Mortality Risk and Real Optimal Asset Allocation For Pension Funds

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Mortality risk and real optimal asset allocation for pension funds[*]

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Abstract

We analyze the problem of real optimal asset allocation for a pension fund maximising the expected CRRA utility of its real disposable wealth. The financial horizon of the analysis coincides with the random death time of a representative subscriber. We consider a very general setting where there exists a stochastic investment opportunity set together with stochastic contributions and pensions and we derive a quasi-explicit solution. When the market price of risk is independent of the state variables we are also able to compute a closed-form solution. Numerical simulations provide useful practical guidelines regarding the optimal investment strategy.

1 Introduction

As recalled by James and Vittas (1999), when a new system of pension funds is activated in a country, policy-makers need to ensure that this system is efficient, or at least does not collapse. Early failures of participating institutions through misuse of funds is likely to cause a major setback to the whole reform program and to discourage older workers to join the new system.

Even if old age pensions will not arise for many years, both accumulation and decumulation phases need to be well organized and efficient to guarantee full success of a new pension system. Indeed, the failure or not of a new pension system depends on its ability to use whatever capital has been amassed at the end of the active life of covered workers to supply them with a reasonably sufficient regular income.

The importance of finding a suitable way for managing both phases is further supported by the figures provided by the United Nations population division (2002). In 2000, about 0.6 billion people, 10 percent of world population, were over 60 years of age. By 2025, the number will nearly double to reach 1.18 billion, around 15 percent of the world population, and by 2050, the number will be 1.96 billion, around 21.1 percent of world population. Almost half of the world population is now in countries which are under the replacement rate fertility level (less than 2.1 children per woman). The life expectancy has been rising too. It is estimated that worldwide life expectancy at birth for men will rise from 61 in 1998 to 67 in 2025, and for women from 65 to 72. In high income countries, life expectancy of women may soon be above 80 years of age.

Furthermore, as it can be seen in Figures 1 [1] and 2 [2] for the United States and the United Kingdom, respectively, dollars and pounds invested in the pension fund industry are significant. In the United States, the total amount of contributions to pension and welfare funds has been increasing at an average rate of about 4.2% per year from 1987 to 2001, while, during the same period, the total benefits paid by pension and welfare funds have been increasing of about 6.7% per year. In 2001 contributions were around 567 billions of dollars while pensions were around 923 billions of dollars. In the United Kingdom, the amount of assets held in English pension fund portfolios has been growing of around 14% per year passing from 4.6 billions of pounds in 1987 to more than 713 billions of pounds in 2001. The need to find an optimal rule for allocating such a huge amount of money among financial assets clearly arises.

[Fig. 1 here]
[Fig. 2 hete]

Up to now, most countries that have undertaken a reform of their pension system have primarily focused on the accumulation phase and paid less attention

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[1] The US data have been downloaded from the "National Income and Product Accounts Tables" provided by the BEA National Economic Accounts (http://www.bea.gov/).

[2] The UK data have been downloaded from the "National Statistics" website (http://www.statistics.gov.uk/). Crown copyright material is reproduced with the permission of the Controller of HMSO.

to the decumulation phase. This is also true in the academic literature (see for instance, Deelstra *et al.*, 2000, Boulier *et al.*, 2001, and Battocchio and Menoncin, 2002). There are only few works dealing with the decumulation phase (see for instance, Charupat and Milevsky, 2002, and Battocchio *et al.*, 2003).

In this work, we present a closed-form solution for optimal asset allocation during the accumulation and decumulation phases under mortality risk and inflation risk. The main difficulty in designing a dedicated framework for pension funds is the presence of nontradeable endowment processes. Some closed-form solutions without any nontradeable income sources have already been derived in the literature. After the seminal papers of Merton (1969, 1971), we mainly refer to the works of Kim and Omberg (1996), Chacko and Viceira (1999), Deelstra *et al.* (2000), Boulier *et al.* (2001), and Wachter (2002). In these papers the market structure is as follows: (i) there exists only one stochastic state variable (the riskless interest rate or the risk premium) following the Vasiček (1977) model or the Cox *et al.* (1985) model, (ii) there exists only one risky asset, (iii) a discount bond may exist. Some works consider a complete financial market (Deelstra *et al.*, 2000, Boulier *et al.*, 2001, and Wachter, 2002) while others deal with an incomplete market (Kim and Omberg, 1996, and Chacko and Viceira, 1999).

In a very general setting, El Karoui and Jeanblanc-Picqué (1998) analyze the case of a constrained investor who cannot borrow against the future and whose wealth cannot therefore be negative. They show that the optimal constrained solution consists in investing a part of the wealth in the unconstrained strategy and spending the remainder for financing an American put written on the free wealth, in order to provide an insurance against the constraint.

Also Cuoco (1997) offers an existence result for the optimal portfolio for a constrained investor who is endowed with a stochastic labor income flow. The type of constraint he analyzes is sufficiently general for describing the case of: (i) nontradeable assets (i.e. incomplete markets), (ii) short-sale constraints, (iii) buying constraints, (iv) portfolio-mix constraints, and (v) minimal capital requirements. The last three constraints are relevant for banks and other financial institutions whose portfolios are affected by regulation of an authority (like a central bank).

In this work, instead, we assume that the pension fund is able to borrow against its prospective mathematical reserve. Thus, its objective is to maximize the intertemporal utility of its real wealth, augmented by the expected value of all the future contributions and diminished by the expected value of all the future pensions (let us call this modified wealth as "disposable wealth"). Furthermore, the objective function takes the form of a HARA (Hyperbolic Absolute Risk Aversion index) utility function which coincides with the power of fund disposable wealth. This assumption allows us to reach a closed-form solution that will be used in numerical simulations to provide useful guidelines for optimal asset allocation in a pension fund context.

For a theoretical justification of risk aversion for banks and other institutional investors the reader is referred to Koehn and Santomero (1980), Kim and

Santomero (1988), and Keeley (1990).

We use the setting described in Merton (1990) and Bodie *et al.* (1992) where a non-financial income flow is taken into account. Furthermore, in order to obtain a closed form solution, we assume that the stochastic contributions and pensions can be spanned in the financial market. Note however, that the case of a pension fund is different from the standard case of an investor having labor income. Indeed, the revenues (contributions) and expenses (pensions) of the fund must be linked by a condition ("feasibility condition") guaranteeing that it is profitable to contract for both the subscriber and the pension fund. To further enrich our framework, we introduce a deterministic profit sharing rule. This means that a proportion of the fund nominal surplus (i.e. the difference between the managed wealth and the contributions) is redistributed to the members, who thus share profits induced by the exposure to the risky assets.

Moreover, the link between contributions and pensions can be established inside one of the two following frameworks: the so-called defined-benefit pension plan (hereafter DB) or the so-called defined-contribution pension plan (hereafter DC). In a DB plan benefits are fixed in advance by the sponsor and contributions are initially set and subsequently adjusted in order to maintain the fund in balance. In a DC plan contributions are fixed and benefits depend on the returns on the fund portfolio. In particular, DC plans allow contributors to know, at each time, the value of their retirement accounts. Historically, fund managers have mainly proposed DB plans, which are definitely preferred by workers. Indeed, the financial risks associated with DB plans are supported by the plan sponsor rather than by individual members of the plan. Nowadays, most of the proposed pension plans are based on DC schemes involving a considerable transfer of risks to workers. Accordingly, DC pension funds provide contributors with a service of savings management, even if they do not guarantee any minimum performance. As we have already highlighted, only contributions are fixed in advance, while the final state of the retirement account depends fundamentally on the administrative and financial skills of the fund manager. Therefore, an efficient financial management is essential to gain contributor trust.

The continuous time model studied in this paper is able to describe both DB and DC pension plans since we take into account two different stochastic variables for contributions and pensions. Note that we do not require one of them to be necessarily deterministic. In order to reduce the model to a pure DB plan it is sufficient to equate the diffusion term of pensions to zero, while in a pure DC plan it is the diffusion term of contributions which must be equated to zero.

An approach related to ours is presented in Sundaresan and Zapatero (1997) where the asset allocation for a pension plan is analyzed from the point of view of a firm which must pay both wages (before its workers retire) and pensions (after they retire). In this case the link between the accumulation and the decumulation phases is provided by equating the total expected value of wages and pensions paid with the total expected value of worker productivity (according to the usual economic rule equating the optimal wage with the marginal product of labor). Furthermore, Sundaresan and Zapatero (1997) solve for an optimal

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stopping time for workers who must decide when to retire. In this paper, instead, we suppose the pension date to be determined by law and thus, to be an exogenous deterministic time. Also Rudolf and Ziemba (2003) study a framework which is very similar to ours. Nevertheless, the points that distinguish our work are the following ones: (i) we analyze the inflation impact on the optimal asset allocation, (ii) we study the effect of the mortality risk by explicitly taking into account a mortality law, (iii) we disentangle the impact on the asset allocation of the accumulation and decumulatio phases.

In the case of pension funds, the optimal asset-allocation problem involves a rather long period, generally from 20 to 40 years. Therefore, the usual nominal setting cannot provide a fully adequate paradigm for yielding a good benchmark to pension fund managers. For this reason we explicitly introduce inflation risk in our analysis as in the extension of Merton's problem presented by Brennan and Xia (2002). As a consequence, the objective for the pension fund corresponds to the maximization of the expected utility of its real disposable wealth, namely the sum between its real wealth and its prospective mathematical reserve. Indeed, the contributions to the pension fund are akin to the labor income for a single investor, and thus the objective function should take into account not only the fund real wealth but also the expected value of future incomes. This expected value coincides with the so-called "human capital" when a labour income for the investor-consumer is present (see Bodie *et al.*, 1992) and with the prospective mathematical reserve of the actuarial literature in our case.

In our setting the usual discount factor in the optimization problem is replaced by an actuarial discount factor taking into account the mortality risk. The introduction of this risk complicates the solution of the Hamilton-Jacobi-Bellman equation. Basically, the solution cannot be led back to a parabolic partial differential equation as in Zariphopoulou (2001). Young and Zariphopoulou (2002), like Merton (1990, Theorem 5.6) and Richard (1975), assume that the event of death at each instant of time is an independent Poisson process. If the parameter characterizing this process is constant, then the age of death is an exponentially distributed random variable. Instead, in this work we follow the approach after Charupat and Milevsky (2002) where a more general surviving probability is considered. Furthermore, we present an explicit solution of the Hamilton-Jacobi-Bellman equation when the market price of risk is independent of the state variables.

This paper generalizes the nominal setting presented in Battocchio *et al.* (2003). These authors deal with a very simple market structure with only stocks, a constant riskless interest rate, and deterministic contributions and pensions. Our model is much more general since we take into account: (i) a set of stochastic investment opportunities, (ii) a set of risky assets whose prices follow general Itô processes, (iii) a stochastic riskless interest rate, (iv) a stochastic inflation risk, (v) stochastic contributions and pensions. Features (i) and (ii) allow us to account for most common market structures existing in the literature. For instance, the works after Deelstra *et al.* (2000), Boulier *et al.* (2001), and Battocchio and Menoncin (2002), only take into account the case of a stochastic interest rate and two risky assets: a bond and a stock.

Along the paper we consider agents trading continuously in a frictionless, arbitrage-free and complete market.

The paper is structured as follows. In Section 2 we first outline the general economic background and give the stochastic differential equations describing the dynamics of asset prices, state variables, contributions, and pensions. Then, we determine the evolution of the fund real wealth, and present the objective function to be maximized. The impact of a profit sharing rule between the fund and its members is further analyzed. In Section 3 the optimal portfolio allocation is computed. There we also present our main result: a closed-form solution of the problem if the financial market is complete and the market price of risk is independent of the state variables. Section 4 provides a numerical illustration based on a simple market structure. Practical implications for pension fund investment strategy are discussed in detail. Section 5 concludes.

2 The model

2.1 The nominal financial market

In the following, asset prices depend on a set of s stochastic state variables whose dynamics are described by the multivariate stochastic differential equation

$$\underset{s\times1}{dX} = \underset{s\times1}{f\left(X,t\right)}dt + \underset{s\times m}{g\left(X,t\right)'}\underset{m\times1}{dW}, \quad X\left(t_0\right) = X_0, \tag{1}$$

where W is a $m-$dimensional Wiener process, and the prime denotes transposition. The drift and diffusion terms $f\left(X,t\right)$ and $g\left(X,t\right)$ are supposed to satisfy the usual Lipschitz conditions guaranteeing that Equation (1) has a unique strong solution (see Karatzas and Shreve, 1991). Furthermore, f and g are $\mathcal{F}_t$-measurable, where $\mathcal{F}_t$ is the $\sigma-$algebra through which the Wiener processes are measured on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All processes below are supposed to satisfy the same properties as those stated for Equation (1). Values of all state variables are known at the initial date t_0 and are equal to the non-stochastic variable X_0.

In the literature on optimal portfolio allocation there generally exists only one state variable coinciding either with the riskless interest rate (e.g. Deelstra *et al.*, 2000, and Boulier *et al.*, 2001) or with a random market price of risk (e.g. Kim and Omberg, 1996, and Chacko and Viceira, 1999). Furthermore, when a closed-form solution is found, this state variable follows the dynamics used in the Vasiček (1977) model or the Cox *et al.* (1985) model. Our framework is much more general since we deal with a set of state variables and we do not specify any particular functional form for the drift and the diffusion terms in (1). The particular case of a riskless interest rate following a Vasiček (1977) model or a Cox *et al.* (1985) model can be easily obtained by putting $X = r$, $f = a_r\left(b_r - r\right)$, and $g = \sigma_r$ or $g = \sigma_r\sqrt{r}$, respectively.

On the financial market there are n risky assets and one riskless asset whose (nominal) prices are driven by the stochastic differential equations

$$\underset{n \times 1}{dS} = \underset{n \times 1}{\mu(S, X, t)}dt + \underset{n \times m}{\Sigma(S, X, t)'}\underset{m \times 1}{dW}, \quad S(t_0) = S_0, \tag{2}$$

$$dG = Gr(X, t)\, dt, \quad G(t_0) = G_0, \tag{3}$$

where $r(X, t)$ is the instantaneous riskless interest rate. The set of risk sources (W) for the risky assets coincides with the one used for the state variables. This assumption is not restrictive because of potential handling of various situations via the matrices g and Σ.

