APPROXIMATION AND CALIBRATION OF SHORT-TERM IMPLIED VOLATILITIES UNDER JUMP-DIFFUSION STOCHASTIC VOLATILITY

Alexey MEDVEDEV and Olivier SCAILLET[a] [1]

[a] HEC Genève and Swiss Finance Institute, Université de Genève, 102 Bd Carl Vogt, CH - 1211 Genève 4, Suisse.

medvedev@hec.unige.ch, scaillet@hec.unige.ch

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Abstract

We derive a closed-form asymptotic expansion formula for option implied volatility under a two-factor jump-diffusion stochastic volatility model when time-to-maturity is small. Based on numerical experiments we describe the range of time-to-maturity and moneyness for which the approximation is accurate. We further propose a simple calibration procedure of an arbitrary parametric model to short-term near-the-money implied volatilities. An important advantage of our approximation is that it is free of the unobserved spot volatility. Therefore, the model can be calibrated on option data pooled across different calendar dates in order to extract information from the dynamics of the implied volatility smile. An example of calibration to a sample of S&P500 option prices is provided. We find that jumps are significant. The evidence also supports an affine specification for the jump intensity and Constant-Elasticity-of-Variance for the dynamics of the return volatility.

Key words: Option pricing, stochastic volatility, asymptotic approximation, jump-diffusion.

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1 Introduction

The choice of an option pricing model is typically a tradeoff between its flexibility and its analytical tractability. It has long been noted that the Black-Scholes formula does not fully explain actual option prices. The phenomenon of implied volatility smile has led to numerous generalizations of the Black-Scholes model via the introduction of stochastic volatility and jumps. Nowadays, most of the analytically tractable specifications used in practice belong to the class of affine jump-diffusion models (Duffie et al. (2000)). The vast majority of the empirical literature employs two-factor affine models (see e.g. Andersen et al. (2002), Bates (2000a,b), Bates (2003), Broadie et al. (2004), Eraker et al. (2003), Eraker (2004), Pan (2002)).

The affine models admit closed-form option pricing at the expense of imposing strong parametric restrictions. In the context of the two-factor model, the variance follows a square root process (the so-called Heston (1993) specification), and the jump intensity is a linear function of the variance. The affine specification is typically assumed on the ground of its analytical tractability, but has been challenged on the empirical ground. Indeed, papers attempting to go beyond the affine specification by considering more flexible models tend to reject the Heston specification against the more general Constant-Elasticity-of-Variance (CEV) specification (see e.g. Jones (2003), Aït-Sahalia and Kimmel (2004)).

Several approximations has been introduced to deal with analytically non-tractable models. They aim at avoiding the need to solve complex partial differential equations (PDEs) or running lengthy Monte-Carlo simulations. Lewis (2000) derives the asymptotic expansion of implied volatility assuming small volatility of volatility. Lee (2001) obtains a similar result assuming slow mean-reversion of the volatility. Alternatively, Fouque et al. (2000) study fast mean-reverting volatility. Hagan et al. (2000) derive an approximate formula for implied volatilities in the context of a CEV-type model assuming small volatility. Backus et al. (1997) and Zhang and Xiang (2006) develop pricing formulae using a quadratic approximation of the implied volatility smile.

This paper contributes to this literature by proposing a new closed-form approximation for implied volatility based on its asymptotic expansion under a two-factor jump-diffusion stochastic volatility model. The approximation is similar in spirit to the formulas obtained in Lewis (2000) and Lee (2001). Both authors deal with stochastic volatility but without jumps. The extension of their approaches to the jump-diffusion case is not clear. The main advantage of our result is that it is derived under a more general specification incorporating jumps.

Alternative approximations of implied volatilities are proposed by Backus et al. (1997) and Zhang and Xiang (2006). Using asymptotics for the density function and assuming a quadratic implied volatility smile they establish relationships between the return distribution and the implied volatility. The advantage of their approaches is that they do not depend on model specification. However, they only allow to identify risk-neutral conditional distribution of returns and not the process governing their dynamics.

