</sup>	-0.015 10 <sup>7</sup>	8.354 10 <sup>7</sup>	-0.023 10 <sup>7</sup>
11	7.019 10 <sup>7</sup>	0	8.365 10 <sup>12</sup>	0
12	7.019 10 <sup>7</sup>	-0.005 10 <sup>7</sup>	8.346 10 <sup>7</sup>	-0.013 10 <sup>7</sup>
13	6.941 10 <sup>7</sup>	-0.069 10 <sup>7</sup>	8.301 10 <sup>7</sup>	-0.083 10 <sup>7</sup>
14	7.001 10 <sup>7</sup>	-0.008 10 <sup>7</sup>	8.350 10 <sup>7</sup>	-0.018 10 <sup>7</sup>
15	7.033 10 <sup>7</sup>	-0.004 10 <sup>7</sup>	8.359 10 <sup>7</sup>	-0.019 10 <sup>7</sup>
16	6.975 10 <sup>7</sup>	0	8.332 10 <sup>7</sup>	0
17	6.928 10 <sup>7</sup>	-0.101 10 <sup>7</sup>	8.283 10 <sup>7</sup>	-0.103 10 <sup>7</sup>
18	7.017 10 <sup>7</sup>	0	8.364 10 <sup>7</sup>	0
19	7.033 10 <sup>7</sup>	0	8.380 10 <sup>7</sup>	0
20	7.033 10 <sup>7</sup>	0	8.380 10 <sup>7</sup>	0
21	7.033 10 <sup>7</sup>	0	8.380 10 <sup>7</sup>	0
22	7.033 10 <sup>7</sup>	0	8.380 10 <sup>7</sup>	0
23	7.029 10 <sup>7</sup>	-0.002 10 <sup>7</sup>	8.382 10 <sup>7</sup>	-0.003 10 <sup>7</sup>
24	6.962 10 <sup>7</sup>	-0.033 10 <sup>7</sup>	8.328 10 <sup>7</sup>	-0.038 10 <sup>7</sup>
25	7.033 10 <sup>7</sup>	0	8.380 10 <sup>7</sup>	0
26	6.928 10 <sup>7</sup>	-0.080 10 <sup>7</sup>	8.291 10 <sup>7</sup>	-0.087 10 <sup>7</sup>
27	6.928 10 <sup>7</sup>	-0.101 10 <sup>7</sup>	8.283 10 <sup>7</sup>	-0.103 10 <sup>7</sup>
28	6.931 10 <sup>7</sup>	-0.095 10 <sup>7</sup>	8.278 10 <sup>7</sup>	-0.096 10 <sup>7</sup>
29	7.007 10 <sup>7</sup>	0	8.351 10 <sup>7</sup>	0
30	6.928 10 <sup>7</sup>	-0.087 10 <sup>7</sup>	8.291 10 <sup>7</sup>	-0.086 10 <sup>7</sup>
31	6.992 10 <sup>7</sup>	0	8.338 10 <sup>7</sup>	0
32	6.972 10 <sup>7</sup>	-0.0001 10 <sup>7</sup>	8.328 10 <sup>7</sup>	-0.0002 10 <sup>7</sup>
33	6.999 10 <sup>7</sup>	0	8.346 10 <sup>7</sup>	0
34	7.033 10 <sup>7</sup>	0	8.379 10 <sup>7</sup>	0
35	6.928 10 <sup>7</sup>	-0.101 10 <sup>7</sup>	8.284 10 <sup>7</sup>	-0.102 10 <sup>7</sup>

## A Proof of Proposition 1.

### A.1 First derivative of Value at Risk.

We only provide derivation of  $\partial_\varepsilon VaR_3$ . The formula for  $\partial_\varepsilon VaR_2$  can be obtained from  $\partial_\varepsilon VaR_3$  by simply considering a sequence of random variables  $(Y_n)$  which tends to zero almost surely and by applying a limit theorem (such as the theorem of dominated convergence). The expression for  $\partial_\varepsilon VaR_1$  is a direct consequence of Property 1 in GLS (2000).

We may rewrite the equation defining  $VaR_3$  as:

$$\begin{aligned}
\alpha &= P[L_3 > VaR_3] \\
&= P[X - \varepsilon Z > VaR_3, D = 0] \\
&\quad + P[X - \varepsilon Z > VaR_3, Y + \varepsilon Z \leq 0, D = 1] \\
&\quad + P[X + Y > VaR_3, Y + \varepsilon Z > 0, D = 1] \\
&\equiv T_1 P[D = 0] + T_2 P[D = 1] + T_3 P[D = 1],
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= \int \left[ \int \left\{ \int_{VaR_3 + \varepsilon z}^{+\infty} f_{(X,Y,Z)|0}(x, y, z) dx \right\} dy \right] dz, \\
T_2 &= \int \left[ \int_{-\infty}^{-\varepsilon z} \left\{ \int_{VaR_3 + \varepsilon z}^{+\infty} f_{(X,Y,Z)|1}(x, y, z) dx \right\} dy \right] dz, \\
T_3 &= \int \left[ \int_{-\varepsilon z}^{+\infty} \left\{ \int_{VaR_3 - y}^{+\infty} f_{(X,Y,Z)|1}(x, y, z) dx \right\} dy \right] dz,
\end{aligned}$$

denoting by  $f_{U|1}$ , resp.  $f_{U|0}$ , the density of any random vector  $U$  with respects to the Lebesgue measure, conditionally to  $D = 1$ , resp.  $D = 0$ . To ease reading, set  $p = P[D = 1]$ .

A simple calculation provides

$$\begin{aligned}
\frac{\partial T_1}{\partial \varepsilon} &= - \int (\partial_\varepsilon VaR_3 + z) \left\{ \int f_{(X,Y,Z)|0}(VaR_3 + \varepsilon z, y, z) dy \right\} dz, \\
\frac{\partial T_2}{\partial \varepsilon} &= - \int z \left\{ \int_{VaR_3 + \varepsilon z}^{+\infty} f_{(X,Y,Z)|1}(x, -\varepsilon z, z) dx \right\} dz \\
&\quad - \int (\partial_\varepsilon VaR_3 + z) \left\{ \int_{-\infty}^{-\varepsilon z} f_{(X,Y,Z)|1}(VaR_3 + \varepsilon z, y, z) dy \right\} dz, \\
\frac{\partial T_3}{\partial \varepsilon} &= \int z \left\{ \int_{VaR_3 + \varepsilon z}^{+\infty} f_{(X,Y,Z)|1}(x, -\varepsilon z, z) dx \right\} dz \\
&\quad - \partial_\varepsilon VaR_3 \int \left\{ \int_{-\varepsilon z}^{+\infty} f_{(X,Y,Z)|1}(VaR_3 - y, y, z) dy \right\} dz.
\end{aligned}$$