We have already mentioned that we will work in a complete market setting. Thus, the rank of the matrix Σ must be maximum and the number of risk sources (its rows) cannot be higher than the number of risky assets (its columns). If there are less assets than risk sources $(n < m)$ on the market, then the market cannot be complete. In this work we assume that $n = m$ and that the rank of the matrix Σ is maximum. This means that we suppose the following assumption to hold in our work.

Assumption 1 *The asset diffusion matrix is invertible (i.e. $\exists \Sigma^{-1}$).*

The set of risky assets S may contain stocks and bonds. Hence, our market structure given by Equations (1), (2), and (3) also accounts for the typical (and simpler) market structure generally taken into account in the literature about pension fund asset allocation (e.g., Deelstra *et al.*, 2000, Boulier *et al.*, 2001, and Battocchio and Menoncin, 2002). Such a structure contains: (i) a single state variable given by the riskless interest rate, (ii) a stock, (iii) a bond, and (iv) a money market account. For instance, the model after Boulier *et al.* (also studied in Battocchio and Menoncin, 2002), can be represented as follows:

$$\begin{cases} dr = a_r(b_r - r)\, dt - \sigma_r dW_r, \\ dS = S(r + \sigma_1\lambda_1 + \sigma_2\lambda_r)\, dt + S\sigma_1 dW_S + S\sigma_2 dW_r, \\ dB = B\left(r + \left(1 - e^{a_r(H-t)}\right)\frac{\lambda_r\sigma_r}{a_r}\right)dt + B\left(1 - e^{a_r(H-t)}\right)\frac{\sigma_r}{a_r}dW_r, \\ dG = Grdt, \end{cases}$$

where all parameters take positive values and B is the value of a bond with maturity H.

2.2 The inflation

As in Menoncin (2002) the vector X is supposed to contain the consumption price process P whose behavior can be represented as

$$\frac{dP}{P} = \mu_\pi(X, P, t)\, dt + \underset{1 \times m}{\sigma_\pi(X, P, t)'}\underset{m \times 1}{dW}, \quad P(t_0) = 1. \tag{4}$$

The initial value of the price consumption index is conventionally put equal to 1 without loss of generality because prices can always be normalized. For the sake of generality we do not specify any particular form for the drift and the diffusion coefficients of this process (see below for particular choices of inflation dynamics).

Even if the inflation risk is generally neglected in the literature about optimal asset allocation, we prefer to explicitly introduce it in our framework since we study the investment strategy for a pension fund. This kind of institutional investor has typically a long financial horizon which corresponds to the expected lifetime of the subscriber. It is true that the inflation risk can be passed over by the short run investor, but this is no more reasonable for an investor having a long expected financial horizon, more than 40 years. Actually, during such a length of time, consumption prices can vary a lot. For instance, as clearly indicated by Table 1, the US inflation rate has been widely fluctuating during the past 45 years. Inflation risk can thus heavily affect investment decision. This explains why the objective function of the pension fund is based on real quantities.

[Table 1 here]

We recall that Cox *et al.* (1985) propose the following stochastic differential equation for the price level

$$dP = P\pi dt + P\sigma_P\sqrt{\pi}dW_P,$$

where σ_P is a constant and π is the inflation rate which is supposed to behave according to one of the two following stochastic differential equations:

$$
\begin{aligned}
d\pi &= k_1\pi\left(\bar{\pi}_1 - \pi\right)dt + \sigma_1\pi^{\frac{3}{2}}dW_\pi, \\
d\pi &= k_2\left(\bar{\pi}_2 - \pi\right)dt + \sigma_2\sqrt{\pi}dW_\pi,
\end{aligned}
$$

where k_i, $\bar{\pi}_i$, and σ_i, $i \in \{1,2\}$ are all positive constant.

Brennan and Xia (2002) and Munk *et al.* (2003) use a simpler framework where

$$
\begin{aligned}
dP &= P\pi dt + P\sigma_P dW_P, \\
d\pi &= k\left(\bar{\pi} - \pi\right)dt + \sigma_\pi dW\pi.
\end{aligned}
$$

In all these models W_P and W_π may be correlated. Nevertheless, we recall that a set of correlated Wiener processes can always be transformed into a set of uncorrelated Wiener processes through a Cholesky decomposition. Hence, we will always deal with uncorrelated risks in our analysis.

As Brennan and Xia (2002) recall, dP/P can be interpreted as the realized rate of inflation while π can be thought of as the expected rate of inflation. The authors also argue that if the expected rate of inflation is not observable but must be inferred from observation of the price level itself then the change in the expected rate of inflation will be perfectly correlated with the realized rate

of inflation. This last case is the only one compatible with the hypothesis of a complete financial market (for further explanation we refer the interested reader to Brennan and Xia, 2002).

In a complete market setting, our general model is able to account for all the particular specifications recalled above. Here, the drift of the price level is assumed to depend on a set of state variables that may contain also the inflation rate.

2.3   The contributions and the pensions

As already mentioned the lifetime of a pension fund can be conveniently divided into two different phases: (i) the accumulation phase (hereafter APh) when the subscribers pay the contributions to the fund, and (ii) the decumulation phase (hereafter DPh) when the fund pays back these contributions in the form of pensions. We suppose here that the retirement date T is a deterministic and exogenous variable.

A first contribution to the analysis of the optimal asset allocation during the APh and the DPh has been given by Charupat and Milevsky (2002) who show that, under particular conditions, the investment strategy is the same during the two phases. Nevertheless, Battocchio *et al.* (2003), under similar conditions, find different portfolios for APh and DPh. The difference can be explained as follows. Charupat and Milevsky solve for two different dynamic optimization problems: (i) during the APh the pension fund maximizes the expected utility of its terminal wealth, while (ii) during the DPh a similar problem (with a stochastic time horizon) is solved by the pensioner who wants to optimize his consumption path. Instead, in Battocchio *et al.* (2003) the optimal allocation is simultaneously solved in both phases from the point of view of the pension fund which must manage some wealth even after the retirement of its subscribers.

Here, we use the latter approach and we define both accumulated contributions and accumulated pensions through the same stochastic variable L,[3] whose dynamic behavior is given by the following stochastic differential equation:

$$dL = \mu_L\left(L, X, t\right) dt + \underset{1\times m}{\Lambda\left(L, X, t\right)'} \underset{m\times 1}{dW}, \quad L\left(t_0\right) = L_0, \qquad (5)$$

where

$$
\begin{aligned}
\mu_L\left(L, X, t\right) &= \mu_A\left(L, X, t\right) \mathbb{I}_{t<T} + \mu_D\left(L, X, t\right)\left(1 - \mathbb{I}_{t<T}\right), \\
\Lambda\left(L, X, t\right) &= \Lambda_A\left(L, X, t\right) \mathbb{I}_{t<T} + \Lambda_D\left(L, X, t\right)\left(1 - \mathbb{I}_{t<T}\right),
\end{aligned}
$$

---

[3] Observe that we indicate with L the accumulated contribution and pension process. This means that $L\left(t\right)$ is the sum of all contributions paid until time t, reduced by the amount of all pensions paid until time t. Accordingly, the balance of subscriber account between time t_1 and time t_2 $\left(t_2 > t_1\right)$ can be represented as $L\left(t_2\right) - L\left(t_1\right) = \int_{t_1}^{t_2} dL\left(t\right)$. Analogously, the contribution instantaneously paid (or the pension instantaneously received) between time t and time $t + dt$ is given by $dL\left(t\right)$.

and $\mathbb{I}_\mathcal{C}$ is the indicator function whose value is 1 when condition $\mathcal{C}$ holds and 0 otherwise. Thus, we suppose that the drift and diffusion terms of L can suddenly change once the retirement date T is reached. Obviously, the two drift components should have opposite signs. The first component μ_A is positive because contributions increase the fund wealth, and the second component μ_D is negative since pension payments decrease fund wealth.

In Battocchio *et al.* (2003) L is deterministic ($\Lambda_A = \Lambda_D = \mathbf{0}$), and μ_A, μ_D are two constants. Instead, in Battocchio and Menoncin (2002) a stochastic contribution process is analyzed but the problem is limited to the accumulation phase.

The stochastic behavior of L during the APh comes from the randomness of the subscriber wages which contributions are paid on. Instead, during the DPh, L can be stochastic because pensioners can choose to receive a *fixed immediate annuity* or a *variable immediate annuity* (or a combination of them) as stated in Charupat and Milevsky (2002).

The m risk sources which affect the behaviour of L are the same as those affecting the behaviour of asset prices (in a complete market). This means that, in our framework, the contributions and pensions can be spanned. We will use this property in the following subsection for deriving the so-called feasibility condition linking contributions and pensions.

Our model is able to account for both a defined contribution (DC) and a defined benefit (DB) pension plan:

1. in a DC plan contributions μ_A are previously fixed and pensions μ_D are chosen according to a "feasibility condition". In this case, since contributions are not stochastic, it is sufficient to put $\Lambda_A = \mathbf{0}$;

2. in a DB plan pensions μ_D are fixed in advance and contributions μ_A must be paid according to a "feasibility condition". Thus, pensions are not stochastic, and we put $\Lambda_D = \mathbf{0}$.

Furthermore, our framework is general enough to accommodate three most common forms of old age retirement benefits (i.e. μ_D). We recall here the main characteristic of these three forms (for a more comprehensive analysis the reader is referred to James and Vittas, 1999).

1. Lump sum payments.[4] In this case $\mu_D = 0$, for $t > T$, and $\Lambda_D = \mathbf{0}$. The lump sum payments are easy to operate as they do not require any of the complex calculations involved in scheduled withdrawals and annuities (see the two following points). In fact, since the value of all μ_D after T is zero, there is no need to take into account the mortality risk after T. Nevertheless, it happens that some workers use part of their lump sums to purchase annuities. In most OECD countries, company pension schemes allow partial commutation of future benefits into a lump sum. This varies

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[4] These are extensively used in many countries: Australia, New Zealand, South Africa, Malaysia, Singapore, and Sri Lanka.

between 25% and 33% of the discounted present value of benefits. Available evidence suggests that most workers opt for this facility.

2. Scheduled (or programmed) withdrawals.[5] In this case the drift component μ_D is not weighted by the survival probability since, in the event of early death, remaining account balances are inherited by dependents, accommodating a bequest motive. Unfortunately, these withdrawals are exposed to fluctuating payments as a result of the volatility of pension fund returns (i.e. $\Lambda_D \neq \mathbf{0}$). In Latin American countries, scheduled withdrawals are recalculated each year on the basis of the remaining life expectancy of the family of covered workers and a stipulated rate of return. By regulation, the rate of return is equal to the average real return achieved by the pension fund concerned over the past 10 years. The life table to be used is also set by the regulators.

3. Life annuities. In this case $\mu_D(L, X, t) = \mu_D(L, t)$ and $\Lambda_D = \mathbf{0}$. That is, life annuities are paid until subscriber death time and they do not depend on fund performances. Among countries with mandatory second pillars, only Switzerland and Bolivia impose the use of annuities. Eastern European countries are also leaning towards compulsory annuitization. Compulsory annuitization is often advocated in order to avoid the problems caused by adverse selection. If it were not compulsory, only subscribers who know to have a long life expectancy would choose it. Accordingly, the annuity market would be greater and better developed if all workers were forced to purchase an annuity.

2.4   The fund real wealth

If we denote by $w(t) \in \mathbb{R}^{n \times 1}$ and $w_G(t) \in \mathbb{R}$ the number of risky assets and the number of riskless asset held in the portfolio, respectively, then the fund nominal wealth R_N can be written as

$$R_N(t) = w(t)' S + w_G(t) G, \tag{6}$$

since the fund invests the total value of its portfolio at each instant t.

The Itô differential of (6) is

$$dR_N = \underbrace{w' dS + w_G dG}_{dR_1} + \underbrace{dw'(S + dS) + dw_G G}_{dR_2},$$

where two sources of change have been identified: the change in asset values (dR_1) and the change in portfolio composition (dR_2). The self-financing condition implies that this last source of variations must at least finance L made

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[5] These are available in many Latin American countries: Chile, Argentina, El Salvador, Mexico, and Peru, though not in Bolivia where mandatory annuitization is imposed.

of accumulated contributions and pensions. To further enrich our framework, we introduce a deterministic profit sharing rule $\phi(t)$ where $0 \leq \phi(t) < 1$. This means that a proportion $\phi(t)$ of the fund nominal surplus $R_N - L$ is redistributed to the members, who thus share profits induced by the exposure to the risky assets. The proportion may for example be: $\phi(t) = 0$ (no profit sharing) or $\phi(t) = (1 - \mathbb{I}_{t<T}) \bar{\phi}$ with $0 \leq \bar{\phi} < 1$ (constant profit sharing only during the decumulation phase). Hence, the self-financing condition in our case must ensure that the changes in portfolio composition dR_2 must finance L as well as the percentage $\phi(t)$ of fund surplus paid to the members.