After deriving the asymptotic expansion of implied volatility we focus on its application to calibration and model testing. The calibration formula is free of any latent (unobserved) variable in the sense that we do not need to estimate the value of the unobserved spot volatility. Hence it can be applied on implied volatility data pooled across different calendar dates in order to extract information from the dynamics of the implied volatility smile. We run several numerical experiments to determine the range of option characteristics where the formula yields an accurate approximation. Then we provide an example of calibration to a sample of S&P500 option price data. We find that jumps are significant. The evidence also supports CEV specification of the volatility of volatility and an affine specification of the jump intensity. These empirical results agree with the literature that tends to reject the affine specification for the volatility of volatility (Jones (2003), Aït-Sahalia and Kimmel (2004)), and advocates introducing jumps in returns (Bakshi et al. (1997), Bates (2000a) e.g.). We obtain these results however with a totally different approach.

The paper is organized as follows. Section 2 recalls the concept of implied volatility, and outlines the model setup. In Section 3 we develop the main theoretical results. In Section 4 we assess the accuracy of the calibration formula. In Section 5 we perform the calibration to S&P500 option prices. Section 6 contains some concluding remarks. Appendices gather proofs and technical details.

2 Black-Scholes implied volatility

Throughout the paper we deal with a two-factor jump-diffusion stochastic volatility model. The joint dynamics of the stock price and its volatility under the pricing measure is given by:

$$
\begin{aligned}
dS_t &= (r-\delta-\mu(\sigma_t))S_t dt + \sigma_t S_t dW_t^{(1)} + S_t dJ_t, \\
d\sigma_t &= a(\sigma_t)dt + b(\sigma_t)\left(\rho dW_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)}\right),
\end{aligned}
\tag{1}
$$

where $W_t^{(1)}$ and $W_t^{(2)}$ are two independent standard Brownian motions, and J_t is an independent Poisson jump process. The risk-free interest rate r, the dividend yield δ, and the correlation ρ are assumed constant. The *expected jump size* $E(\Delta J)$ is constant, but the jump intensity $\lambda_t = \lambda(\sigma_t)$ may depend on the volatility in a deterministic way. Hence, the *expected jump* $\mu(\sigma_t) = \lambda(\sigma_t)E(\Delta J)$ may depend on the volatility as well. All functions are assumed to be differentiable. The time-homogeneous specification (1) is general enough to host most parametric models actually used in practice. We do not allow for jumps in volatility since they do not impact the asymptotics used in this paper (see the proof of Proposition 3).

The Black-Scholes implied volatility (or simply the implied volatility) $IV_t(K,T)$ of, let say, a European call option with *maturity date* $T > t$ and *strike price* $K > 0$ is defined as the value of the volatility parameter in the Black-Scholes formula such that the Black-Scholes price coincides with the actual option price $C_t(K,T)$:

$$
\begin{aligned}
C_t(K,T) &\equiv C^{BS}(X_t, K, \tau, IV_t) \\
&= Ke^{-r\tau}\left[e^{X_t}N(d_1) - N(d_2)\right],
\end{aligned}
\tag{2}
$$

where

$$
d_1 = \frac{X_t}{IV_t\sqrt{\tau}} + \frac{1}{2}IV_t\sqrt{\tau}, \quad d_2 = \frac{X_t}{IV_t\sqrt{\tau}} - \frac{1}{2}IV_t\sqrt{\tau}, \quad N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{s^2}{2}}ds,
$$

while $\tau = T - t$ is the option *time-to-maturity*, $X_t = \log\left(S_t e^{(r-\delta)\tau}/K\right)$ is the option *moneyness*.

To further characterize the implied volatility, recall that the price of a European contingent claim is equal to the risk-neutral expectation of its final payoff discounted at the risk-free rate r. In the case of the call option we have:

$$
C_t(K,T) = e^{-r(T-t)}E_t\left[S_T - K\right]_+ . \tag{3}
$$

Using the definition of the implied volatility and moneyness we can rewrite (3) as:

$$e^{X_t} N\left(d_1(X_t, IV_t, \tau)\right) - N\left(d_2(X_t, IV_t, \tau)\right) = E_t\left[e^{X_T} - 1\right]_+, \tag{4}$$

Under Model (1) the joint dynamics of X_t and σ_t is a time-homogeneous Markov process. Hence, the expectation on the right hand side of (4) is a deterministic function of X_t, τ, and σ_t. As a consequence, the implied volatility can be characterized by:

$$IV_t(K, T) = \mathbf{I}\left(X_t, \tau; \sigma_t\right), \tag{5}$$

where $\mathbf{I}$ is a deterministic function.