This yields

$$\begin{aligned}
0 &= (1-p) \frac{\partial T_1}{\partial \varepsilon} + p \cdot \frac{\partial T_2}{\partial \varepsilon} + p \cdot \frac{\partial T_3}{\partial \varepsilon} \\
&= -\partial_\varepsilon VaR_3 \\
&\quad \times \int \left\{ (1-p) \int f_{(X,Y,Z)|0}(VaR_3 + \varepsilon z, y, z) dy \right. \\
&\quad \quad + p \int_{-\infty}^{-\varepsilon z} f_{(X,Y,Z)|1}(VaR_3 + \varepsilon z, y, z) dy \\
&\quad \quad \left. + p \int_{-\varepsilon z}^{+\infty} f_{(X,Y,Z)|1}(VaR_3 - y, y, z) dy \right\} dz \\
&+ \int z \left\{ -(1-p) \int f_{(X,Y,Z)|0}(VaR_3 + \varepsilon z, y, z) dy \right. \\
&\quad \quad \left. - p \int_{-\infty}^{-\varepsilon z} f_{(X,Y,Z)|1}(VaR_3 + \varepsilon z, y, z) dy \right\} dz.
\end{aligned}$$

Clearly,

$$f_{(L_3,Z)|0}(l, z) = f_{(X,Z)|0}(l + \varepsilon z, z), \quad (25)$$

and

$$f_{(L_3,Y,Z)|1}(l, y, z) = f_{(X,Y,Z)|1}(l + \varepsilon z, y, z) \mathbf{1}\{y + \varepsilon z < 0\} + f_{(X,Y,Z)|1}(l - y, y, z) \mathbf{1}\{y + \varepsilon z \geq 0\}. \quad (26)$$

Thus, we deduce

$$\begin{aligned}
0 &= -\partial_\varepsilon VaR_3 \\
&\quad \times \int \left\{ (1-p) \int f_{(L_3,Y,Z)|0}(VaR_3, y, z) dy \right. \\
&\quad \quad \left. + p \int_{-\infty}^{-\varepsilon z} f_{(L_3,Y,Z)|1}(VaR_3, y, z) dy + p \int_{-\varepsilon z}^{+\infty} f_{(L_3,Y,Z)|1}(VaR_3, y, z) dy \right\} dz \\
&+ \int z \left\{ -(1-p) \int f_{(L_3,Y,Z)|0}(VaR_3, y, z) dy - p \int_{-\infty}^{-\varepsilon z} f_{(L_3,Y,Z)|1}(VaR_3, y, z) dy \right\} dz \\
&= -\partial_\varepsilon VaR_3 \int \int f_{(L_3,Y,Z)}(VaR_3, y, z) dy dz + \int z \left\{ -(1-p) \int f_{(L_3,Y,Z)|0}(VaR_3, y, z) dy \right. \\
&\quad \quad \left. - p \int_{-\infty}^{-\varepsilon z} f_{(L_3,Y,Z)|1}(VaR_3, y, z) dy \right\} dz \\
&= -\partial_\varepsilon VaR_3 \cdot f_{L_3}(VaR_3) - (1-p) \int \int z f_{(L_3,Y,Z)|0, L_3=VaR_3}(VaR_3, y, z) dy dz \cdot f_{L_3|0}(VaR_3) \\
&\quad - p \int \int_{-\infty}^{-\varepsilon z} z f_{(L_3,Y,Z)|1, L_3=VaR_3}(VaR_3, y, z) dy dz f_{L_3|1}(VaR_3).
\end{aligned}$$



We get

$$\begin{aligned} \partial_\varepsilon VaR_3 &= \frac{(-p)f_{L_3|1}(VaR_3)}{f_{L_3}(VaR_3)} \int \int_{-\infty}^{-\varepsilon z} z f_{(L_3, Y, Z)|1, L_3=VaR} (VaR_3, y, z) dy dz \\ &\quad - \frac{(1-p)f_{L_3|0}(VaR_3)}{f_{L_3}(VaR_3)} \int \int z f_{(L_3, Y, Z)|0, L_3=VaR_3} (VaR_3, y, z) dy dz, \quad (27) \end{aligned}$$

and Bayes' rule gives the stated result.  $\square$

## A.2 First derivative of Expected Shortfall.

The expression for  $\partial_\varepsilon ES_1$  obtains directly from Proposition 1 in Scaillet (2004). Hence we only provide derivation of  $\partial_\varepsilon ES_3$  since  $\partial_\varepsilon ES_2$  can be deduced from  $\partial_\varepsilon ES_3$ .

$$\begin{aligned} ES_3 &= E[L_3|L_3 > VaR_3] = \frac{1}{\alpha} E[L_3 \mathbf{1}\{L_3 > VaR_3\}] \\ &= \frac{(1-p)}{\alpha} E[L_3 \mathbf{1}\{L_3 > VaR_3\}|D=0] + \frac{p}{\alpha} E[L_3 \mathbf{1}\{L_3 > VaR_3, Y + \varepsilon Z \geq 0\}|D=1] \\ &\quad + \frac{p}{\alpha} E[L_3 \mathbf{1}\{L_3 > VaR_3, Y + \varepsilon Z < 0\}|D=1] \\ &\equiv \frac{(1-p)}{\alpha} \tilde{T}_1 + \frac{p}{\alpha} \tilde{T}_2 + \frac{p}{\alpha} \tilde{T}_3. \end{aligned}$$

By direct differentiation, we get

$$\begin{aligned} \partial_\varepsilon \tilde{T}_1 &= - \int z \mathbf{1}\{x - \varepsilon z > VaR_3\} f_{(X, Z)|0}(x, z) dx dz - VaR_3 \partial_\varepsilon VaR_3 \int f_{(X, Z)|0}(VaR_3 + \varepsilon z, z) dz \\ &\quad - VaR_3 \int z f_{(X, Z)|0}(VaR_3 + \varepsilon z, z) dz \\ &= - \int z \mathbf{1}\{l > VaR_3\} f_{(L_3, Z)|0}(l, z) dl dz - VaR_3 \partial_\varepsilon VaR_3 \int f_{(L_3, Z)|0}(VaR_3, z) dz \\ &\quad - VaR_3 \int z f_{(L_3, Z)|0}(VaR_3, z) dz. \end{aligned}$$