Thus, the dynamic budget constraint can be written as

$$dR_N = w'dS + w_G dG + dL - \phi d(R_N - L),$$

where $d(R_N - L)$ is the differential of the nominal surplus. Accordingly, the fund nominal wealth follows the differential equation

$$
\begin{aligned}
dR_N &= \frac{1}{1+\phi} w'dS + \frac{1}{1+\phi} w_G dG + dL \\
&= \left(\frac{1}{1+\phi} (w'\mu + w_G Gr) + \mu_L \right) dt + \left(\frac{1}{1+\phi} w'\Sigma' + \Lambda' \right) dW.
\end{aligned}
$$

Now, the fund is supposed to maximize the expected value of a suitable function of its real wealth that is defined as the ratio between the nominal wealth and the price level (R_N/P). By applying Itô's Lemma we have[6]

$$
\begin{aligned}
dR &= -\frac{R_N}{P} (\mu_\pi - \sigma'_\pi \sigma_\pi) dt + \frac{1}{P} \left(\frac{1}{1+\phi} w'\mu + \frac{1}{1+\phi} w_G Gr + \mu_L \right) dt \\
&\quad -\frac{1}{P} \left(\frac{1}{1+\phi} w'\Sigma' \sigma_\pi + \Lambda' \sigma_\pi \right) dt + \frac{1}{P} \left(\frac{1}{1+\phi} w'\Sigma' + \Lambda' - R_N \sigma'_\pi \right) dW,
\end{aligned}
$$

which can be written, after substituting for the value of $w_G G$ given in Equation (6), as

$$dR = (R\rho + w'M + k) dt + (w'\Gamma' + K' - R\sigma'_\pi) dW, \tag{7}$$

where

$$
\begin{aligned}
\Gamma &\equiv \frac{1}{P} \frac{1}{1+\phi} \Sigma, \quad M \equiv \frac{1}{P} \frac{1}{1+\phi} (\mu - Sr - \Sigma' \sigma_\pi), \\
k &\equiv \frac{1}{P} (\mu_L - \Lambda' \sigma_\pi), \quad K \equiv \frac{1}{P} \Lambda, \quad \rho \equiv \frac{1}{1+\phi} r - \mu_\pi + \sigma'_\pi \sigma_\pi.
\end{aligned}
$$

---

[6] We recall that the Jacobian of the real wealth is:

$$\nabla_{R_N, P} R = \begin{bmatrix} \frac{1}{P} & -\frac{R_N}{P^2} \end{bmatrix}',$$

while its Hessian is:

$$\nabla^2_{R_N, P} R = \begin{bmatrix} 0 & -\frac{1}{P^2} \\ -\frac{1}{P^2} & 2\frac{R_N}{P^3} \end{bmatrix}.$$

We see that the market parameters μ, Σ, and r are all multiplied by $(1 + \phi)^{-1}$. Nevertheless, the profit sharing rule $\phi(t)$ does not alter in any way the algebraic solution of the optimization problem. In fact, the fund real wealth always follows a stochastic differential equation having the same form as in (7). This means that there exists no deterministic profit sharing rule that alters the structure of the fund optimal portfolio (while, of course, the amount of wealth that must be invested in each asset is modified by ϕ).

2.5 The feasibility condition

Obviously, there exists a link between contributions and pensions that makes the writing of the contract convenient for both the pension fund and the worker. In Battocchio *et al.* (2003) a feasible condition is found by imposing that the expected discounted value of all contributions is equal to the expected discounted value of all pensions. A similar condition is imposed in Josa-Fombellida and Rincón-Zapatero (2001). Also Sundaresan and Zapatero (1997) present what they call a "fairness rule". This fairness rule equates the total amount of salaries and pensions paid by a firm to its workers to the total amount of workers marginal productivity.

In order to write down a suitable feasibility condition in our framework, the balance of all contributions and pensions can be interpreted as an asset (and it can be priced like an asset since both contributions and pensions can be spanned). From the point of view of the fund member, this particular asset has the following form: until time T the instantaneous amount μ_A must be paid in order to have the right, after T, to receive the instantaneous amount μ_D. Since subscribing a pension contract has to be convenient for both the pension fund and the worker, the initial balance of all contributions and pensions has to be zero. In other words, the expected discounted flow of all pensions must compensate the expected discounted flow of all contributions (in absolute value). Only in this way, we can obtain what Sundaresan and Zapatero (1997) call a "fairness rule" and what Battocchio *et al.* (2003) call a "feasibility condition".

It is well known that, in a complete market, an asset can be priced by taking the expectation of its discounted cashflows under a suitable probability measure. In the usual nominal approach, asset prices are martingales if they are evaluated under the risk neutral probability and measured in terms of the riskless asset which can be thought of as the *numéraire* of the economy.[7] When inflation risk enters the analysis, the suitable *numéraire* making asset prices martingales, is the riskless asset value computed in real terms. In order to show this property, let us introduce a probability measure called "real risk neutral probability" which can be defined as follows (see also Menoncin, 2002, Definition 1).

---

[7] Formally, this property can be written as

$$\mathbb{E}_t^{\mathbb{Q}_N}\left[\frac{S(T)}{G(T)}\right] = \frac{S(t)}{G(t)},$$

where the expectation is taken with respect to the (nominal) risk neutral probability $\mathbb{Q}_N$.

Definition 1 *Given Equations (1), (2), (3), and the historical probability* $\mathbb{P}$, *under Assumption 1 a "real risk neutral probability"* $\mathbb{Q}$ *satisfies*

$$d\mathbb{Q} = \exp\left(-\int_{t_0}^{H} \xi' dW_t - \frac{1}{2}\int_{t_0}^{H} \xi' \xi dt\right) d\mathbb{P},$$

if

$$\mathbb{E}\left[e^{\frac{1}{2}\int_{t_0}^{H} \xi' \xi dt}\right] < \infty.$$

Then

$$dW^{\mathbb{Q}} = \xi dt + dW,$$

is a Wiener process with respect to $\mathbb{Q}$, *where*

$$\xi = \Gamma'^{-1} M \equiv \Sigma'^{-1}\left(\mu - Sr - \Sigma' \sigma_\pi\right),$$

is the (unique) real market price of risk.

We just recall that the nominal market price of risk is usually defined as

$$\xi_N = \Sigma'^{-1}\left(\mu - Sr\right),$$

and so the real market price of risk coincides with the nominal market price of risk after deducting the inflation risk

$$\xi = \xi_N - \sigma_\pi.$$

Furthermore, neither the real nor the nominal market prices of risk are affected by the profit sharing rule (ϕ).

If we suppose that Σ is positive definite then, given the form of ξ, we can conclude that:

1. when the inflation risk is positively correlated with asset prices (i.e. the elements of σ_π are positive) then the real market price of risk is lower than the nominal one;

2. when the inflation risk is negatively correlated with asset prices (i.e. the elements of σ_π are negative) then the real market price of risk is higher than the nominal one.

Under the real risk neutral probability, all the asset prices have the same return which coincides with the return on the riskless asset (in real term). In fact, a straightforward application of Itô's lemma allows us to write

$$
\begin{aligned}
d\hat{G} &\equiv d\left(\frac{G}{P}\right) = \frac{G}{P}\left(r - \mu_\pi + \sigma'_\pi \sigma_\pi + \sigma'_\pi \xi\right) dt - \frac{G}{P}\sigma'_\pi dW^{\mathbb{Q}}, \\
d\hat{S} &\equiv d\left(\frac{S}{P}\right) = \frac{S}{P}\left(r - \mu_\pi + \sigma'_\pi \sigma_\pi + \sigma'_\pi \xi\right) dt + \frac{1}{P}\left(\Sigma' - S\sigma'_\pi\right) dW^{\mathbb{Q}}.
\end{aligned}
$$

Accordingly, it is evident that the suitable *numéraire* for the real market is the riskless asset value computed in real terms.

Now, we come back to the problem of determining the feasibility condition. The suitable measure of contributions and pensions and the suitable interest rate for discounting them are both suggested by the differential equation of real wealth computed under the probability $\mathbb{Q}$:

$$dR = \left(R\left(\rho + \sigma'_\pi \xi\right) + k - K'\xi\right)dt + \left(w'\Gamma' + K' - R\sigma'_\pi\right)dW^{\mathbb{Q}}. \tag{8}$$

Hence in the feasibility condition we must compute, under $\mathbb{Q}$, the expected value of $k - K'\xi$ discounted at the interest rate $\rho + \sigma'_\pi \xi$. So, if we indicate with τ the stochastic death time, the feasibility condition can be written as

$$0 = \mathbb{E}^{\mathbb{Q},\tau}_{t_0}\left[\int_{t_0}^\tau \left(k - K'\xi\right)e^{-\int_{t_0}^s \left(\rho + \sigma'_\pi \xi\right)d\theta}ds\right], \tag{9}$$

where the expectation is taken with respect to the martingale equivalent measure $\mathbb{Q}$ and to the death time τ. In a nominal framework, where $\sigma_\pi = \mathbf{0}$, $\mu_\pi = 0$, and $P(t) = 1, \forall t$, the feasibility condition is

$$0 = \mathbb{E}^{\mathbb{Q},\tau}_{t_0}\left[\int_{t_0}^\tau \left(\mu_L - \Lambda'\xi_N\right)e^{-\frac{1}{1+\phi}\int_{t_0}^s r d\theta}ds\right], \tag{10}$$

where we compute the expected value of the sum of all the future contributions and pensions diminuished by their price of risk and discounted with a modified riskless interest rate for taking into account the profit sharing rule. This is the case analyzed in Battocchio *et al.* (2003) where contributions, pensions, and the riskless interest rate are not stochastic (i.e. $dL = \mu_L(t)dt$, and $\Lambda = \mathbf{0}$). Furthermore, in their model there is no profit sharing (i.e. $\phi = 0$). Thus, the feasibility condition is simply[8]

$$0 = \mathbb{E}^\tau_{t_0}\left[\int_{t_0}^\tau \mu_L(s)e^{-r(s-t_0)}ds\right].$$

Condition (9) implies that at t_0 (when the contract is written) the real discounted value of all contributions received must compensate the real discounted value of all pensions paid. The expected value in Condition (9) is computed with respect to the joint distribution of all the involved stochastic processes and τ. Now, we use the independence between τ and the joint distribution of the other processes for writing (as in Charupat and Milevsky, 2002)

$$\mathbb{E}^{\mathbb{Q},\tau}_{t_0}\left[\int_{t_0}^\tau \left(k - K'\xi\right)e^{-\int_{t_0}^s \left(\rho + \sigma'_\pi \xi\right)d\theta}ds\right]$$

$$= \mathbb{E}^{\mathbb{Q},\tau}_{t_0}\left[\int_{t_0}^\infty \mathbb{I}_{s<\tau}\left(k - K'\xi\right)e^{-\int_{t_0}^s \left(\rho + \sigma'_\pi \xi\right)d\theta}ds\right]$$

$$= \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\int_{t_0}^\infty \left({}_{s-t_0}p_{t_0}\right)\left(k - K'\xi\right)e^{-\int_{t_0}^s \left(\rho + \sigma'_\pi \xi\right)d\theta}ds\right],$$

---

[8] The probability measure $\mathbb{Q}$ does not play any role since both μ_L and r are two deterministic functions.

where $\mathbb{I}_{t<\tau}$ is the indicator function for the event that death occurs after t, and $_{t-t_0}p_{t_0}$ is the conditional probability that an individual of age t_0 will survive for another $t - t_0$ years.[9] Accordingly, during our work we will use the following definition.

Definition 2 *A pair of contribution and pension rates (μ_A, μ_D) is said to be feasible if it satisfies Equation (9).*

Before showing the form of the objective function in the following section, let us have a closer look at the feasibility condition. Given the value of k and K defined in (7), it can also be written as

$$\mathbb{E}_{t_0}^{\mathbb{Q},\tau} \left[\int_{t_0}^{\tau} (k - K'\xi) \, e^{- \int_{t_0}^{s} (\rho + \sigma_\pi' \xi) d\theta} ds \right]$$
$$= \mathbb{E}_{t_0}^{\mathbb{Q},\tau} \left[\int_{t_0}^{\tau} e^{- \int_{t_0}^{s} (\rho + \sigma_\pi' \xi) d\theta} d \left(\frac{L(s)}{P(s)} \right) \right]$$
$$- \mathbb{E}_{t_0}^{\mathbb{Q},\tau} \left[\int_{t_0}^{\tau} L(s) \, e^{- \int_{t_0}^{s} (\rho + \sigma_\pi' \xi) d\theta} d \left(\frac{1}{P(s)} \right) \right],$$

where the second term in the right hand side captures the modification in the nominal condition due to the introduction of inflation. In fact, when there is no inflation risk, then $P(t)$ does not change over time and $d(1/P) = 0$. So, the second term disappears.

2.6 The objective function

Our problem is akin to the asset allocation problem when a labor income is present. Indeed, the contributions to the pension fund can be assimilated to the labor income of a single investor. As Merton (1990) and Bodie *et al.* (1992) show, an investor endowed with a non-financial income flow behaves as if his wealth were augmented by the expected discounted value of all his future incomes.

Even if the single investor can be restricted in his ability to borrow against his future income (the constrained consumption-investment problem is studied in El Karoui and Jeanblanc-Picqué, 1998), an investment fund or a pension fund are generally less constrained. Thus, the objective function we consider

---

[9] Formally $_{t-t_0}p_x = \exp\left(- \int_x^{x+t-t_0} \lambda(s) \, ds\right)$, where $\lambda(s)$ is the *instantaneous hazard rate*. As Merton (1990, Section 18.2) underlines, $\lambda(t)$ takes the usual interpretation of the force measuring the probability that the person will die between t and $t + dt$.

does not contain only the fund real wealth but also the expected value of all future incomes. This expected value coincides with the prospective mathematical reserve. This reserve denoted by $\Delta(t)$ can be written as

$$\Delta(t) = \mathbb{E}_t^{\mathbb{Q},\tau} \left[\int_t^\tau (k - K'\xi) e^{-\int_t^s (\rho + \sigma'_\pi \xi) d\theta} ds \right], \tag{11}$$

and the feasibility condition implies $\Delta(t_0) = 0$. In fact, the right hand side of the feasibility condition (9) is nothing but the prospective mathematical reserve computed in t_0.

Finally, if we assume that the fund is characterized by a HARA utility function,[10] the fund will maximize the expected value of

$$U(R,t) = \frac{1}{1-\beta} (R(t) + \Delta(t))^{1-\beta}, \tag{12}$$

which is strictly increasing and concave with respect to R for $\beta > 0$. In the financial literature (see for instance Merton, 1990, Section 6.4) a utility function of the form (12) is known as a "state-dependent" utility. In fact, it depends on the wealth as well as on other state variables (in this case contributions and pensions).

The marginal utility corresponding to (12) is

$$\frac{\partial U}{\partial R} = (R(t) + \Delta(t))^{-\beta},$$

and, since $\beta > 0$, then $R(t)$ will never fall below the value $-\Delta(t)$. This means that the fund can borrow against its prospective mathematical reserve. From this point of view we can think of the sum $R + \Delta$ as a "disposable wealth".

We underline that the introduction of a bequest function is not necessary in our framework. We just need the limit of the indirect utility function to go to zero while the time goes to infinity. In a finite horizon framework Cuoco (1997) argues that the existence result of an optimal portfolio with labor income "does not require the somewhat artificial (although customary) introduction of a bequest function for final wealth with infinite marginal utility at zero."

---

[10] For a theoretical justification of risk aversion for banks and other institutional investors the reader is referred to Koehn and Santomero (1980), Kim and Santomero (1988), and Keeley (1990).