It is important to note that the function $\mathbf{I}(X, \tau; \sigma)$ does not depend on the risk-free rate and the dividend yield. Hence we can safely assume $r = \delta = 0$ when working with implied volatilities instead of option prices. This point is easily deduced from (4) taking into account that the dynamics of X_t does not depend neither on r nor on δ. In applications we should however take into account both the interest rate and the dividend yield to convert strike price K into moneyness X.

The function $\mathbf{I}(X, \tau; \sigma)$ is generally not available in closed-form except for few special cases. In the case of the stochastic volatility model (1) without jumps it is known that the *at-the-money* implied volatility converges to the spot volatility as time-to-maturity shrinks to zero (see e.g. Ledoit et al. (2002)):

$$\sigma = \lim_{\tau \to 0} \mathbf{I}(0, \tau; \sigma) = \mathbf{I}(0, 0; \sigma).$$

Taking into account that the implied volatility is known in the limit it is reasonable to consider its Tailor series expansion of $\mathbf{I}(X, \tau; \sigma)$ around $X = \tau = 0$:

$$\mathbf{I}(X, \tau; \sigma) = \sigma + \frac{\partial \mathbf{I}(0, 0; \sigma)}{\partial X} X + \frac{\partial \mathbf{I}(0, 0; \sigma)}{\partial \tau} \tau + \frac{1}{2} \frac{\partial \mathbf{I}^2(0, 0; \sigma)}{\partial X^2} X^2 + \ldots \tag{6}$$

This power series is defined only if the function $\mathbf{I}$ is well-behaved for small X and τ, which is not the case if jumps are present (see more about this in the next section). This difficulty can be overcome by introducing an alternative parameterization of implied volatility. Indeed, let us define the *moneyness degree* θ by scaling the moneyness:

$$\theta \equiv \frac{X}{\sigma\sqrt{\tau}}, \qquad \text{and} \qquad I(\theta, \tau; \sigma) \equiv \mathbf{I}(\sigma\theta\sqrt{\tau}, \tau; \sigma).$$

Then by substituting $X = \sigma\theta\sqrt{\tau}$ in (6) we can rewrite $\mathbf{I}(\sigma\theta\sqrt{\tau}, \tau; \sigma)$ as:

$$\begin{aligned} I(\theta, \tau; \sigma) &= \sigma + \left[\sigma\theta \frac{\partial \mathbf{I}(0,0;\sigma)}{\partial X}\right]\sqrt{\tau} + \left[\frac{\sigma^2\theta^2}{2}\frac{\partial \mathbf{I}^2(0,0;\sigma)}{\partial X^2} + \frac{\partial \mathbf{I}(0,0;\sigma)}{\partial \tau}\right]\tau + ... \\ &= \sigma + I_1(\theta;\sigma)\sqrt{\tau} + I_2(\theta;\sigma)\tau + ... \end{aligned} \tag{7}$$

To interpret Expansion (7) observe first that under the model without jumps the moneyness degree θ is a natural measure of the option moneyness for small τ. Indeed, ignoring terms of order τ and higher, $\log(S_T/K)$ is approximately normally distributed with mean X_t and standard deviation $\sigma_t\sqrt{\tau}$ under the risk-neutral measure. Therefore the moneyness degree θ, being the ratio of the mean to the standard deviation, measures the likelihood of, let say, a call option to be in-the-money at the expiration.

Expansion (7) can be viewed as a short-maturity expansion of the implied volatility with moneyness degree θ being fixed. A measure similar to θ with at-the-money implied volatility or average volatility instead of the spot volatility is widely used in the empirical literature as a measure of option moneyness (see Bates (2000a), Carr and Wu (2003b)).

To conclude this section, we can see that in the absence of jumps the terms I_m, $m = 1, 2, ..$, in Expansion (7) are polynomials in θ. Under the general version of Model (1) with jumps Expansion (7) appears to be also valid; however, the asymptotic terms are no longer polynomials. This will be shown in the next section.

3 Asymptotic expansion of implied volatility

In this section we present the main theoretical result of the paper, namely the asymptotic expansion of implied volatility. We begin with a pure diffusion stochastic volatility model without jumps before turning our attention towards jump-diffusion models.

3.1 Pure diffusion case

The next proposition contains our expansion result for implied volatilities in the pure diffusion case.