Moreover,

$$\begin{aligned} \partial_\varepsilon \tilde{T}_2 &= -VaR_3 \partial_\varepsilon VaR_3 \int \mathbf{1}\{y + \varepsilon z > 0\} f_{(X, Y, Z)|1}(VaR_3 - y, y, z) dy dz \\ &\quad + \int (x - \varepsilon z) z \mathbf{1}\{x - \varepsilon z > VaR_3\} f_{(X, Y, Z)|1}(x, -\varepsilon z, z) dx dz, \end{aligned}$$

and

$$\begin{aligned}
\partial_\varepsilon \tilde{T}_3 &= - \int z \mathbf{1}\{x - \varepsilon z > VaR_3, y + \varepsilon z < 0\} f_{(X,Y,Z)|1}(x, y, z) dx dy dz \\
&\quad - VaR_3 \int (\partial_\varepsilon VaR_3 + z) \mathbf{1}\{y + \varepsilon z < 0\} f_{(X,Y,Z)|1}(VaR_3 + \varepsilon z, y, z) dy dz \\
&\quad - \int (x - \varepsilon z) z \mathbf{1}\{x - \varepsilon z > VaR_3\} f_{(X,Y,Z)|1}(x, -\varepsilon z, z) dx dz.
\end{aligned}$$

Note that the last term of  $\partial_\varepsilon \tilde{T}_2$  is the opposite of the last term of  $\partial_\varepsilon \tilde{T}_3$ . After summing all terms, we get

$$\begin{aligned}
\alpha \partial_\varepsilon ES_3 &= -(1-p) \int z \mathbf{1}\{l > VaR_3\} f_{(L_3,Z)|0}(l, z) dl dz \\
&\quad - (1-p) VaR_3 \partial_\varepsilon VaR_3 \int f_{(L_3,Z)|0}(VaR_3, z) dz \\
&\quad - (1-p) VaR_3 \int z f_{(L_3,Z)|0}(VaR_3, z) dz \\
&\quad - p VaR_3 \partial_\varepsilon VaR_3 \int \mathbf{1}\{y + \varepsilon z > 0\} f_{(X,Y,Z)|1}(VaR_3 - y, y, z) dy dz \\
&\quad - p \int z \mathbf{1}\{x - \varepsilon z > VaR_3, y + \varepsilon z < 0\} f_{(X,Y,Z)|1}(x, y, z) dx dy dz \\
&\quad - p VaR_3 \partial_\varepsilon VaR_3 \int \mathbf{1}\{y + \varepsilon z < 0\} f_{(X,Y,Z)|1}(VaR_3 + \varepsilon z, y, z) dy dz \\
&\quad - p VaR_3 \int z \mathbf{1}\{y + \varepsilon z < 0\} f_{(X,Y,Z)|1}(VaR_3 + \varepsilon z, y, z) dy dz \\
&\equiv S_1 + \dots + S_7.
\end{aligned}$$

Note that we get by (26):

$$S_2 + S_4 + S_6 = -VaR_3 \partial_\varepsilon VaR_3 f_{L_3}(VaR_3),$$

which simplifies with

$$S_3 + S_7 = VaR_3 \partial_\varepsilon VaR_3 f_{L_3}(VaR_3),$$

because of (27). Thus

$$\begin{aligned}
\alpha \partial_\varepsilon E S_3 &= -(1-p) \int \mathbf{1}\{l > VaR_3\} z f_{(L_3, Z)|0}(l, z) dl dz \\
&\quad - p \int z \mathbf{1}\{l > VaR_3, y + \varepsilon z < 0\} f_{(L_3, Y, Z)|1}(l, y, z) dl dy dz \\
&= -p E[Z \mathbf{1}\{L_3 > VaR_3, Y + \varepsilon Z < 0\} | D = 1, L_3 > VaR_3] P[L_3 > VaR_3 | D = 1] \\
&\quad - (1-p) E[Z | D = 0, L_3 > VaR_3] P[L_3 > VaR_3 | D = 0] \\
&= -E[Z \mathbf{1}\{Y + \varepsilon Z < 0\} | D = 1, L_3 > VaR_3] P[D = 1 | L_3 > VaR_3] \alpha \\
&\quad - E[Z | D = 0, L_3 > VaR_3] P[D = 0 | L_3 > VaR_3] \alpha,
\end{aligned}$$

proving the result.  $\square$

## B Proof of Proposition 2.

### B.1 First derivative of Value at Risk.

Again we only need to explicit  $\partial_\varepsilon VaR_5$ . We may rewrite the equation defining  $VaR_5$  as:

$$\begin{aligned}
\alpha &= P[L_5 > VaR_5] \\
&= (1-p) P[X > VaR_5 | D = 0] \\
&\quad + p P[X > VaR_5, Y + \varepsilon Z < 0 | D = 1] \\
&\quad + p P[X + Y + \varepsilon Z > VaR_5, Y + \varepsilon Z > 0 | D = 1] \\
&\equiv (1-p) \mathcal{T}_1 + p \mathcal{T}_2 + p \mathcal{T}_3,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{T}_1 &= \int \mathbf{1}\{x > VaR_5\} f_{X|0}(x) dx, \\
\mathcal{T}_2 &= \int \mathbf{1}\{x > VaR_5, y + \varepsilon z < 0\} f_{(X, Y, Z)|1}(x, y, z) dx dy dz, \\
\mathcal{T}_3 &= \int \mathbf{1}\{x + y + \varepsilon z > VaR_5, y + \varepsilon z > 0\} f_{(X, Y, Z)|1}(x, y, z) dx dy dz.
\end{aligned}$$

A simple computation yields

$$\begin{aligned}
0 &= -(1-p)f_{X|0}(VaR_5)\partial_\varepsilon VaR_5 - p\partial_\varepsilon VaR_5 \int \mathbf{1}\{y + \varepsilon z < 0\}f_{(X,Y,Z)|1}(VaR_5, y, z) dy dz \\
&\quad -p \int z\mathbf{1}\{x > VaR_5\}f_{(X,Y,Z)|1}(x, -\varepsilon z, z) dx dz \\
&\quad -p \int (\partial_\varepsilon VaR_5 - z)\mathbf{1}\{y + \varepsilon z > 0\}f_{(X,Y,Z)|1}(VaR_5 - y - \varepsilon z, y, z) dy dz \\
&\quad +p \int z\mathbf{1}\{x > VaR_5\}f_{(X,Y,Z)|1}(x, -\varepsilon z, z) dx dz.
\end{aligned}$$

Clearly, we have

$$f_{(L_5,Y,Z)|0}(l, y, z) = f_{(X,Y,Z)|0}(l, y, z), \quad (28)$$

and

$$f_{(L_5,Y,Z)|1}(l, y, z) = \mathbf{1}\{y + \varepsilon z > 0\}f_{(X,Y,Z)|1}(l - y - \varepsilon z, y, z) + \mathbf{1}\{y + \varepsilon z < 0\}f_{(X,Y,Z)|1}(l, y, z). \quad (29)$$

Thus, we can rewrite

$$\begin{aligned}
0 &= -\partial_\varepsilon VaR_5 \cdot f_{L_5}(VaR_5) \\
&\quad +p \int z\mathbf{1}\{y + \varepsilon z > 0\}f_{(L_5,Y,Z)|1}(VaR_5, y, z) dy dz.
\end{aligned} \quad (30)$$