3 The optimal portfolio

The optimization problem for a pension fund can be written as[11]

$$\begin{cases} \max_w \mathbb{E}_{t_0} \left[\int_{t_0}^{\infty} f(t) \frac{1}{1-\beta} \left(R + \Delta(z,t) \right)^{1-\beta} dt \right] \\ \text{s.t.} \begin{bmatrix} dz \\ dR \end{bmatrix} = \begin{bmatrix} \mu_z \\ R\rho + w'M + k \end{bmatrix} dt + \begin{bmatrix} \Omega' \\ w'\Gamma' + K' - R\sigma'_\pi \end{bmatrix} dW, \\ R(t_0) = R_0, \quad z(t_0) = z_0, \quad \forall t_0 < t < H, \end{cases} \quad (13)$$

where

$$\underset{(s+n+1+l)\times 1}{z} \equiv \begin{bmatrix} X \\ G \\ S \\ L \end{bmatrix}, \quad \underset{(s+n+1+l)\times 1}{\mu_z} \equiv \begin{bmatrix} f \\ Gr \\ \mu \\ \mu_L \end{bmatrix}, \quad \underset{(s+n+1+l)\times k}{\Omega'} \equiv \begin{bmatrix} g' \\ \mathbf{0} \\ \Sigma' \\ \Lambda \end{bmatrix},$$

and $f(t)$ is the actuarial discount factor

$$f(t) = \left({}_{t-t_0}p_{t_0} \right) e^{-\rho(t-t_0)}.$$

The Hamiltonian of Problem (13) is

$$\begin{aligned} \mathcal{H} &= f(t) \frac{1}{1-\beta} \left(R + \Delta(z,t) \right)^{1-\beta} + \mu_z' J_z + J_R \left(R\rho + w'M + k \right) \\ &\quad + \frac{1}{2} \operatorname{tr} \left(\Omega' \Omega J_{zz} \right) + \left(w'\Gamma' + K' - R\sigma'_\pi \right) \Omega J_{zR} \\ &\quad + \frac{1}{2} J_{RR} \left(w'\Gamma'\Gamma w + 2w'\Gamma'K - 2Rw'\Gamma'\sigma_\pi + K'K - 2RK'\sigma_\pi + R^2\sigma'_\pi\sigma_\pi \right), \end{aligned}$$

and $J(z, R, t)$ is the value function solving the optimization problem. The subscripts on J indicate partial derivatives.

The first order conditions on $\mathcal{H}$ give the following (implicit) optimal asset allocation:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial w} &= J_R M + \Gamma' \Omega J_{zR} + J_{RR} \left(\Gamma'\Gamma w + \Gamma'K - R\Gamma'\sigma_\pi \right) = 0 \\ &\Rightarrow w^* = \underbrace{R\Gamma^{-1}\sigma_\pi - \Gamma^{-1}K}_{w^*_{(1)}} \underbrace{- \frac{J_R}{J_{RR}} \left(\Gamma'\Gamma \right)^{-1} M}_{w^*_{(1)}} \underbrace{- \frac{1}{J_{RR}} \Gamma^{-1}\Omega J_{zR}}_{w^*_{(2)}}. \quad (14) \end{aligned}$$

After substituting w^* into the Hamiltonian, the Hamilton-Jacobi-Bellman (hereafter HJB) equation is

$$\begin{aligned} 0 &= J_t + f(t) \frac{1}{1-\beta} \left(R + \Delta(z,t) \right)^{1-\beta} + \mu_z' J_z + J_R R \left(\rho + \sigma'_\pi \xi \right) \\ &\quad + J_R \left(k - K'\xi \right) - \frac{1}{2} \frac{J_R^2}{J_{RR}} \xi'\xi - \frac{J_R}{J_{RR}} \xi'\Omega J_{zR} \\ &\quad + \frac{1}{2} \operatorname{tr} \left(\Omega'\Omega J_{zz} \right) - \frac{1}{2} \frac{1}{J_{RR}} J_{zR}' \Omega'\Omega J_{zR}, \end{aligned}$$

---

[11] With a slight twist of language, we will call z the "state variables" associated to Problem (13). Note that the literature sometimes gathers the original state variables X under the name "investment opportunity set" (see for instance Merton, 1990).

whose transversality condition is

$$\lim_{t \to \infty} J(z, R, t) = 0.$$

Since it is well known that the value function usually inherits its functional form from the utility function, we try

$$J(R, z, t) = \frac{1}{1-\beta} F(z, t) (R + \Delta(z, t))^{1-\beta},$$

where $F(z, t)$ is a function whose value must be determined. This yields the following system of PDEs:

$$
\begin{aligned}
0 &= F_t + f(t) + \left(\mu_z' + \frac{1-\beta}{\beta} \xi' \Omega \right) F_z + \frac{1}{2} \frac{1-\beta}{\beta} F \xi' \xi + \frac{1}{2} \operatorname{tr}(\Omega' \Omega F_{zz}) \\
&\quad + \frac{1}{2} \frac{1-\beta}{\beta} \frac{1}{F} F_z' \Omega' \Omega F_z + F(1-\beta)(\rho + \sigma_\pi' \xi), \\
0 &= \Delta_t + \left(\mu_z' - \xi' \Omega \right) \Delta_z + \frac{1}{2} \operatorname{tr}(\Omega' \Omega \Delta_{zz}) - \Delta(\rho + \sigma_\pi' \xi) + (k - K' \xi).
\end{aligned}
$$

The second one is solved by the value of $\Delta(z, t)$ in (11). This can be easily checked via the Feynman-Kač Theorem. Consequently, we can formulate an implicit solution of optimal portfolio as in the following proposition.

Proposition 1 *Under Assumption 1 the portfolio solving Problem (13) is*

$$w^* = w^*_{(1)} + w^*_{(2)} + w^*_{(3)},$$

where

$$
\begin{aligned}
w^*_{(1)} &\equiv \Gamma^{-1} (R \sigma_\pi - K), \\
w^*_{(2)} &\equiv \frac{1}{\beta} (R + \Delta) (\Gamma' \Gamma)^{-1} M, \\
w^*_{(3)} &\equiv \frac{1}{\beta} (R + \Delta) \frac{1}{F} \Gamma^{-1} \Omega F_z - \Gamma^{-1} \Omega \frac{\partial \Delta}{\partial z},
\end{aligned}
$$

and $F(z, t)$ solves

$$
\begin{cases}
0 = F_t + f(t) + \left(\mu_z' + \frac{1-\beta}{\beta} \xi' \Omega \right) F_z + \frac{1}{2} \operatorname{tr}(\Omega' \Omega F_{zz}) \\
\qquad + (1-\beta) \left(\rho + \sigma_\pi' \xi + \frac{1}{2} \frac{1}{\beta} \xi' \xi \right) F + \frac{1}{2} \frac{1-\beta}{\beta} \frac{1}{F} F_z' \Omega' \Omega F_z, \\
\lim_{t \to \infty} F(z, t) = 0.
\end{cases}
\tag{15}
$$

Since values of $L(t)$ differ during the APh and the DPh, also $\Delta(t)$ does. Thus, we confirm the result presented in Battocchio *et al.* (2003) where different asset allocations are found in the two phases.

The functions $F(z,t)$ and $\Delta(z,t)$ that affect the second and the third optimal portfolio components, are the only portfolio parts depending on the retirement date T. Thus, a study of their behaviour with respect to T can catch how a modification in the retirement date (as planned nowadays in some European countries) affects the investment strategies of pension funds.

>From Proposition 1 we can see that the optimal portfolio is actually formed by five funds. This "high" number of funds with respect to the usual number of three or four generally found in the literature, can be explained because of the presence of the contributions and pensions processes.

Let us analyze the portfolio components closer.

The first part $w^*_{(1)}$ minimizes the instantaneous variance of fund wealth and is potentially formed by a combination of all available assets. It is easy to check this minimization property by computing the first (and second) derivative of the wealth diffusion term in Equation (7).

The second component $w^*_{(2)}$ is the growth-optimal fund that remains unchanged with respect to the Merton's classical case but where the wealth level R is augmented by the prospective mathematical reserve Δ.

The third optimal portfolio component $w^*_{(3)}$ arises because of the need to hedge against the changes in the state variables values. In fact, it contains the derivatives of two particular functions with respect to z. The value of $w^*_{(3)}$ can be split into two components: (i) one depending on the derivative of function $F(z,t)$ with respect to z (we will show in the following subsections some particular cases where the value of $F(z,t)$ can be computed in closed form), and (ii) the other one depending on the changes of the mathematical prospective reserve with respect to the state variables (this component disappears when there are no contributions nor pensions).

As Bajeux-Besnainou *et al.* (2001) notice, when there exist one stock and one bond, and the single state variable is the riskless interest rate, the hedging fund contains the bond only. These authors show that when the two fund separation fails (as in our case), portfolio weights of risky assets are not the same for investors with different risk aversions. In fact, since the first and third funds of our structure may contain all assets, the composition of the optimal risky portfolio also vary for investors with different risk aversions.

Finally, the fifth fund is given by the riskless asset whose value is computed, by difference, from the static budget constraint.

3.1 Deterministic time horizon: a quasi-explicit solution

When the actuarial discount factor $f(t)$ is always nil but for one value of t (let us say H) which coincides with a deterministic time horizon, then we fall in the case where the investor maximizes the expected value of his final wealth in H. In this case, the differential equation (15) can be simplified by putting

$$F(z,t) = h(z,t)^{\beta},$$

which yields

$$0 = h_t + \left(\mu'_z + \frac{1-\beta}{\beta}\xi'\Omega\right)h_z + \frac{1}{2}\operatorname{tr}\left(\Omega'\Omega h_{zz}\right) + \frac{1-\beta}{\beta}\left(\rho + \sigma'_\pi\xi + \frac{1}{2}\frac{1}{\beta}\xi'\xi\right)h, \tag{16}$$

whose boundary condition is

$$h\left(z, H\right) = 1.$$

The solution to Equation (16) can be represented through the Feynman-Kač Theorem as follows

$$h\left(z, t\right) = \mathbb{E}_{t_0}^{Z}\left[\exp\left\{\int_t^H \frac{1-\beta}{\beta}\left(\rho + \sigma'_\pi\xi + \frac{1}{2}\frac{1}{\beta}\xi'\xi\right)ds\right\}\right], \tag{17}$$

where

$$dZ = \left(\mu_z + \frac{1-\beta}{\beta}\Omega'\xi\right)dt + \Omega'dW, \quad Z\left(t\right) = z.$$

Once the computed value of $h\left(z, t\right)$ is substituted in Proposition 1 we can state what follows.

Proposition 2 *Under Assumption 1, and if $f\left(t\right) = 0$, $\forall t \neq H$, the portfolio maximizing the expected utility of investor's final wealth (in H) is*

$$w^* = w^*_{(1)} + w^*_{(2)} + w^*_{(3)},$$

where

$$\begin{aligned}
w^*_{(1)} &\equiv \Gamma^{-1}\left(R\sigma_\pi - K\right), \\
w^*_{(2)} &\equiv \frac{1}{\beta}\left(R + \Delta\right)\left(\Gamma'\Gamma\right)^{-1}M, \\
w^*_{(3)} &\equiv \left(R + \Delta\right)\frac{1}{h}\Gamma^{-1}\Omega h_z - \Gamma^{-1}\Omega\frac{\partial\Delta}{\partial z},
\end{aligned}$$

and $h\left(z, t\right)$ is defined in (17).

>From the optimal asset allocation presented in Proposition 2 it is easy to check that our result exactly matches the result presented in Brennan and Xia (2002) when there are neither contributions nor withdrawals from the managed fund. If we put $\Delta = 0$ and $K = \mathbf{0}$ in Proposition 2 we have

$$w^* = \Gamma^{-1}R\sigma_\pi + \frac{1}{\beta}R\left(\Gamma'\Gamma\right)^{-1}M + R\frac{1}{h}\Gamma^{-1}\Omega h_z.$$

Now, since the matrices Γ and M are divided by the consumption price level P, we can define

$$\Gamma \equiv \frac{1}{P}\widetilde{\Gamma}, \quad M \equiv \frac{1}{P}\widetilde{M},$$

21

and, if $\widetilde{M}$, $\widetilde{\Gamma}$, and Ω do not depend on P (as in Brennan and Xia, 2002), we can finally write

$$R_N^{-1} w^* = \widetilde{\Gamma}^{-1} \sigma_\pi + \frac{1}{\beta} \left(\widetilde{\Gamma}' \widetilde{\Gamma} \right)^{-1} \widetilde{M} + \frac{1}{h} \widetilde{\Gamma}^{-1} \Omega h_z,$$

where also the function $h(z,t)$ does not depend on P. In fact, under the previous hypotheses ξ does not depend on P and the value of $h(z,t)$ shown in Equation (17) neither.

Thus, we are able to find the same type of result as Lemma 1 of Brennan and Xia (2002), namely that the optimal proportions of wealth invested in risky assets (i.e. $R_N^{-1} w^*$) is independent of both real wealth and price level. Nevertheless, their result heavily depend on the choice of both the utility function and the functional form chosen for the drift and diffusion terms of assets, price level, and inflation.

3.2 An explicit solution

Now we go back to the general case where $f(t) \neq 0$. It is easy to check that when neither the market price of risk ξ nor the real riskless interest rate $\rho + \sigma_\pi' \xi$ depend on the state variables z, then there exists a solution to PDE (15) for $F(z,t)$ which does not depend on z. A simple market structure where these hypotheses hold is described in the example developed in the next section.

Assumption 2 *Neither the market price of risk nor the real riskless interest rate depend on the state variables (i.e. $\frac{\partial \xi}{\partial z} = \mathbf{0}$, and $\frac{\partial}{\partial z} (\rho + \sigma_\pi' \xi) = \mathbf{0}$).*

We acknowledge that the assumption of a deterministic market price of risk may seem relatively strong with respect to the actual market structure. Nevertheless, it is often used in the literature because it simplifies computations a lot, and allows to obtain a closed form solution. This is the case, for instance, in Boulier *et al.* (2001) and Brennan and Xia (2002). Furthermore, our analysis where also the mortality risk is introduced, needs also the assumption of a deterministic real riskless interest rate. Our exact solution can be thought of as a benchmark that can give some practical insights for the actual investments of pension fund managers.