Proposition 1 *In Model (1) without jumps assume that for any level of σ the implied volatility function admits a Taylor series representation in some neighborhood of $X = \tau = 0$:*

$$\mathbf{I}(X, \tau; \sigma) = \sum_{n,m=0}^{\infty} \frac{\partial^{(n+m)}}{\partial X^n \partial \tau^m} \mathbf{I}(0, 0; \sigma) X^n \tau^m \ , \tag{8}$$

then the implied volatility function I has the following asymptotics:

$$I(\theta, \tau; \sigma) = \sigma + I_1(\theta; \sigma)\sqrt{\tau} + I_2(\theta; \sigma)\tau + O(\tau\sqrt{\tau}), \tag{9}$$

where I_1 and I_2 are functions of the moneyness degree θ and the spot volatility σ only:

$$I_1(\theta; \sigma) = -\frac{\rho b \theta}{2}, \tag{10}$$

$$\begin{aligned} I_2(\theta; \sigma) \quad &= \quad \left(-\frac{5}{12} \frac{\rho^2 b^2}{\sigma} + \frac{1}{6} \frac{b^2}{\sigma} + \frac{1}{6} \rho^2 b b' \right) \theta^2 \\ &\quad + \frac{a}{2} + \frac{\rho b \sigma}{4} + \frac{1}{24} \frac{\rho^2 b^2}{\sigma} + \frac{1}{12} \frac{b^2}{\sigma} - \frac{1}{6} \rho^2 b b', \end{aligned} \tag{11}$$

with $a = a(\sigma)$, $b = b(\sigma)$, and b' denotes the derivative of b w.r.t. σ.

Proof. See Appendix A. ■

In the statement of the proposition we have made the assumption that the implied volatility is "well-behaved" near maturity. In particular, this induces that the implied volatility does not explode as time-to-maturity shrinks to zero. This is typically assumed in the literature dealing with diffusion type models of option prices; Schönbucher (1999) refers to it as the no-bubble constraint; Brace et al. (2001), and Brace et al. (2002) call it the feedback condition.

Proposition 1 states that the asymptotics are such that the implied volatility is equal to the spot volatility plus two correction factors whose forms are explicit functions of the moneyness degree θ and the spot volatility σ. If we limit ourselves to the first order approximation we can see that a non-zero volatility of volatility b induces a linear structure in the moneyness degree θ (see (10)). This structure is independent of the choice of the risk-neutral measure since the volatility drift a does not turn up in I_1. This is quite intuitive. Indeed, if time-to-maturity is small enough then the volatility "does not have time to change much". Hence, the volatility risk cannot have a first order effect on the option price.

Proposition 1 delivers asymptotics of implied volatilities, which are nothing else but prices of options quoted on the volatility scale. The next proposition shows how asymptotics of option prices themselves can be obtained from the asymptotics of implied volatilities.

Proposition 2 *Let us assume that we can write the implied volatility function I as:*

$$I(\theta, \tau; \sigma) = \sigma + I_1(\theta; \sigma)\sqrt{\tau} + I_2(\theta; \sigma)\tau + I_3(\theta; \sigma)\tau\sqrt{\tau} + O(\tau^2) \tag{12}$$

for some functions I_1, I_2, I_3 of the moneyness degree θ and the spot volatility σ, then the price of the call option has the following asymptotics:

$$C(\theta, \sigma, \tau) = K\left[\Lambda_1(\theta; \sigma)\sqrt{\tau} + \Lambda_2(\theta; \sigma)\tau + \Lambda_3(\theta; \sigma)\tau\sqrt{\tau} + O(\tau^2)\right], \tag{13}$$

where the explicit relationships between Λ_1, Λ_2, Λ_3, and I_1, I_2 are given at the end of Appendix B.

Proof. See Appendix B. ■

Observe that although the Λ's do not depend on I_3, assuming that the expansion of I has the form (12) is necessary. Indeed, if we replace the sum of the last expansion term $I_3(\theta; \sigma)\tau\sqrt{\tau}$ and the error $O(\tau^2)$ with $O(\tau\sqrt{\tau})$ in (12) then we obtain a weaker assumption not strictly sufficient to justify (13).

3.2 Jump-diffusion case

In this section we derive the expansion for the jump-diffusion stochastic volatility model (1).

3.2.1 Constant intensity

Let us first assume that the jump intensity is constant. In that framework it is easier to first characterize the asymptotics of option prices (see the proof of the next proposition for further explanations) before characterizing the asymptotics of implied volatilities.