We deduce

$$\partial_\varepsilon VaR_5 \frac{f_{L_5}(VaR_5)}{f_{L_5|1}(VaR_5)} = P[D = 1]E[Z\mathbf{1}\{Y + \varepsilon Z > 0\}|D = 1, L_5 = VaR_5],$$

and finally

$$\partial_\varepsilon VaR_5 = P[D = 1|L_5 = VaR_5]E[Z\mathbf{1}\{Y + \varepsilon Z > 0\}|D = 1, L_5 = VaR_5],$$

by Bayes' rule.  $\square$

## B.2 First derivative of Expected Shortfall.

We only provide derivation of  $\partial_\varepsilon ES_5$ . As previously,

$$\begin{aligned}
ES_5 &= E[L_5 | L_5 > VaR_5] = \frac{1}{\alpha} E[L_5 \mathbf{1}\{L_5 > VaR_5\}] \\
&= \frac{1-p}{\alpha} E[L_5 \mathbf{1}\{L_5 > VaR_5\} | D=0] + \frac{p}{\alpha} E[L_5 \mathbf{1}\{L_5 > VaR_5, Y + \varepsilon Z < 0\} | D=1] \\
&\quad + \frac{p}{\alpha} E[L_5 \mathbf{1}\{L_5 > VaR_5, Y + \varepsilon Z \geq 0\} | D=1] \\
&= \frac{1-p}{\alpha} \int x \mathbf{1}\{x \geq VaR_5\} f_{X|0}(x) dx \\
&\quad + \frac{p}{\alpha} \int x \mathbf{1}\{x \geq VaR_5, y + \varepsilon z < 0\} f_{(X,Y,Z)|1}(x, y, z) dx dy dz \\
&\quad + \frac{p}{\alpha} \int (x + y + \varepsilon z) \mathbf{1}\{(x + y + \varepsilon z) \geq VaR_5, y + \varepsilon z \geq 0\} f_{(X,Y,Z)|1}(x, y, z) dx dy dz \\
&\equiv \frac{1-p}{\alpha} \tilde{\mathcal{T}}_1 + \frac{p}{\alpha} \tilde{\mathcal{T}}_2 + \frac{p}{\alpha} \tilde{\mathcal{T}}_3.
\end{aligned}$$

Here,

$$\begin{aligned}
\partial_\varepsilon \tilde{\mathcal{T}}_1 &= -VaR_5 \partial_\varepsilon VaR_5 f_{X|0}(VaR_5), \\
\partial_\varepsilon \tilde{\mathcal{T}}_2 &= -VaR_5 \partial_\varepsilon VaR_5 \int \mathbf{1}\{y + \varepsilon z < 0\} f_{(X,Y,Z)|1}(VaR_5, y, z) dy dz \\
&\quad - \int z x \mathbf{1}\{x > VaR_5\} f_{(X,Y,Z)|1}(x, -\varepsilon z, z) dx dz, \\
\partial_\varepsilon \tilde{\mathcal{T}}_3 &= \int z \mathbf{1}\{x + y + \varepsilon z > VaR_5, y + \varepsilon z \geq 0\} f_{(X,Y,Z)|1}(x, y, z) dx dy dz \\
&\quad - \int (\partial_\varepsilon VaR_5 - z) VaR_5 \mathbf{1}\{y + \varepsilon z > 0\} f_{(X,Y,Z)|1}(VaR_5 - y - \varepsilon z, y, z) dy dz \\
&\quad + \int z x \mathbf{1}\{x > VaR_5\} f_{(X,Y,Z)|1}(x, -\varepsilon z, z) dx dz.
\end{aligned}$$

We get by recalling Equation (30):

$$\begin{aligned}
\alpha \partial_\varepsilon ES_5 &= -VaR_5 \partial_\varepsilon VaR_5 f_{L_5}(VaR_5) \\
&\quad + p \int z \mathbf{1}\{l > VaR_5, y + \varepsilon z > 0\} f_{(L_5,Y,Z)|1}(l, y, z) dl dy dz \\
&\quad + p VaR_5 \int z \mathbf{1}\{y + \varepsilon z > 0\} f_{(L_5,Y,Z)|1}(VaR_5, y, z) dl dy dz \\
&= p \int z \mathbf{1}\{l > VaR_5, y + \varepsilon z > 0\} f_{(L_5,Y,Z)|1}(l, y, z) dl dy dz.
\end{aligned}$$

Thus, by applying Bayes' rule, we get

$$\begin{aligned}\partial_\varepsilon ES_5 &= \frac{P[D = 1]}{P[L_5 > VaR_5]} \int z \mathbf{1}\{l > VaR_5, y + \varepsilon z > 0\} f_{(L_5, Y, Z)|1}(l, y, z) dl dy dz \\ &= P[D = 1 | L_5 > VaR_5] E[Z \mathbf{1}\{Y + \varepsilon Z > 0\} | L_5 > VaR_5, D = 1],\end{aligned}$$

which proves the result.  $\square$

## C Proof of Proposition 3.

Since the calculations are very similar to previous ones, we prove the results for  $L'_5$  only and we omit unnecessary details.

### C.1 First derivative of Value at Risk.

We may rewrite the equation defining  $VaR'_5$  as:

$$\begin{aligned}\alpha = P[L'_5 > VaR'_5] &= (1 - p)P[X > VaR'_5 | D = 0] + pP[X > VaR'_5, Y + \varepsilon Z < 0 | D = 1] \\ &\quad + pP[X + Y + \varepsilon Z > VaR'_5, Y + \varepsilon Z \in [0, M] | D = 1] \\ &\quad + pP[X + M > VaR'_5, Y + \varepsilon Z > M | D = 1].\end{aligned}$$

Simple computations and some simplifications yield

$$\begin{aligned}0 &= -(1 - p)f_{X|0}(VaR'_5)\partial_\varepsilon VaR'_5 - p\partial_\varepsilon VaR'_5 \int \mathbf{1}\{y + \varepsilon z < 0\} f_{(X, Y, Z)|1}(VaR'_5, y, z) dy dz \\ &\quad - p \int (\partial_\varepsilon VaR'_5 - z) \mathbf{1}\{y + \varepsilon z \in [0, M]\} f_{(X, Y, Z)|1}(VaR'_5 - y - \varepsilon z, y, z) dy dz \\ &\quad - p\partial_\varepsilon VaR'_5 \int \mathbf{1}\{y + \varepsilon z > M\} f_{(X, Y, Z)|1}(VaR'_5 - M, y, z) dy dz.\end{aligned}$$

Clearly, we have

$$\begin{aligned}f_{(L'_5, Y, Z)|1}(l, y, z) &= \mathbf{1}\{y + \varepsilon z \in [0, M]\} f_{(X, Y, Z)|1}(l - y - \varepsilon z, y, z) \\ &\quad + \mathbf{1}\{y + \varepsilon z < 0\} f_{(X, Y, Z)|1}(l, y, z) \\ &\quad + \mathbf{1}\{y + \varepsilon z > M\} f_{(X, Y, Z)|1}(l - M, y, z).\end{aligned}$$