So, under Assumption 2 the function $F(t)$ must verify

$$\begin{cases} 0 = F_t + f(t) + (1-\beta) \left(\rho + \sigma_\pi' \xi + \frac{1}{2} \frac{1}{\beta} \xi' \xi \right) F, \\ \lim_{t \to \infty} F(z,t) = 0, \end{cases}$$

whose solution is given by

$$F(t) = \int_t^\infty f(s) \, e^{\int_t^s (1-\beta) \left(\rho + \sigma_\pi' \xi + \frac{1}{2} \frac{1}{\beta} \xi' \xi \right) d\theta} ds.$$

Accordingly, the result stated in Proposition 1 can be explicitly computed as in the following corollary.

Corollary 1 *Under Assumptions 1 and 2, the optimal portfolio solving Problem (13) is*

$$w^* = w^*_{(1)} + w^*_{(2)} + w^*_{(3)},$$

where

$$
\begin{aligned}
w^*_{(1)} &\equiv \Gamma^{-1}\left(R\sigma_\pi - K\right), \\
w^*_{(2)} &\equiv \frac{1}{\beta}\left(R + \Delta\right)\left(\Gamma'\Gamma\right)^{-1} M, \\
w^*_{(3)} &\equiv -\Gamma^{-1}\Omega\frac{\partial\Delta}{\partial z}.
\end{aligned}
$$

3.3 The time horizon

As shown in Menoncin (2002) for a CARA investor, when a background risk is taken into account, the optimal portfolio hedging component is the only component depending on the time horizon. Instead, in our framework, we use a CRRA utility function whose argument contains the function $\Delta(z,t)$ and, so, depends on the (stochastic) time horizon τ. This means that, in some sense, the risk aversion depends on the time horizon as well.

Let us further elaborate on the case presented in Corollary 1 where an explicit solution for the optimal portfolio is found. Although the time horizon does not affect the optimal portfolio first component $w^*_{(1)}$, it does affect the other two components where the function $\Delta(z,t)$ appears.

The second so-called speculative component is directly proportional to Δ and so its behaviour with respect to time is governed by the behaviour of $\partial\Delta/\partial t$. Thus, if the real balance between future contributions and pensions follows the same path as the nominal balance (i.e. it decreases till the retirement date T and then increases), then also the speculative portfolio component becomes lower and lower as time reaches T and increases again after. This means that the speculative activity of the pension fund (i.e. investment in risky assets) must decrease when the pension date approaches. Thus, investments should be concentrated on the riskless asset in order to provide a safer revenue for paying the pensions whose payments approach. Finally, after T, when the death probability becomes higher and higher and the pensions start being paid, the speculative investments increase since the relative weight of the total amount of future pensions that must be faced by the fund becomes lower and lower. Nevertheless, we underline that the speculative component $w^*_{(2)}$ in Corollary 1 also contains the wealth level R. Thus, the effect of the reduction in Δ could be compensated by the increase in the managed wealth. In fact, after T, since the weight of the risky assets starts increasing also the portfolio return

becomes higher and the wealth should grow at a higher rate. This could imply a lower reduction in the relative weight of risky assets than those we have presented above. This behaviour will be observed on a numerical simulation in the following section.

The changes in the third optimal portfolio component $w^*_{(3)}$ due to time are less easy to investigate since the term $w^*_{(3)}$ contains the derivative of Δ with respect to the state variables z. Thus, we refer the reader to the following section where a numerical simulation is carried out.

Finally, observe that our setting can be used to catch the impact of an extension of the retirement date (T) in terms of loss of expected utility (see the current political debate in Europe).

4   An example

In this section we fully solve the investment problem for a simplified market structure, and aim to supply the reader with an effective prescription on how to allocate a nominal wealth between three assets.

We take into account a market structure similar to the one presented in Battocchio and Menoncin (2002). More precisely, we consider two independent risk sources: one for the interest rate (W_r), and one for the stock (W_S):

$$dW = \left[\begin{array}{cc} dW_r & dW_S \end{array} \right]'.$$

This market structure can be summarized through the following stochastic differential equations:

$$
\begin{cases}
\left[\begin{array}{c} dr \\ \frac{dP}{P} \end{array} \right] = \left[\begin{array}{c} \eta \left(\bar{r} - r \right) \\ r + m_\pi \end{array} \right] dt + \left[\begin{array}{cc} -\sigma_r & 0 \\ \sigma_{\pi r} & \sigma_{\pi S} \end{array} \right] dW, \\
dG = Grdt, \\
\left[\begin{array}{c} dS \\ dB \end{array} \right] = \left[\begin{array}{c} S \left(r + m_S \right) \\ B \left(r + a_K \sigma_r \zeta \right) \end{array} \right] dt + \left[\begin{array}{cc} S\sigma_{Sr} & S\sigma_S \\ Ba_K\sigma_r & 0 \end{array} \right] dW, \\
dL = \left(u \mathbb{I}_{t<T} - v \left(1 - \mathbb{I}_{t<T} \right) \right) dt,
\end{cases}
\tag{18}
$$

where

$$a_K = \frac{1 - e^{-\eta T_K}}{\eta},$$

and all parameters are real constants.

The economic structure described by this system has the following characteristics.

1. The riskless interest rate r follows a mean reverting process. The strength of the mean reversion effect is measured by η while $\bar{r}$ is the mean rate (as in Vasiček, 1977).

2. The drift of the realized rate of inflation (dP/P) is supposed to follow the riskless interest rate. The constant term m_π can be positive or negative. Nevertheless, in a economic peaceful period, the inflation rate is less than the nominal interest rate, thus $m_\pi < 0$. We have supposed that the price process is affected by the risk sources of both the interest rate and the stock.

3. There exists one bond whose price is derived from the behavior of the riskless interest rate (see Vasiček, 1977). The bond is supposed to have a constant time-to-maturity equal to T_K and a constant price of risk ζ. As shown in Boulier *et al.* (2001) there exists a suitable combination of this bond and the riskless asset which is able to replicate a constant maturity bond. Thus, the market structure with a constant time-to-maturity bond can always be reformulated as a market structure with a constant maturity bond and vice-versa.

4. There exists one stock whose price is affected by the interest rate risk source and by a risk source of its own. Furthermore, its revenue is supposed to be higher than the riskless interest rate (with m_S strictly positive) to avoid arbitrage opportunities.

5. We suppose, for the sake of simplicity, a deterministic nominal contribution flow (u) and a deterministic nominal pension flow (v). This is the same assumption as in Battocchio *et al.* (2003), but we underline that the stochastic nature of both the consumption price index and the riskless interest rate makes our analysis much more complex.

6. We recall that the presence of a profit sharing rule $\phi(t)$ modifies the functions r, μ, and Σ. Thus, the value of ρ is affected by ϕ via the modification on the riskless interest rate. Given the form of P process the value of ρ is given by

$$\rho = -\frac{\phi}{1+\phi}r - m_\pi + \sigma_{\pi r}^2 + \sigma_{\pi S}^2.$$

Nevertheless, we have previously shown that our model displays a closed form solution if Assumption 2 holds, that is if the value of ρ does not depend on the state variables. Since r is a state variable in our example, then Assumption 2 holds only if there is no profit sharing. Thus, during our example, we will assume $\phi(t) = 0$, $\forall t \geq t_0$.

The fundamental matrices as defined in (7) and (13) are as follows:

$$
\underset{2\times 2}{\Gamma'} = \frac{1}{P}\begin{bmatrix} S & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} \sigma_{Sr} & \sigma_S \\ a_K\sigma_r & 0 \end{bmatrix},
$$

$$
\underset{2\times 1}{M} = \frac{1}{P}\begin{bmatrix} S & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} m_S - \sigma_{Sr}\sigma_{\pi r} - \sigma_S\sigma_{\pi S} \\ a_K\sigma_r(\zeta - \sigma_{\pi r}) \end{bmatrix},
$$

$$
\underset{2\times 4}{\Omega} = \begin{bmatrix} -\sigma_r & P\sigma_{\pi r} & S\sigma_{Sr} & Ba_K\sigma_r & 0 \\ 0 & P\sigma_{\pi S} & S\sigma_S & 0 & 0 \end{bmatrix}.
$$

Accordingly, the market price of risk has the following value:

$$\xi \equiv \Gamma'^{-1} M = \left[\begin{array}{c} \zeta - \sigma_{\pi r} \\ \frac{1}{\sigma_S} \left(m_S - \sigma_S \sigma_{\pi S} - \sigma_{Sr} \zeta \right) \end{array} \right], \tag{19}$$

which does not depend on the state variables L, P, r, or time.

In the next subsections we present the mortality law we are going to use and we explicitly compute the feasibility condition in order to determine suitable pairs (u, v).

4.1   The mortality law

For our example we assume that the remaining lifetime of a member follows the popular Gompertz-Makeham distribution. Thus, the probability to be alive in t for an individual aged of t_0 is given by[12]

$$_{t-t_0} p_{t_0} = \exp \left\{ -\lambda \left(t - t_0 \right) + e^{\frac{t_0 - m}{b}} \left(1 - e^{\frac{t - t_0}{b}} \right) \right\}, \tag{20}$$

where λ is a positive constant measuring accidental deaths linked to non-age factors, while m and b are modal and scaling parameters of the distribution, respectively. When b tends to infinity we have the exponential distribution of the form

$$_{t-t_0} p_{t_0} = e^{-\lambda(t-t_0)},$$

whose force of mortality (λ) is constant.

[Fig. 3 here]

The behaviour of Function (20) through time is shown in Figure 3 where we have supposed $\lambda = 0$ (the so-called pure Gompertz case). The parameter values for (m, b) are $(88.18, 10.5)$ for males (solid line) and $(92.63, 8.78)$ for females (dashed line). These values are presented in Milevsky (2001) where the author prices all annuities using the Individual Annuity Mortality (IAM) 2000 table, dynamically adjusted using scale G, published by the Society of Actuaries.

We can observe that the survival probability till 50 years for males and 60 years for females is very high and close to 1. Then there is a sudden decrease and we reach a probability of surviving till 100 years that is almost zero for males but still positive for females.

4.2   The feasibility condition

In our example the volatility of pensions and contributions are put equal to zero (i.e. $K = \mathbf{0}$). Thus, the feasibility condition in (9) can be written as

$$0 = \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\int_{t_0}^{\infty} \left(_{s-t_0} p_{t_0} \right) k \left(s \right) e^{- \int_{t_0}^{s} \left(r - \mu_\pi + \sigma_\pi' \sigma_\pi + \sigma_\pi' \xi \right) d\theta} ds \right],$$

---

[12]It can be immediately checked that $_0 p_{t_0} = 1$, $_\infty p_{t_0} = 0$.

and, since $k(t) = (u\mathbb{I}_{t<T} - v(1 - \mathbb{I}_{t<T}))/P(t)$ it can be further simplified as

$$\frac{v}{u} = \frac{\int_{t_0}^{T} (_{s-t_0}p_{t_0}) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\frac{1}{P(s)} e^{-\int_{t_0}^{s} (r-\mu_\pi+\sigma'_\pi\sigma_\pi+\sigma'_\pi\xi)d\theta} \right] ds}{\int_{T}^{\infty} (_{s-t_0}p_{t_0}) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\frac{1}{P(s)} e^{-\int_{t_0}^{s} (r-\mu_\pi+\sigma'_\pi\sigma_\pi+\sigma'_\pi\xi)d\theta} \right] ds}.$$

A straightforward application of Itô's lemma and Girsanov's theorem, allows us to write

$$\frac{1}{P(s)} = \frac{1}{P(t)} e^{-\int_t^s \left(\mu_\pi - \frac{1}{2}\sigma'_\pi\sigma_\pi - \sigma'_\pi\xi\right)d\theta - \int_t^s \sigma'_\pi dW^{\mathbb{Q}}},$$

and so the ratio v/u simplifies to

$$\frac{v}{u} = \frac{\int_{t_0}^{T} (_{s-t_0}p_{t_0}) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\frac{1}{P(t)} e^{-\int_{t_0}^{s} \left(r+\frac{1}{2}\sigma'_\pi\sigma_\pi\right)d\theta - \int_{t_0}^{s} \sigma'_\pi dW^{\mathbb{Q}}} \right] ds}{\int_{T}^{\infty} (_{s-t_0}p_{t_0}) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\frac{1}{P(t)} e^{-\int_{t_0}^{s} \left(r+\frac{1}{2}\sigma'_\pi\sigma_\pi\right)d\theta - \int_{t_0}^{s} \sigma'_\pi dW^{\mathbb{Q}}} \right] ds}.$$

The computations of the expectations can be found in Appendix A where we show that the feasible ratio v/u can be written as

$$\frac{v}{u} = \frac{\int_{t_0}^{T} (_{s-t_0}p_{t_0}) e^{\Phi(r(t_0),s,t_0)} ds}{\int_{T}^{\infty} (_{s-t_0}p_{t_0}) e^{\Phi(r(t_0),,s,t_0)} ds}, \tag{21}$$

where $\Phi(r(t_0),s,t_0)$ is defined in (23), and $(_{s-t_0}p_{t_0})$ in (20). Unfortunately, we cannot find a closed form solution for the feasible set of pairs (u,v). Note however that the values of feasible v and u are positively correlated. When higher contributions are paid, the fund can afford to supply its members with higher pensions, and vice-versa.

If we assume that the death probability follows an exponential distribution (i.e. b tends to infinity in (20)) and the interest rate r is deterministic and constant (i.e. it coincides with $\widetilde{r}$ in Appendix A where $\sigma = \mathbf{0}$ and η tends to infinity) Condition (21) can be simplified:

$$\frac{v}{u} = \frac{\int_{t_0}^{T} e^{-(\lambda+r)(s-t_0)} ds}{\int_{T}^{\infty} e^{-(\lambda+r)(s-t_0)} ds} = e^{(\lambda+r)(T-t_0)} - 1.$$

This simply shows that the value of pensions (v) is given by the capitalized value of the contributions (u) where the rate of capitalization is given by the sum of the riskless interest rate (r) and the force of mortality (λ). Under stochastic interest rates, the value of the optimal ratio v/u given in (21) can be computed only through numerical solutions. The behaviour of v/u for the pure Gompertz case (i.e. $\lambda = 0$) with respect to T and t_0 for both males and females is summarized in Figure 4, where the following values of the interest rate parameters are taken: $\eta = 0.2$, $\bar{r} = 0.05$, $\sigma_r = 0.01$, $\sigma_{\pi r} = 0.016$, $\zeta = 0.46$, and $r_0 = 0.03$.[13] In the simulations we will show in the course of our work, we will assume $t_0 = 25$

---

[13] More explanations for the choice of these values will be supplied in the following subsections.

and $T = 65$. This means that a member starts contributing when he is 25 and he retires when he is 65. For those values of the parameters, the feasible ratio v/u is given by 10.97, as reported in Table 2 where all the parameter vales are summarized.

[Fig. 4 here]

>From Figure 4 we see that the feasible ratio v/u is positively correlated with time of retirement T and negatively correlated with the age at which the member enters the fund. This behavior is very intuitive and does not require extensive discussion. Once the pension date T has been fixed, the later a member enters the fund, the lower the pension he will be able to accumulate. On the contrary, given the initial age t_0, the later the pension date T, the longer the accumulation phase and the higher the pensions that will be paid.

Since females enjoy a longer lifetime than males, their feasible ratio v/u is lower. Indeed, the pension must be paid to a female worker for a longer period of time. From the two pictures, we can see that the pension rate v paid to females should be around 80% less than the one paid to males for u hold fixed.

4.3   The optimal portfolio

In this subsection we exhibit a simple solution for the optimal asset allocation under the market structure of the example presented above. Given the result stated in Corollary 1, the optimal portfolio can be written as

$$
\begin{aligned}
w_{(1)}^* &\equiv R\Gamma^{-1}\sigma_\pi, \\
w_{(2)}^* &\equiv \frac{1}{\beta}\left(R+\Delta\right)\left(\Gamma'\Gamma\right)^{-1}M, \\
w_{(3)}^* &\equiv -\Gamma^{-1}\Omega\frac{\partial\Delta}{\partial z}.
\end{aligned}
$$

Now, the value of Δ (as in Equation (11)) must be computed. As shown in the previous subsection we can write

$$
\begin{aligned}
\Delta\left(t\right) &= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^\infty \left(_{s-t}p_t\right)ke^{-\int_t^s\left(r-\mu_\pi+\sigma_\pi'\sigma_\pi+\sigma_\pi'\xi\right)d\theta}ds\right] \\
&= \int_t^\infty \left(_{s-t}p_t\right)\left(u\mathbb{I}_{s<T}-v\left(1-\mathbb{I}_{s<T}\right)\right) \\
&\quad \times\mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{P\left(t\right)}e^{-\int_t^s\left(r+\frac{1}{2}\sigma_\pi'\sigma_\pi\right)d\theta-\int_t^s\sigma_\pi'dW^{\mathbb{Q}}}\right]ds \\
&= \frac{1}{P\left(t\right)}\int_t^\infty \left(_{s-t}p_t\right)e^{\Phi(r(t),s,t)}\left(u\mathbb{I}_{s<T}-v\left(1-\mathbb{I}_{s<T}\right)\right)ds,
\end{aligned}
$$

where $\Phi\left(r\left(t\right),s,t\right)$ is defined in (23) in Appendix A.

Consequently, Δ can be split into two different equations according to the value of t:

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1. when $t < T$ we have

$$
\begin{aligned}
\Delta\left(t\right) &= \frac{1}{P\left(t\right)} u \int_{t}^{T} \left(_{s-t}p_t\right) e^{\Phi(r(t),s,t)} ds \\
&\quad - \frac{1}{P\left(t\right)} v \int_{T}^{\infty} \left(_{s-t}p_t\right) e^{\Phi(r(t),s,t)} ds,
\end{aligned}
$$

2. when $t \geq T$ we have

$$
\Delta\left(t\right) = -\frac{1}{P\left(t\right)} v \int_{t}^{\infty} \left(_{s-t}p_t\right) e^{\Phi(r(t),s,t)} ds.
$$

The third optimal portfolio component is then given by

$$
-\Gamma^{-1}\Omega\frac{\partial\Delta}{\partial z}
$$

$$
= \begin{bmatrix} 0 & -\frac{P^2}{S\sigma_S}\sigma_{\pi S} & -P & 0 & 0 \\ \frac{P}{a_K B} & -P^2\frac{\sigma_{\pi r}\sigma_S - \sigma_{Sr}\sigma_{\pi S}}{a_K\sigma_r\sigma_S B} & 0 & -P & 0 \end{bmatrix} \begin{bmatrix} \Delta_r \\ \Delta_P = -\frac{\Delta}{P} \\ \Delta_S\,(=0) \\ \Delta_B\,(=0) \\ \Delta_L\,(=0) \end{bmatrix}
$$

$$
= P \begin{bmatrix} S & 0 \\ 0 & B \end{bmatrix}^{-1} \begin{bmatrix} \frac{\sigma_{\pi S}}{\sigma_S}\Delta \\ \frac{1}{a_K}\Delta_r + \frac{\sigma_{\pi r}\sigma_S - \sigma_{Sr}\sigma_{\pi S}}{a_K\sigma_r\sigma_S}\Delta \end{bmatrix},
$$

where the subscripts on Δ indicate partial derivatives.

After substituting all the simplifications done above, the optimal portfolio can be written as

$$
\begin{bmatrix} \frac{S}{P}w_S^* \\ \frac{B}{P}w_B^* \end{bmatrix} \tag{22}
$$

$$
= \begin{bmatrix} \frac{R+\Delta}{\beta\sigma_S^2}\left(m_S + \sigma_{\pi S}\sigma_S\left(\beta-1\right) - \sigma_{Sr}\zeta\right) \\ \frac{R+\Delta}{a_K\beta\sigma_S^2\sigma_r}\left(-\sigma_{Sr}m_S + \sigma_S\left(\beta-1\right)\left(\sigma_S\sigma_{\pi r} - \sigma_{Sr}\sigma_{\pi S}\right) + \left(\sigma_S^2 + \sigma_{Sr}^2\right)\zeta\right) + \frac{\Delta_r}{a_K} \end{bmatrix}.
$$

Finally, as it is easy to check, the optimal real wealth is given by

$$
\begin{aligned}
dR &= \left(R\left(\rho + \xi'\sigma_\pi\right) + \frac{1}{\beta}\left(R+\Delta\right)\xi'\xi + \xi'\begin{bmatrix} \sigma_r\Delta_r + \sigma_{\pi r}\Delta \\ \sigma_{\pi S}\Delta \end{bmatrix} + k\right) dt \\
&\quad + \left(\frac{1}{\beta}\left(R+\Delta\right)\xi' + \begin{bmatrix} \sigma_r\Delta_r + \sigma_{\pi r}\Delta \\ \sigma_{\pi S}\Delta \end{bmatrix}'\right) dW.
\end{aligned}
$$

It is interesting to draw a parallel between the market structure studied in Bajeux-Besnainou $et\ al.$ (2001) and the one presented here. Even if these authors do not take into account any contributions nor pensions, the comparison can be easily made since contributions and pensions are not stochastic in our example. In both models there exist a riskless asset, a stock, and a bond. The

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main difference is that in Bajeux-Besnainou *et al.* the only state variable is the riskless interest rate while in our example we also have another state variable: the inflation rate. The model presented in Bajeux-Besnainou *et al.* is based on the following market structure:

$$
\begin{aligned}
dr &= a_r \left(b_r - r \right) dt - \sigma_r dW_r, \\
dG &= Grdt, \\
\frac{dS}{S} &= \left(r + m_S \right) dt + \sigma_1 dW_S + \sigma_2 dW_r, \\
\frac{dB}{B} &= \left(r + a_K \sigma_r \zeta \right) dt + a_K \sigma_r dW_r.
\end{aligned}
$$

They find that only the bond and the riskless asset play a role in hedging the risk source W_r. Our result, presented in (22), gives the same insight since only the optimal bond allocation w_B^* and, by difference, also the optimal riskless asset allocation w_G^* contain the term Δ_r which measures the changes in the expected value of all contributions and pensions due to the fluctuations in the riskless interest rate r.

4.4   The parameter values

For the interest rate that follows a Vasiček structure a complete estimation of the parameters (i.e. the volatility, the mean interest rate, and the strength of the mean reverting effect) can be found in Babbs and Nowman (1998, 1999) who construct zero-coupon yields. Their first work analyzes the data of the United States while their second work estimates the parameters for a set of European countries and Japan. The main results are summarized in Table 2. For the (different) sample periods the reader is referred to Babbs and Nowman (1998, 1999).

Given the values exposed in Table 2, we have chosen the values $\bar{r} = 0.05$, $\eta = 0.2$, and $\sigma_r = 0.01$.

Once the interest rate parameters have been estimated, the bond drift and diffusion terms can be obtained by fixing a market price of risk ζ and a maturity T_K. The longest maturity taken into account in Babbs and Nowman (1998, 1999) is 10 years and so we put $T_K = 10$. Now, since the interest rate on a bond with 10 years maturity is around 7%, then we can compute ζ by solving

$$
0.07 = \bar{r} + \frac{1 - e^{-\eta T_K}}{\eta} \sigma_r \zeta,
$$

which immediately gives $\zeta = 0.46$.

[Table 2 here]

The risk premium on the stock (m_S) is chosen after the analysis of Mehra and Prescott (1985). They found that the risk premium was approximately 0.06

for the United States of America during the period 1889-1978. Thus, we put $m_S = 0.06$. The standard deviation of the market return was about 0.2 for the same period. Hence, we put $\sqrt{\sigma_{Sr}^2 + \sigma_S^2} = 0.2$. In this case we have one degree of freedom. To the best of our knowledge, there are no works dedicated to disentangle σ_{Sr} and σ_S, that is the stock's own volatility component and the stock volatility due to the changes in the riskless interest rate. We have decided to give more weight to the σ_S component by putting $\sigma_S = 0.19$ and $\sigma_{Sr} = 0.06$.

As regards inflation, we refer to the US data summarised in Table 1. From these values we infer that the inflation risk premium m_π has to be negative (around -0.01). In fact, the inflation rate is lower than the riskless interest rate. In our model, the inflation volatility is given by $\sqrt{\sigma_{\pi r}^2 + \sigma_{\pi S}^2 + \sigma_\pi^2}$ which is equated to 0.026. In this case we have two degrees of freedom. We suppose half of the inflation volatility is explained by the inflation own diffusion term σ_π. Thus we put $\sigma_\pi = 0.013$. Instead, the effect of interest rate volatility and stock volatility are supposed to be equal $\sigma_{\pi r} = \sigma_{\pi S} = 0.016$. Finally, we assume a member enters the pension fund when he is $t_0 = 25$ and retires when he is $T = 65$. We simulate the behaviour of the optimal fund portfolio during a management period of 60 years (i.e., for the member, from the age of 25 till the age of 85). Parameter values (which are also consistent with the numerical analysis presented by Boulier et $al.,$ 2001) are gathered in Table 3. All initial values are supposed to be equal: $S(0) = B(0) = G(0) = L(0) = P(0) = 1$. Since we have chosen to put $u = 1$ (and $L(0) = 1$) we should not choose a too high level of initial wealth. Indeed, a very high value for $R(0)$ would mitigate the effect of contributions and pensions on the optimal wealth and portfolio composition. On the other hand, the initial value of wealth should not be too low either since we risk to end up with a negative real wealth because of the pension payments. After running some preliminary experiments, we have chosen $R(0) = 20$ which allows: (i) to keep the optimal real wealth positive in simulations, and (ii) to make effective the impact of contributions and pensions on wealth.

The leve of risk aversion (β) is put equal to 10 accordingly to the analysis of Mehra and Prescott (1985). Even if it seems quite high from the pint of view of a pension fund, this level of risk aversion is consistent with the use of a HARA non-time separable utility function.

[Table 3 here]

4.5 The numerical simulation

In this subsection we follow Stojanovic (2003) where the reader can find a complete treatment of simulation techniques relevant to financial mathematics. In particular, his Chapter 7 is dedicated to the study of optimal portfolio rules.

Results of a typical simulation are plotted in Figures 5 and 6. The graphs represent the behaviour of the optimal portfolio compositions and of the optimal wealth both in real and nominal terms, respectively.

Other simulated paths are qualitatively similar in the sense that the behavior of the portfolio weights are analogous, and their values are very close. The figure represents the case of a male worker. The case of a female member presents the same optimal percentages, but the real wealth then decreases less rapidly during the decumulation phase (because of the lower level of v). The difference of mortality risk between males and females does not seem to dramatically affect the asset allocation of a pension fund.

[Fig. 5 here]
[Fig. 6 here]

On the x-axis the time is measured as $t - t_0$. We recall that, in our example, the member enters the fund at $t_0 = 25$ while the simulation ends when the member is 85. As shown in Figure 3, the survival probability after 85 years becomes very small and so we can think of the end of simulation as the probable death time. As already mentioned, the initial wealth level ($R(0) = 20$) has been chosen in order to make more apparent the role of contributions and pensions in the evolution of the optimal wealth. In Figure 5 we show the behavior of asset allocations in percentage of the real wealth and the optimal real wealth itself. Instead, in Figure 6, the optimal nominal wealth and the nominal amounts of wealth invested in each asset are represented.

The investments in the riskless asset and the bond evolve in an opposite way. The weight of the former increases while the weight of the latter decreases as time goes on. The opposite behavior of riskless and bond percentages can be explained as follows. The riskless asset supplies the investor with a short run hedge while the bond offers a long run hedge (in our example $T_K = 10$ years). During the first years, the need of a long run hedge is higher than the need for a short run hedge, and the demand for bond will thus be also higher. After a while the need of a long run hedge diminishes and short run hedge gets priority treatment. The riskless asset is gradually substituted for the bond.

The weight of the stock does not change a lot (it remains around the value of 19%). The stock has typically a speculative role in the optimal portfolio, without supplying investors with any particular form of hedging. Actually, this hedging need is satisfied by the riskless asset and the bond whose optimal percentages change more widely through years. After fifty years of management, the optimal percentage of wealth invested in stock falls sharply from a value of around 19% to a value of around 16%.