Thus, we can rewrite

$$0 = -\partial_\varepsilon VaR'_5 \cdot f_{L'_5}(VaR'_5) + p \int z \mathbf{1}\{y + \varepsilon z \in [0, M]\} f_{(L'_5, Y, Z)|1}(VaR'_5, y, z) dy dz, \quad (31)$$

so the stated result.  $\square$

## C.2 First derivative of Expected Shortfall.

As previously,

$$\begin{aligned}
ES'_5 &= \frac{1-p}{\alpha} E[L'_5 \mathbf{1}\{L'_5 > VaR'_5\} | D=0] + \frac{p}{\alpha} E[L'_5 \mathbf{1}\{L'_5 > VaR'_5, Y + \varepsilon Z < 0\} | D=1] \\
&\quad + \frac{p}{\alpha} E[L'_5 \mathbf{1}\{L'_5 > VaR'_5, Y + \varepsilon Z \in [0, M]\} | D=1] \\
&\quad + \frac{p}{\alpha} E[L'_5 \mathbf{1}\{L'_5 > VaR'_5, Y + \varepsilon Z > M\} | D=1] \\
&= \frac{1-p}{\alpha} \int x \mathbf{1}\{x \geq VaR'_5\} f_{X|0}(x) dx \\
&\quad + \frac{p}{\alpha} \int x \mathbf{1}\{x \geq VaR'_5, y + \varepsilon z < 0\} f_{(X,Y,Z)|1}(x, y, z) dx dy dz \\
&\quad + \frac{p}{\alpha} \int (x + y + \varepsilon z) \mathbf{1}\{(x + y + \varepsilon z) \geq VaR'_5, y + \varepsilon z \in [0, M]\} f_{(X,Y,Z)|1}(x, y, z) dx dy dz \\
&\quad + \frac{p}{\alpha} \int (x + M) \mathbf{1}\{(x + M) \geq VaR'_5, y + \varepsilon z > M\} f_{(X,Y,Z)|1}(x, y, z) dx dy dz \\
&\equiv \frac{1-p}{\alpha} \tilde{T}'_1 + \frac{p}{\alpha} \tilde{T}'_2 + \frac{p}{\alpha} \tilde{T}'_3 + \frac{p}{\alpha} \tilde{T}'_4.
\end{aligned}$$

Here,

$$\begin{aligned}
\partial_\varepsilon \tilde{T}'_1 &= -VaR'_5 \partial_\varepsilon VaR'_5 f_{X|0}(VaR'_5), \\
\partial_\varepsilon \tilde{T}'_2 &= -VaR'_5 \partial_\varepsilon VaR'_5 \int \mathbf{1}\{y + \varepsilon z < 0\} f_{(X,Y,Z)|1}(VaR'_5, y, z) dy dz \\
&\quad - \int z x \mathbf{1}\{x > VaR'_5\} f_{(X,Y,Z)|1}(x, -\varepsilon z, z) dx dz, \\
\partial_\varepsilon \tilde{T}'_3 &= \int z \mathbf{1}\{x + y + \varepsilon z > VaR'_5, y + \varepsilon z \in [0, M]\} f_{(X,Y,Z)|1}(x, y, z) dx dy dz \\
&\quad - \int (\partial_\varepsilon VaR'_5 - z) VaR'_5 \mathbf{1}\{y + \varepsilon z \in [0, M]\} f_{(X,Y,Z)|1}(VaR'_5 - y - \varepsilon z, y, z) dy dz \\
&\quad + \int z x \mathbf{1}\{x > VaR'_5\} f_{(X,Y,Z)|1}(x, -\varepsilon z, z) dx dz \\
&\quad - \int z(x + M) \mathbf{1}\{x + M > VaR'_5\} f_{(X,Y,Z)|1}(x, M - \varepsilon z, z) dx dz, \\
\partial_\varepsilon \tilde{T}'_4 &= -VaR'_5 \partial_\varepsilon VaR'_5 \int \mathbf{1}\{y + \varepsilon z > M\} f_{(X,Y,Z)|1}(VaR'_5 - M, y, z) dy dz \\
&\quad + \int z(x + M) \mathbf{1}\{x + M > VaR'_5\} f_{(X,Y,Z)|1}(x, M - \varepsilon z, z) dx dz.
\end{aligned}$$

After some simplifications and invoking Equation (31), we get

$$\begin{aligned}
\alpha \partial_\varepsilon ES'_5 &= -VaR'_5 \partial_\varepsilon VaR'_5 f_{L'_5}(VaR'_5) \\
&\quad + p \int z \mathbf{1}\{l > VaR'_5, y + \varepsilon z \in [0, M]\} f_{(L'_5, Y, Z)|1}(l, y, z) dl dy dz \\
&\quad + p VaR'_5 \int z \mathbf{1}\{y + \varepsilon z \in [0, M]\} f_{(L'_5, Y, Z)|1}(VaR'_5, y, z) dl dy dz \\
&= p \int z \mathbf{1}\{l > VaR'_5, y + \varepsilon z \in [0, M]\} f_{(L'_5, Y, Z)|1}(l, y, z) dl dy dz,
\end{aligned}$$

and finally the result.  $\square$

## D Sensitivity of $VaR$ and $ES$ under the Gaussian assumption.

Let us first recall a useful lemma:

**Lemma 5.** *The law of  $E_1$  conditionally to  $E_2 = e$  is Gaussian with mean*

$$m_{1|2}(e) = m_1 + \frac{\rho\sigma_1}{\sigma_2}(e - m_2),$$

and variance

$$\sigma_{1|2}^2 = \sigma_1^2(1 - \rho^2).$$

Invoking (13), the sensitivity of  $L_U$ 's Value-At-Risk is

$$\partial_\varepsilon VaR_U = E[E_2 \mathbf{1}\{E_2 > 0, D = 1\} | L_U = v],$$

where  $v$  is the VaR level. First, let us calculate the density of  $L_U$ . Actually,  $L_U$  equals 0 with the probability

$$P(L_U = 0) = 1 - p + pP(E_1 < 0, E_2 < 0).$$

Otherwise,  $L_U$  has a density with respects to the Lebesgue measure on  $\mathbb{R}^+$ .