As shown in Figure 6 the nominal amount of wealth invested in each asset increases while the total nominal wealth increases and decreases while the wealth decreases. Furthermore, the percentage of wealth invested in the risky assets (the stock and the bond)[14] decreases through time and, after the retirement date, its negative slope becomes lower (in absolute value). This means that while the retirement date approaches, the need to switch to a riskless portfolio becomes higher and higher. This is due to the increasing need of paying the

---

[14] The behaviour of the ratio $\left(w_S^* + w_B^*\right)/R$ can be easily derived by inverting the graph of w_G^*/R. In fact, $\left(w_S^* + w_B^*\right)/R = 1 - w_G^*/R$.

pensions. While these pensions are being paid, this need becomes lower and so the reduction in the portfolio riskiness.

It can be seen that the return on the pension fund investment is able to keep the fund wealth positive during 20 years of pension payments even if these pensions are more than ten times the contributions.

5 Conclusion

In this work we have analyzed the real asset allocation problem of a pension fund. Our framework is akin to those taking into account a nontradeable endowment process. Nevertheless, two main differences can be pointed out: (i) the revenues (contributions) and expenses (pensions) of the fund must be linked by a condition ("feasibility condition") guaranteeing that it is profitable to subscribe the pension contract for both the subscriber and the pension fund, (ii) the financial horizon for fund investments is stochastic since it coincides with the subscriber death time. The financial market underlying the study is very general since the set of risky asset values is supposed to be driven by a set of stochastic state variables. Thus, our model encompasses all previous simple models proposed in the literature. Since the financial horizon for pension fund investment is typically long, we also explicitly model the inflation risk.

In such a framework we are able to compute a solution based on a state dependent utility function. Besides, when neither the market price of risk nor the real riskless interest rate depend on the state variables, we are able to compute a closed-form solution. In this case we carry out numerical experiments for a simple market structure where there exist one nominal riskless asset, one stock, and one bond. These numerical experiments have allowed us to provide some practical investment strategy recommendations for pension funds.

A Feasibility condition

Since the riskless interest rate follows the process

$$dr = \eta \left(\bar{r} - r \right) dt + \left[\begin{array}{ccc} -\sigma_r & 0 & 0 \end{array} \right] dW,$$

then under the real risk neutral probability $\mathbb{Q}$ it follows

$$
\begin{aligned}
dr &= \eta \left(\bar{r} - r \right) dt + \left[\begin{array}{ccc} -\sigma_r & 0 & 0 \end{array} \right] \left(dW^{\mathbb{Q}} - \xi dt \right) \\
&= \left(\eta \left(\bar{r} - r \right) - \left[\begin{array}{ccc} -\sigma_r & 0 & 0 \end{array} \right] \xi \right) dt + \left[\begin{array}{ccc} -\sigma_r & 0 & 0 \end{array} \right] dW^{\mathbb{Q}},
\end{aligned}
$$

and, after substituting the value of ξ computed in (19), we have

$$dr = \left(\eta \left(\bar{r} - r \right) - \zeta \sigma_r \sigma_{\pi r} \right) dt + \left[\begin{array}{ccc} -\sigma_r & 0 & 0 \end{array} \right] dW^{\mathbb{Q}}.$$

Accordingly, we can define

$$\widetilde{r} \equiv \bar{r} - \frac{\zeta \sigma_r \sigma_{\pi r}}{\eta},$$

$$\sigma' \equiv \begin{bmatrix} -\sigma_r & 0 & 0 \end{bmatrix},$$

and write

$$dr = \eta \left(\widetilde{r} - r \right) dt + \sigma' dW^{\mathbb{Q}},$$

whose solution is

$$r(t) = \left(r(t_0) - \widetilde{r} \right) e^{-\eta(t-t_0)} + \widetilde{r} + \sigma' \int_{t_0}^t e^{-\eta(t-s)} dW_s^{\mathbb{Q}}.$$

The integral we want to compute is thus

$$-\int_{t_0}^t r(s)\, ds = -\int_{t_0}^t \left(\left(r(t_0) - \widetilde{r} \right) e^{-\eta(s-t_0)} + \widetilde{r} \right) ds - \int_{t_0}^t \int_{t_0}^s e^{-\eta(s-i)} \sigma' dW_i^{\mathbb{Q}} ds,$$

which can be written as

$$-\int_{t_0}^t r(s)\, ds = -\int_{t_0}^t \left(\left(r(t_0) - \widetilde{r} \right) e^{-\eta(s-t_0)} + \widetilde{r} \right) ds - \int_{t_0}^t \int_{t_0}^j e^{-\eta(j-s)} \sigma' ds dW_j^{\mathbb{Q}}.$$

The mean and variance of this integral are then very easy to compute:

$$\begin{aligned}
\mathbb{E}_{t_0}^{\mathbb{Q}} \left[-\int_{t_0}^t r(s)\, ds \right] &= -\int_{t_0}^t \left(\left(r(t_0) - \widetilde{r} \right) e^{-\eta(s-t_0)} + \widetilde{r} \right) ds \\
&= -\left(r(t_0) - \widetilde{r} \right) \int_{t_0}^t e^{-\eta(s-t_0)} ds - \widetilde{r}(t - t_0) \\
&= -\left(r(t_0) - \widetilde{r} \right) \frac{1 - e^{-\eta(t-t_0)}}{\eta} - \widetilde{r}(t - t_0),
\end{aligned}$$

$$\begin{aligned}
\mathbb{V}_{t_0}^{\mathbb{Q}} \left[-\int_{t_0}^t r(s)\, ds \right] &= \mathbb{V}_{t_0}^{\mathbb{Q}} \left[\int_{t_0}^t \int_{t_0}^j e^{-\eta(j-s)} \sigma' ds dW_j^{\mathbb{Q}} \right] \\
&= \int_{t_0}^t \sigma' \sigma \left(\int_{t_0}^j e^{-\eta(j-s)} ds \right)^2 dj \\
&= \sigma' \sigma \int_{t_0}^t \frac{\left(1 - e^{-\eta(j-t_0)} \right)^2}{\eta^2} dj \\
&= \frac{\sigma' \sigma}{2\eta^3} \left(-e^{-2\eta(t-t_0)} + 4e^{-\eta(t-t_0)} + 2\eta(t - t_0) - 3 \right).
\end{aligned}$$

Now we define

$$N(t) \equiv -\int_{t_0}^t r(s)\, ds - \frac{1}{2} \sigma'_\pi \sigma_\pi (t - t_0) - \sigma'_\pi \int_{t_0}^t dW_s^{\mathbb{Q}},$$

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which is a normally distributed sotchastic variable. Thus, in order to compute the expected value of its exponential, we just have to compute its mean and its variance:

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[N\left(t\right)\right] = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[-\int_{t_0}^{t} r\left(s\right)ds\right] - \frac{1}{2}\sigma_\pi' \sigma_\pi \left(t - t_0\right),$$

and

$$\begin{aligned}
\mathbb{V}_{t_0}^{\mathbb{Q}}\left[N\left(t\right)\right] &= \mathbb{V}_{t_0}^{\mathbb{Q}}\left[-\int_{t_0}^{t} r\left(s\right)ds - \sigma_\pi' \int_{t_0}^{t} dW_s^{\mathbb{Q}}\right] \\
&= \mathbb{V}_{t_0}^{\mathbb{Q}}\left[-\int_{t_0}^{t} r\left(s\right)ds\right] + \sigma_\pi' \sigma_\pi \left(t - t_0\right) \\
&\quad + \frac{1}{2}\mathbb{C}_{t_0}^{\mathbb{Q}}\left[-\int_{t_0}^{t} r\left(s\right)ds, -\sigma_\pi' \int_{t_0}^{t} dW_s^{\mathbb{Q}}\right],
\end{aligned}$$

shwere the covariance term can be simplified as follows:

$$\begin{aligned}
&\mathbb{C}_{t_0}^{\mathbb{Q}}\left[-\int_{t_0}^{t} r\left(s\right)ds, -\sigma_\pi' \int_{t_0}^{t} dW_s^{\mathbb{Q}}\right] \\
&= \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\left(-\int_{t_0}^{t} r\left(s\right)ds\right)\left(-\sigma_\pi' \int_{t_0}^{t} dW_s^{\mathbb{Q}}\right)\right] \\
&\quad - \mathbb{E}_{t_0}^{\mathbb{Q}}\left[-\int_{t_0}^{t} r\left(s\right)ds\right] \mathbb{E}_{t_0}^{\mathbb{Q}}\left[-\sigma_\pi' \int_{t_0}^{t} dW_s^{\mathbb{Q}}\right] \\
&= \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\left(\int_{t_0}^{t} r\left(s\right)ds\right)\left(\sigma_\pi' \int_{t_0}^{t} dW_s^{\mathbb{Q}}\right)\right] \\
&= \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\left(\int_{t_0}^{t}\left(\left(r\left(t_0\right) - \widetilde{r}\right)e^{-\eta\left(s - t_0\right)} + \widetilde{r} + \sigma' \int_{t_0}^{s} e^{-\eta\left(s - \tau\right)}dW_\tau^{\mathbb{Q}}\right)ds\right)\left(\int_{t_0}^{t} \sigma_\pi' dW_s^{\mathbb{Q}}\right)\right] \\
&= \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\left(\int_{t_0}^{t}\int_{t_0}^{s} e^{-\eta\left(s - \tau\right)}\sigma' dW_\tau^{\mathbb{Q}}ds\right)\left(\int_{t_0}^{t} \sigma_\pi' dW_s^{\mathbb{Q}}\right)\right].
\end{aligned}$$

Now, since for any deterministic functions $f\left(s\right)$ and $g\left(s\right)$ we have

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\left(\int_{t_0}^{t} f\left(s\right)' dW_s^{\mathbb{Q}}\right)\left(\int_{t_0}^{t} g\left(s\right)' dW_s^{\mathbb{Q}}\right)\right] = \int_{t_0}^{t} f\left(s\right)' g\left(s\right)ds,$$

then we can write

$$\begin{aligned}
&\mathbb{C}_{t_0}^{\mathbb{Q}}\left[-\int_{t_0}^{t} r\left(s\right)ds, -\sigma_\pi' \int_{t_0}^{t} dW_s^{\mathbb{Q}}\right] \\
&= \int_{t_0}^{t}\int_{t_0}^{j} e^{-\eta\left(j - s\right)}\sigma' \sigma_\pi ds dj \\
&= \frac{\sigma' \sigma_\pi}{\eta^2}\left(e^{-\eta\left(t - t_0\right)} + \eta\left(t - t_0\right) - 1\right).
\end{aligned}$$

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Consequently, we have

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[N\left(t\right)\right] = -\left(r\left(t_0\right) - \widetilde{r}\right)\frac{1 - e^{-\eta(t-t_0)}}{\eta} - \left(\widetilde{r} + \frac{1}{2}\sigma'_\pi\sigma_\pi\right)\left(t - t_0\right),$$

$$
\begin{aligned}
\mathbb{V}_{t_0}^{\mathbb{Q}}\left[N\left(t\right)\right] &= -\frac{\sigma'\sigma}{2\eta^3}e^{-2\eta(t-t_0)} + \frac{1}{2\eta^3}\sigma'\left(4\sigma + \eta\sigma_\pi\right)e^{-\eta(t-t_0)} \\
&\quad + \frac{1}{2\eta^2}\left(2\sigma'\sigma + 2\eta^2\sigma'_\pi\sigma_\pi + \eta\sigma'\sigma_\pi\right)\left(t - t_0\right) \\
&\quad - \frac{1}{2\eta^3}\sigma'\left(3\sigma + \eta\sigma_\pi\right).
\end{aligned}
$$

Now, we can go back to

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{N(t)}\right] = e^{\mathbb{E}_{t_0}^{\mathbb{Q}}[N(t)] + \frac{1}{2}\mathbb{V}_{t_0}^{\mathbb{Q}}[N(t)]},$$

and we can write

$$
\begin{aligned}
\Phi\left(r\left(t_0\right), t, t_0\right) &\equiv \mathbb{E}_{t_0}^{\mathbb{Q}}\left[N\left(t\right)\right] + \frac{1}{2}\mathbb{V}_{t_0}^{\mathbb{Q}}\left[N\left(t\right)\right] \qquad\qquad (23)\\
&= -\frac{\sigma'\sigma}{4\eta^3}e^{-2\eta(t-t_0)} + \left(\frac{1}{\eta}\left(r\left(t_0\right) - \widetilde{r}\right) + \frac{1}{4\eta^3}\sigma'\left(4\sigma + \eta\sigma_\pi\right)\right)e^{-\eta(t-t_0)} \\
&\quad + \left(\frac{1}{4\eta^2}\sigma'\left(2\sigma + \eta\sigma_\pi\right) - \widetilde{r}\right)\left(t - t_0\right) \\
&\quad - \frac{1}{\eta}\left(r\left(t_0\right) - \widetilde{r}\right) - \frac{1}{4\eta^3}\sigma'\left(3\sigma + \eta\sigma_\pi\right).
\end{aligned}
$$

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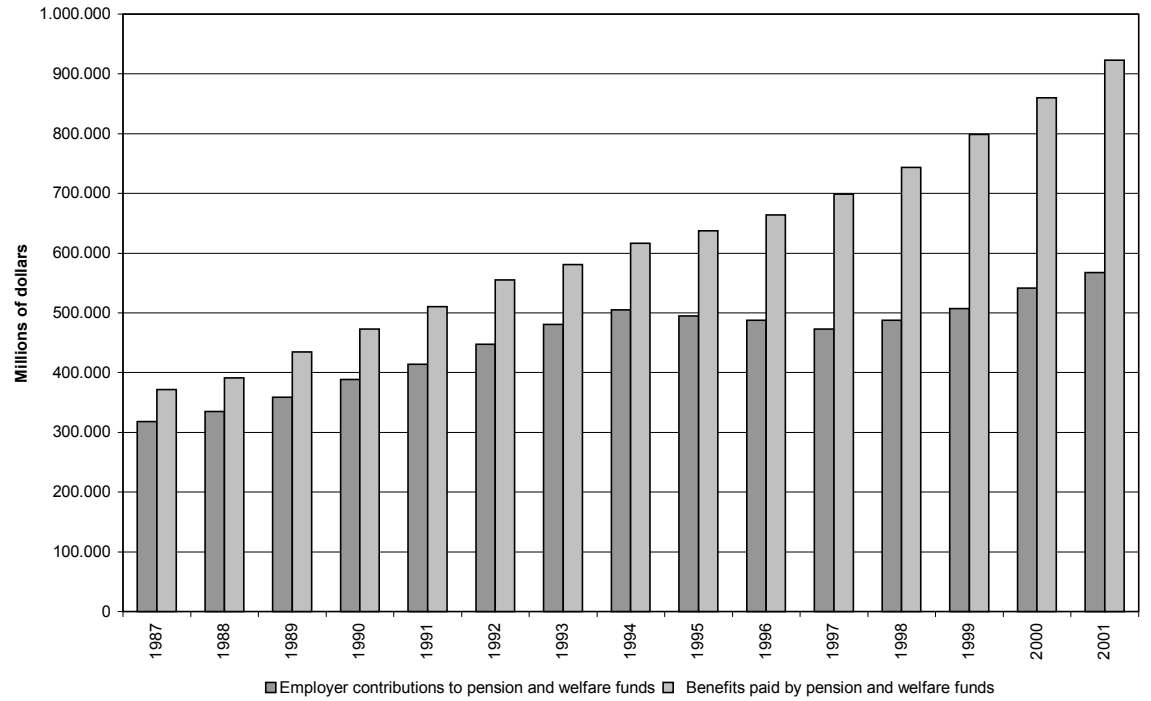Figure 1: Contributions and pensions for US pension and welfare funds