More specifically, consider a measurable real function  $\psi$ . Then

$$\begin{aligned}
E[\psi(L_U)] &= \psi(0)\{1 - p + pP(E_1 < 0, E_2 < 0)\} \\
&\quad + p \int \psi(e)\mathbf{1}\{e > 0\}P(E_2 < 0|E_1 = e)f_{E_1}(e) de \\
&\quad + p \int \psi(\varepsilon e)\mathbf{1}\{e > 0\}P(E_1 < 0|E_2 = e)f_{E_2}(e) de \\
&\quad + p \int \psi(l)\mathbf{1}\{l > 0, e > 0\}f_{(\bar{L}, E_2)}(l, e) de dl,
\end{aligned}$$

where we have set  $\bar{L} = E_1 + \varepsilon E_2$ . By applying successively Lemma 5, we get

$$\begin{aligned}
P(E_2 < 0|E_1 = e) &= \Phi\left(-\left(m_2 + \frac{\rho\sigma_2}{\sigma_1}(e - m_1)\right) \cdot \frac{1}{\sigma_2\sqrt{1 - \rho^2}}\right), \\
P(E_1 < 0|E_2 = e) &= \Phi\left(-\left(m_1 + \frac{\rho\sigma_1}{\sigma_2}(e - m_2)\right) \cdot \frac{1}{\sigma_1\sqrt{1 - \rho^2}}\right).
\end{aligned}$$

Moreover, the random vector  $(\bar{L}, E_2)$  is Gaussian with mean  $m = (m_1 + \varepsilon m_2, m_2)'$  and covariance matrix

$$\bar{\Omega} = \begin{bmatrix} \tau^2 & \nu\tau\sigma_2 \\ \nu\tau\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

with

$$\begin{aligned}
\tau^2 &= \sigma_1^2 + \varepsilon^2\sigma_2^2 + 2\varepsilon\rho\sigma_1\sigma_2, \\
\nu\tau\sigma_2 &= \rho\sigma_1\sigma_2 + \varepsilon\sigma_2^2.
\end{aligned}$$

Thus, by applying Lemma 5 and the symmetry of the Gaussian distribution, we get

$$P(E_2 > 0|\bar{L} = l) = \Phi\left(\frac{1}{\sigma_2\sqrt{1 - \nu^2}}\left[m_2 + \frac{\nu\sigma_2}{\tau}(l - m_1 - \varepsilon m_2)\right]\right).$$

The density of  $L_U$  at  $l > 0$  is then

$$\begin{aligned}
f_{L_U}(l) &= p \frac{P(E_2 < 0|E_1 = l)}{\sigma_1} \phi\left(\frac{(l - m_1)}{\sigma_1}\right) + p \frac{P(E_1 < 0|E_2 = l/\varepsilon)}{\varepsilon\sigma_2} \phi\left(\frac{(l - m_2\varepsilon)}{\varepsilon\sigma_2}\right) \\
&\quad + p \frac{P(E_2 > 0|\bar{L} = l)}{\tau} \phi\left(\frac{(l - m_1 - \varepsilon m_2)}{\tau}\right).
\end{aligned} \tag{32}$$

Moreover, for every  $v > 0$  such that  $f_{L_U}(v) > 0$ ,

$$\begin{aligned}
\partial_\varepsilon VaR_U &= E[E_2 \mathbf{1}\{E_2 > 0, D = 1\} | L_U = v] \\
&= E[E_2 \mathbf{1}\{E_1 > 0, E_2 > 0, D = 1\} | L_U = v] + E[E_2 \mathbf{1}\{E_1 < 0, E_2 > 0, D = 1\} | L_U = v] \\
&= E[E_2 \mathbf{1}\{E_1 > 0, E_2 > 0, D = 1\} | L_U = v] + E\left[\frac{L_U}{\varepsilon} \mathbf{1}\{E_1 < 0, L_U > 0, D = 1\} | L_U = v\right] \\
&= \frac{p}{f_{L_U}(v)} \left[ \int e \mathbf{1}\{v - \varepsilon e > 0, e > 0\} f_{(E_1, E_2)}(v - \varepsilon e, e) de \right. \\
&\quad \left. + \frac{v}{\varepsilon} \int \mathbf{1}\{e < 0\} f_{(E_1, E_2)}\left(e, \frac{v}{\varepsilon}\right) de \right]. \tag{33}
\end{aligned}$$

For convenience, denote  $\tilde{e} = e - m_2$  and  $\tilde{v} = v - m_1 - \varepsilon m_2$ . Let us compute

$$\begin{aligned}
&\int e \mathbf{1}\{v - \varepsilon e > 0, e > 0\} f_{(E_1, E_2)}(v - \varepsilon e, e) de \\
&= \int e \mathbf{1}\{v - \varepsilon e > 0, e > 0\} \exp\left(-\frac{\sigma_2^2[\tilde{v} - \varepsilon \tilde{e}]^2 + \sigma_1^2 \tilde{e}^2 - 2\rho\sigma_1\sigma_2[\tilde{v} - \varepsilon \tilde{e}]\tilde{e}}{2\sigma_1^2\sigma_2^2(1 - \rho^2)}\right) \frac{de}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \\
&= \int e \mathbf{1}\{v - \varepsilon e > 0, e > 0\} \exp\left(-\frac{[\tau\tilde{e} - \nu\tilde{v}\sigma_2]^2 + (1 - \nu^2)\sigma_2^2\tilde{v}^2}{2\sigma_1^2\sigma_2^2(1 - \rho^2)}\right) \frac{de}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \\
&= \phi\left(\frac{\tilde{v}\sqrt{1 - \nu^2}}{\sigma_1\sqrt{1 - \rho^2}}\right) \int e \mathbf{1}\{v - \varepsilon e > 0, e > 0\} \phi\left(\frac{\tau\tilde{e} - \nu\sigma_2\tilde{v}}{\sigma_1\sigma_2\sqrt{1 - \rho^2}}\right) \frac{de}{\sigma_1\sigma_2\sqrt{1 - \rho^2}} \\
&= \frac{1}{\tau^2} \phi\left(\frac{\tilde{v}\sqrt{1 - \nu^2}}{\sigma_1\sqrt{1 - \rho^2}}\right) \cdot \left[ \sigma_1\sigma_2\sqrt{1 - \rho^2} \int_c^d t\phi(t) dt + (m_2\tau + \nu\sigma_2\tilde{v}) \int_c^d \phi(t) dt \right] \\
&= \frac{1}{\tau^2} \phi\left(\frac{\tilde{v}\sqrt{1 - \nu^2}}{\sigma_1\sqrt{1 - \rho^2}}\right) \cdot \left[ \sigma_1\sigma_2\sqrt{1 - \rho^2} (\phi(c) - \phi(d)) + (m_2\tau + \nu\sigma_2\tilde{v}) (\Phi(d) - \Phi(c)) \right], \tag{34}
\end{aligned}$$

by setting

$$c = -\frac{m_2\tau}{\sigma_1\sigma_2\sqrt{1 - \rho^2}} - \frac{\nu\tilde{v}}{\sigma_1\sqrt{1 - \rho^2}}, \quad d = \left(\frac{v}{\varepsilon} - \frac{\nu\sigma_2\tilde{v}}{\tau} - m_2\right) \cdot \frac{\tau}{\sigma_1\sigma_2\sqrt{1 - \rho^2}}.$$

and by denoting  $\phi$ , resp.  $\Phi$ , the pdf, resp. cdf of a standard Gaussian random variable.