■ Employer contributions to pension and welfare funds ☐ Benefits paid by pension and welfare funds

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Figure 2: Total identified assets in English pension funds balance sheet

.
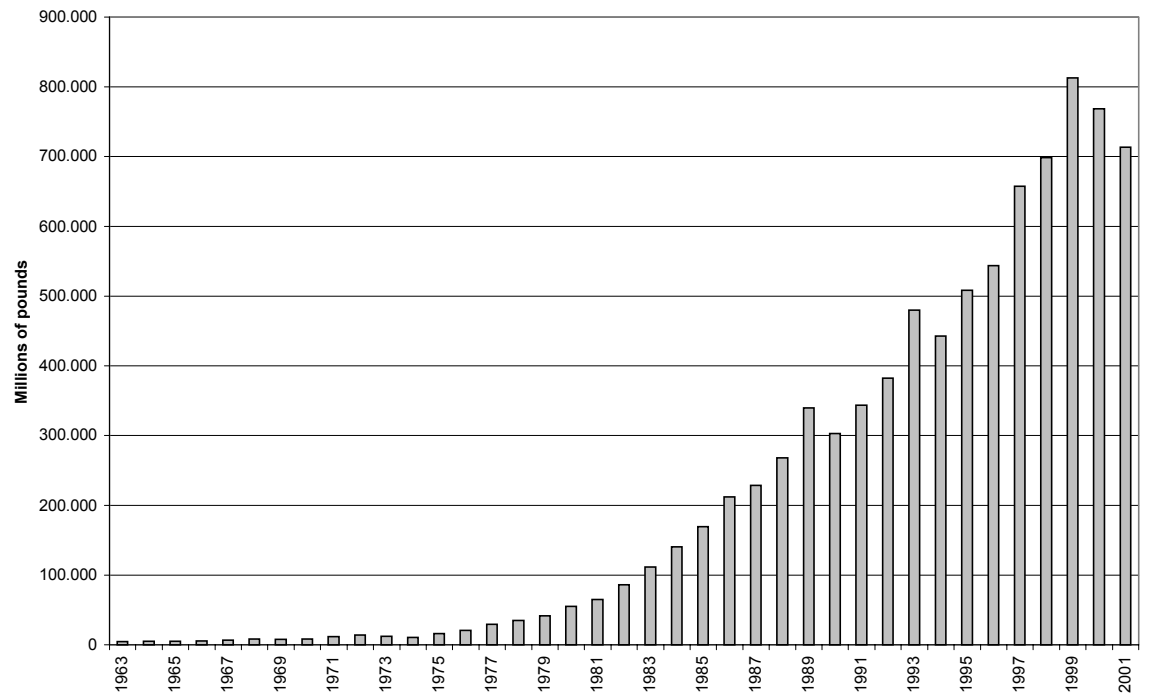
Table 1: Consumption Price Index yearly growth rate for the USA

Period	15/02/1958-15/02/2003
Mean	4.23%
St. dev.	2.58%
Max	12.76%
Min	0.65%

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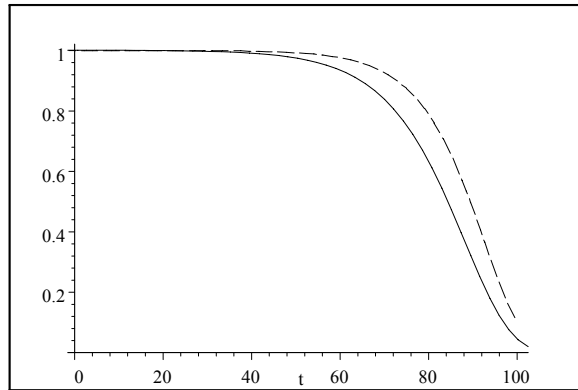Figure 3: Survival probability for an individual aged of 20 (the pure Gompertz case)

Table 2: Interest rate parameters for ten countries

	Mean 3-month rate % ($\bar{r}$)	Mean 10-year rate %	Mean reversion (η)	Standard deviation (σ)
USD	6.170	8.090	0.1908	0.0132
BEF	4.372	7.162	0.1516	0.0096
DKK	5.155	7.774	0.2196	0.0128
FRF	6.870	7.341	0.2604	0.0139
DEM	6.598	7.274	0.2711	0.0150
ITL	9.762	9.919	0.0939	0.0154
JPY	2.600	4.458	0.1238	0.0097
NLG	5.799	7.033	0.3622	0.0091
CHF	4.663	5.357	0.2780	0.0011
GBP	9.249	9.453	0.2256	0.0168

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Figure 4: Feasible ratio v/u

males

females
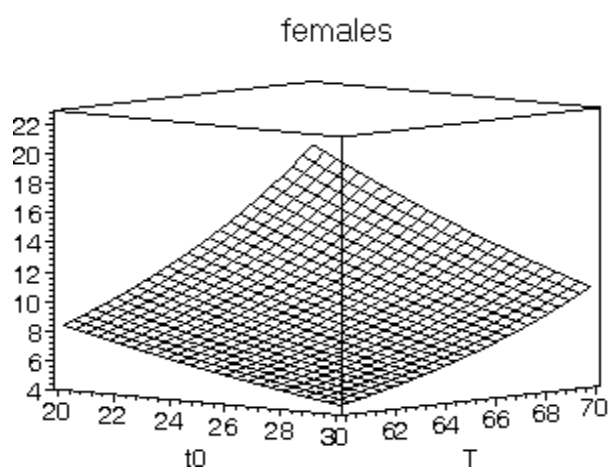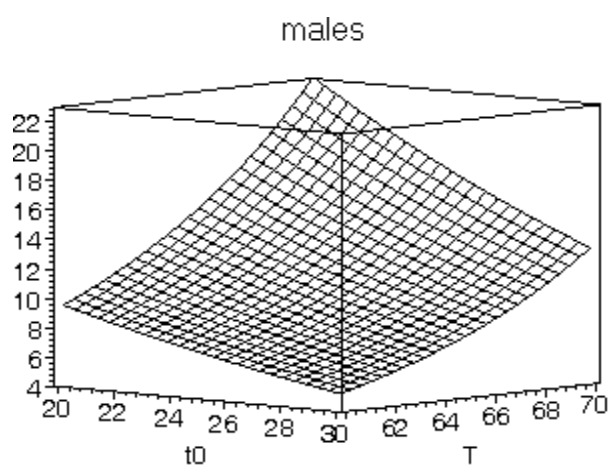
Table 3: Values of parameters

Interest rate			Contrib./Pension process		
Mean reversion, η		0.2	Contribution rate, u		1
Mean rate, $\bar{r}$		0.05	Pension rate, v		10.97
Volatility, σ_r		0.01			
Initial rate, r_0		0.03	**Fix-maturity bond**		
			Maturity, T_K		10
Inflation process			Market price of risk, ζ		0.46
Risk premium, m_π		-0.01			
Interest rate source risk, $\sigma_{\pi r}$		0.016	**Stock**		
Stock source risk, $\sigma_{\pi S}$		0.016	Risk premium, m_S		0.06
Non-hedgeable volatility, σ_π		0.013	Interest rate source risk, σ_{Sr}		0.06
			Stock own volatility, σ_S		0.19
Mortality risk					
Modal parameter, m		88.18	**Demography**		
Scaling parameter, b		10.5	Initial age, t_0		25
Makeham parameter, λ		0	Retirement age, T		65

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Figure 5: Optimal portfolio composition (in percentages) and the real wealth
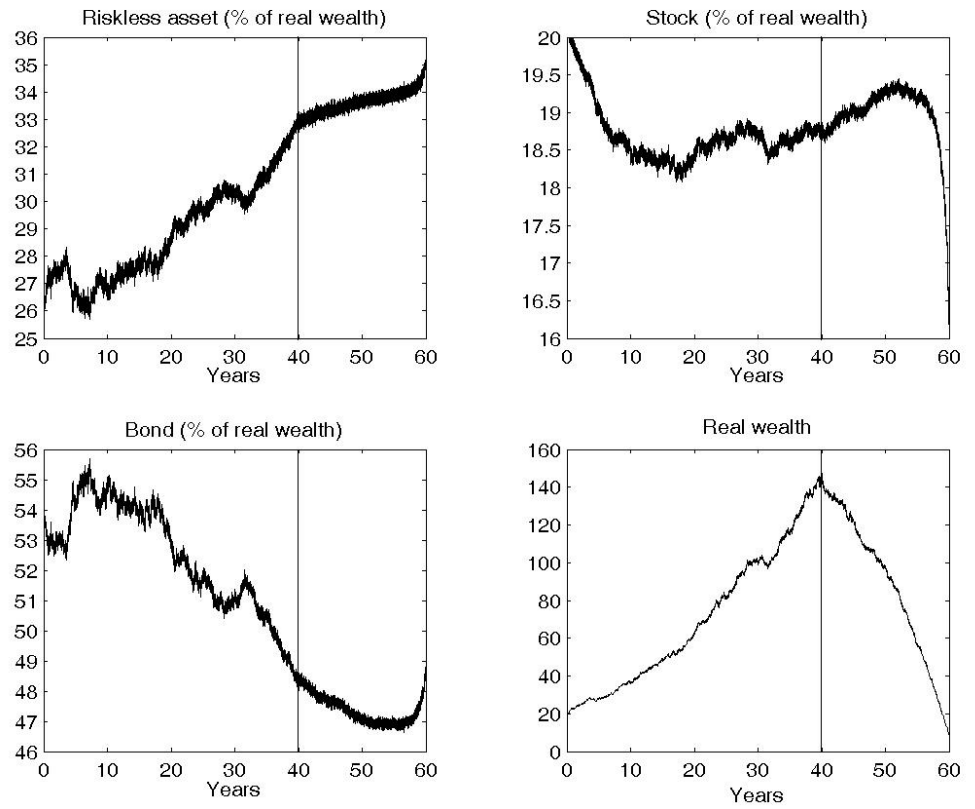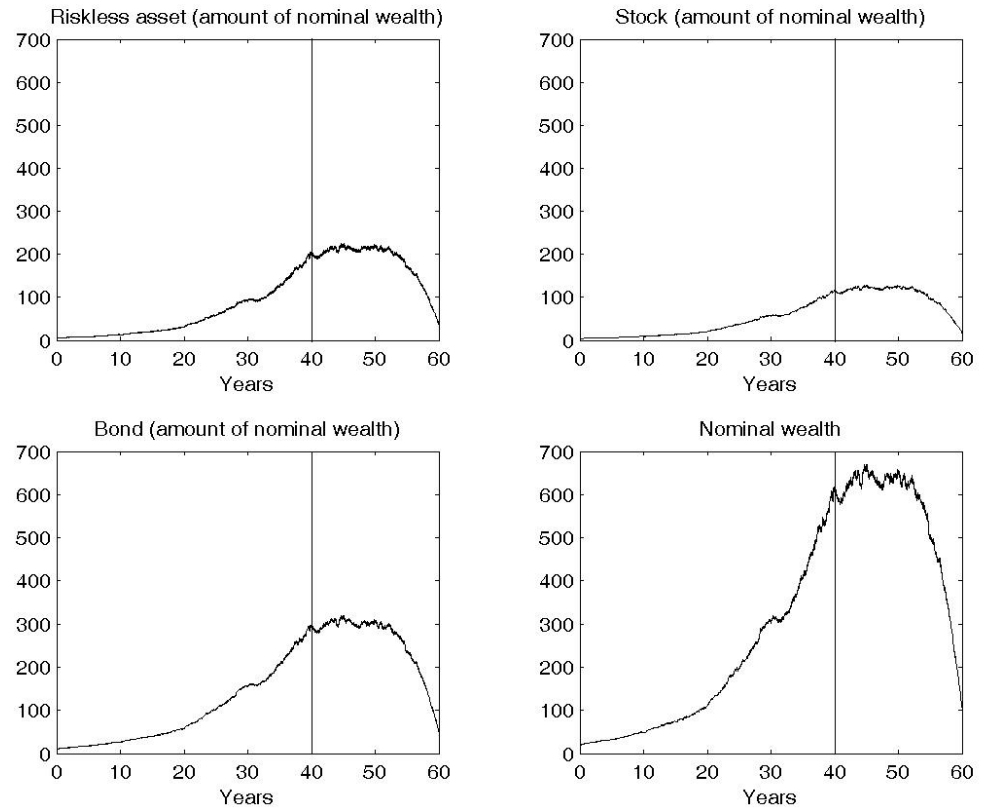
Figure 6: Optimal portfolio composition (in levels) and nominal wealth

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The Institute attempts to be both international and pluridisciplinary. The subjects in its curriculum, the composition of its teaching staff and the diversity of origin of its student body, confer upon it its international character. Professors teaching at the Institute come from all regions of the world, and the approximately 650 students arrive from some 60 different countries. Its international character is further emphasized by the use of both English and French as working languages. Its pluralistic approach - which draws upon the methods of economics, history, law, and political science - reflects its aim to provide a broad approach and in-depth understanding of international relations in general. ***http://heiwww.unige.ch***

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