Moreover,

$$\begin{aligned}
&\int \mathbf{1}\{e < 0\} f_{(E_1, E_2)}\left(e, \frac{v}{\varepsilon}\right) de \\
&= \phi\left(\frac{v/\varepsilon - m_2}{\sigma_2}\right) \int \mathbf{1}\{e < 0\} \exp\left(-\frac{[\sigma_2(e - m_1) - \rho\sigma_1(v/\varepsilon - m_2)]^2}{2\sigma_1^2\sigma_2^2(1 - \rho^2)}\right) \frac{de}{\sqrt{2\pi}\sigma_1\sigma_2\sqrt{1 - \rho^2}} \\
&= \frac{1}{\sigma_2} \phi\left(\frac{v - \varepsilon m_2}{\varepsilon\sigma_2}\right) \Phi\left(-\frac{m_1}{\sigma_1\sqrt{1 - \rho^2}} - \frac{\rho(v - \varepsilon m_2)}{\varepsilon\sigma_2\sqrt{1 - \rho^2}}\right). \tag{35}
\end{aligned}$$

By taking together equations (33), (32), (34) and (35), we find  $\partial_\varepsilon VaR_U$ .

Similar calculations provide  $\partial_\varepsilon VaR_N$  in the netted case. Here, the loss function equals zero with the probability  $1-p+pP(E_1+\varepsilon E_2 < 0)$ . This function has a density with respect to the Lebesgue measure at every positive point  $v$ , and, with the previous notations,

$$f_{L_N}(v) = \frac{p}{\tau} \cdot \phi\left(\frac{v - m_1 - \varepsilon m_2}{\tau}\right) \mathbf{1}\{v > 0\}. \quad (36)$$

Moreover,

$$\begin{aligned} \partial_\varepsilon VaR_N &= E[E_2 \mathbf{1}\{D = 1\} | L_N = v] \\ &= \frac{p}{f_{L_N}(v)} \int e f_{(\bar{L}, E_2)}(v, e) de \\ &= \frac{p}{f_{L_N}(v)} \int e \exp\left(-\frac{[\tau \tilde{e} - \nu \sigma_2 \tilde{v}]^2}{2\sigma_2^2 \tau^2 (1 - \nu^2)}\right) \frac{de}{\sqrt{2\pi\sigma_2\tau\sqrt{1-\nu^2}}} \cdot \phi\left(\frac{\tilde{v}}{\tau}\right) \\ &= \frac{p}{f_{L_N}(v)} \int (\sigma_2\tau\sqrt{1-\nu^2}u + \nu\sigma_2\tilde{v} + m_2\tau) \phi(u) \frac{du}{\tau^2} \cdot \phi\left(\frac{\tilde{v}}{\tau}\right) \\ &= \frac{p}{\tau^2 f_{L_N}(v)} \cdot \phi\left(\frac{\tilde{v}}{\tau}\right) [\nu\sigma_2\tilde{v} + \tau m_2]. \end{aligned} \quad (37)$$

Equations (36) and (37) provide the sensitivity of the VaR for  $L_N$ .  $\square$

Let us remark that the sensitivities of the expected shortfalls of  $L_N$  and  $L_U$  can be calculated in the same way. Unfortunately the formulas are more complicated. An extra integration with respect to  $v$  is in fact needed, and the sensitivities do not admit simple closed forms.

## E Asymptotic normality of VaR sensitivity estimators

Let us consider an i.i.d. sample  $(e_i, l_i)_{i=1, \dots, n}$  of a random vector  $(e, l)$ . The density of  $l$  is denoted by  $f$ . Let  $\hat{v}$  be a statistic which tends to a constant  $v$  and  $n^{1/2}(\hat{v} - v) = O_P(1)$ .

Set

$$\hat{r}(v) = \frac{\sum_{i=1}^n e_i K_h(l_i - \hat{v})}{\sum_{i=1}^n K_h(l_i - \hat{v})},$$

and denote the expectation of  $e_i$  conditionally to  $l_i = v$  by  $r(v)$ , and its conditional variance by

$$\mu(v) = E[e_i^2 | l_i = v] - r(v)^2.$$

**Lemma 6.** *Assume*

- $K$  is an even kernel function,  $\int |t|^3 |K|(t) dt < \infty$ ,  $\lim_{|t| \rightarrow \infty} |t|^3 K(t) = 0$ . It is three times continuously differentiable,  $K'$  and  $K''$  are integrable and  $K'''$  is bounded.
- $E[|e|^p] < \infty$  for every integer  $p$ ,
- $t \mapsto E[e^4 | l = t]$  is bounded in a neighborhood of  $v$ ,
- $nh^5 \rightarrow 0$  and  $nh^{7/2} \rightarrow \infty$ ,
- $r$  and  $f$  are two times continuously differentiable,
- $f$  and  $\mu$  are continuous and strictly positive in a neighborhood of  $v$ .

Then

$$\sqrt{nh} \{\hat{r}(v) - r(v)\} \xrightarrow[n \rightarrow \infty]{law} \mathcal{N} \left( 0, \int K^2 \cdot \mu(v)/f(v) \right).$$

Obviously the limiting behavior of  $\widehat{\partial_\varepsilon VaR}_k$ ,  $k = 4, 5$  is a direct consequence of the previous lemma. For instance, set  $e_i = z_i \mathbf{1}\{z_i > 0, D_i = 1\}$  to get the result for  $\widehat{\partial_\varepsilon VaR}_4$ .

**Proof of Lemma 6**

The estimator  $\hat{r}(v)$  can be decomposed into

$$\hat{r}(v) = \frac{\sum_{i=1}^n e_i K_h(l_i - v)}{\sum_{i=1}^n K_h(l_i - v)} + R_1(v) + \hat{r}(v) R_2(v), \quad (38)$$

with

$$R_1(v) = \frac{n^{-1} \sum_{i=1}^n e_i \{K_h(l_i - \hat{v}) - K_h(l_i - v)\}}{n^{-1} \sum_{i=1}^n K_h(l_i - v)},$$

and

$$R_2(v) = \frac{n^{-1} \sum_{i=1}^n \{K_h(l_i - v) - K_h(l_i - \hat{v})\}}{n^{-1} \sum_{i=1}^n K_h(l_i - v)}.$$

By applying Corollary IV.2 in Bosq and Lecoutre (1987), we get

$$\sqrt{nh} \left\{ \frac{\sum_{i=1}^n e_i K_h(l_i - v)}{\sum_{i=1}^n K_h(l_i - v)} - E[e | l = v] \right\} \xrightarrow[n \rightarrow \infty]{law} \mathcal{N} \left( 0, \int K^2 \cdot \mu(v)/f(v) \right). \quad (39)$$

Moreover,

$$P\left(\sqrt{nh}|R_1(v)| > \varepsilon\right) \leq P\left(\frac{\sqrt{nh}}{n} \left| \sum_{i=1}^n e_i \{K_h(l_i - \hat{v}) - K_h(l_i - v)\} \right| > \varepsilon f(v)/2\right) \\ + P\left(\left| n^{-1} \sum_{i=1}^n K_h(l_i - v) - f(v) \right| > f(v)/2\right) \equiv P_1 + P_2.$$

The second term on the right hand side tends to zero (convergence in probability of the usual kernel estimator of the density function). By a three order expansion of  $K_h$ , we get

$$P_1 \leq P\left(\frac{\sqrt{nh}}{nh} \left| \sum_{i=1}^n e_i \{(K')_h(l_i - v) \cdot (\hat{v} - v)\} \right| > \varepsilon f(v)/6\right) \\ + P\left(\frac{\sqrt{nh}}{nh^2} \left| \sum_{i=1}^n e_i \{(K'')_h(l_i - v) \cdot (\hat{v} - v)^2\} \right| > \varepsilon f(v)/3\right) \\ + P\left(\frac{\sqrt{nh}}{nh^3} \left| \sum_{i=1}^n e_i \{(K''')_h(l_i - v^*) \cdot (\hat{v} - v)^3\} \right| > \varepsilon f(v)\right) \equiv P_{11} + P_{12} + P_{13},$$

where  $|v^* - v| < |\hat{v} - v|$  a.e. Clearly,  $P_{13}$  is zero for  $n$  sufficiently large when  $nh^{7/2} \rightarrow \infty$ .

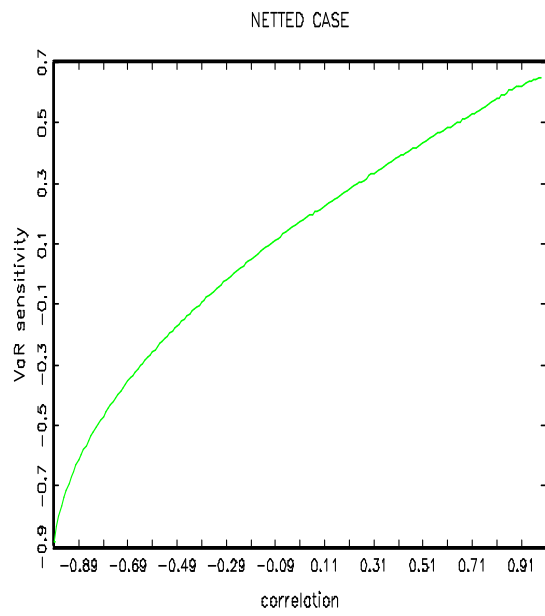
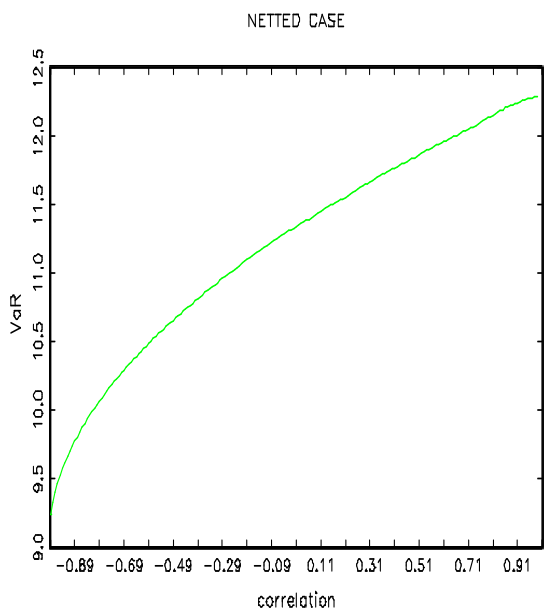
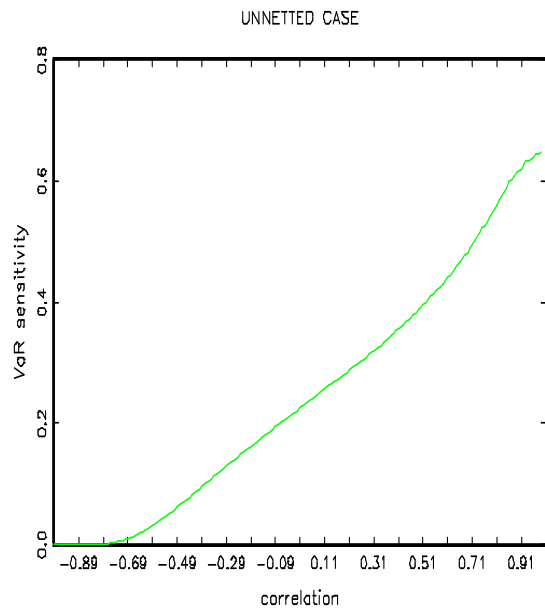
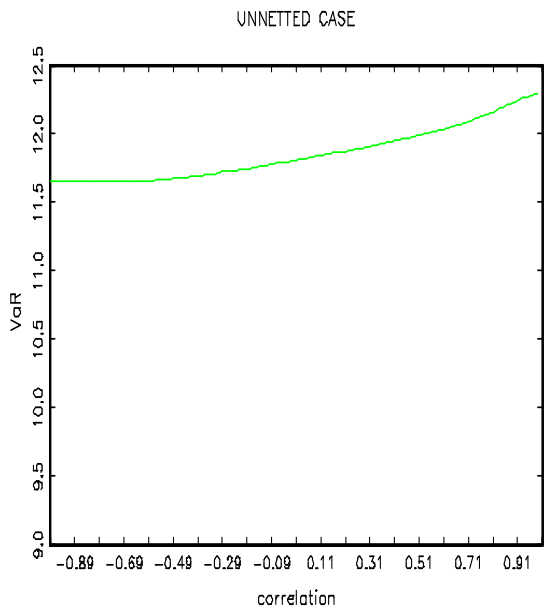
Moreover, for every constant  $\eta > 0$ , there exists a constant  $C > 0$  such that

$$P_{11} \leq P(n^{1/2}|\hat{v} - v| > C) + P\left(\frac{\sqrt{h}}{nh} \left| \sum_{i=1}^n e_i \{(K')_h(l_i - v)\} \right| > \varepsilon f(v)/(6C)\right) \\ \leq \eta + \frac{E[|e_i(K')_h(l_i - v)|]}{\sqrt{Cst.h}} \\ \leq \eta + O(h^{1/2}),$$

by an integration by parts. Thus,  $P_{11}$  tends to zero when  $n \rightarrow \infty$ . Similar calculations provide the same result for  $P_{12}$ . Thus, we deduce

$$R_1(v) = o_P(1). \quad (40)$$

We can handle  $R_2(v)$  similarly. Thus, we get the stated result by combining (38), (39) and (40).  $\square$



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