Proposition 3 *Assume Model (1) with constant jump intensity. If the implied volatility under the pure diffusion model ($\mu \equiv 0$) satisfies (8), then the price of the call option has the following asymptotics:*

$$C(\theta, \tau; \sigma) = K \left[\Gamma_1(\theta; \sigma)\sqrt{\tau} + \Gamma_2(\theta; \sigma)\tau + \Gamma_3(\theta; \sigma)\tau\sqrt{\tau} + O(\tau^2) \right], \tag{14}$$

where Γ_1, Γ_2, Γ_3 are explicit functions defined at the end of Appendix C.

Proof. See Appendix C. ■

Now let us compare Equation (14) with that of Proposition 2. By equalizing corresponding asymptotic terms we may directly obtain the asymptotics of implied volatilities. The result is summarized in Proposition 4.

Proposition 4 *Assume Model (1) with constant jump intensity. If the implied volatility under the pure diffusion model ($\mu \equiv 0$) satisfies (8), then the implied volatility under the jump-diffusion model has the following asymptotics:*

$$I(\theta, \tau; \sigma) = \sigma + I_1(\theta; \sigma)\sqrt{\tau} + I_2(\theta; \sigma)\tau + O(\tau\sqrt{\tau}), \tag{15}$$

where

$$I_1 = -\frac{b\rho}{2}\theta - \mu g + \eta h, \tag{16}$$

$$\begin{aligned} I_2 \;=\; & -\frac{\mu^2}{2\sigma}\theta^2 g^2 - \frac{\eta^2}{2\sigma}\theta^2 h^2 + \frac{\mu\eta}{\sigma}\theta^2 gh \\ & + \left[-\frac{\mu b\rho}{2\sigma}\theta^3 - \frac{\mu\sigma}{2}\theta - \sigma\lambda\theta \right] g \\ & + \left[\frac{\eta b\rho}{2\sigma}\theta^3 + \frac{\eta\sigma}{2}\theta + \sigma\chi\theta \right] h + P(\theta; \sigma), \end{aligned} \tag{17}$$

and P is a quadratic function in θ:

$$\begin{aligned} P(\theta;\sigma) &= \left(-\frac{5}{12}\frac{\rho^2 b^2}{\sigma}+\frac{1}{6}\frac{b^2}{\sigma}+\frac{1}{6}\rho^2 bb' - \frac{1}{2}\frac{\mu b\rho}{\sigma}\right)\theta^2 + \\ &\quad +\frac{a}{2}+\frac{\rho b\sigma}{4}+\frac{\rho b\mu}{2\sigma}+\frac{1}{24}\frac{\rho^2 b^2}{\sigma}+\frac{1}{12}\frac{b^2}{\sigma} \\ &\quad -\frac{1}{6}\rho^2 bb' + \frac{\mu^2}{2\sigma}-\frac{\sigma\mu}{2}-\lambda\sigma, \end{aligned}$$

with $g = N(\theta)/n(\theta)$, $h = 1/n(\theta)$, $\eta = \lambda E\Delta J_+$, $\chi = \lambda \Pr(\Delta J > 0)$, and where $n = n(\theta)$, $N = N(\theta)$ are pdf and cdf of the standard normal distribution.

Let us observe that in the limit as $\tau \to 0$ and θ being fixed the implied volatility converges to spot volatility σ. The fact that the jump component does not contribute to short-term implied volatilities might look surprising. Indeed, for small τ the variance of returns is approximately given by $(\sigma^2 + \lambda E\Delta J^2)\,\tau$, where both components have an impact of the same order. Nevertheless the contributions of these two components are totally different if we consider implied volatilities. To illustrate this let us compare the pure jump model with the pure diffusion model. In the pure Poisson jump model the at-the-money call option price has the leading term $(\lambda S E\Delta J_+)\,\tau$ (cf. Formula (3)), whereas in the pure diffusion model the leading term is given by $\left(\sigma S/\sqrt{2\pi}\right)\sqrt{\tau}$ (cf. Black-Scholes formula). This means that the implied volatility under the pure Poisson jump model converges to zero with the leading term $\left(\lambda\sqrt{2\pi}E\Delta J_+\right)\sqrt{\tau}$ contrary to the pure diffusion case where the implied volatility is equal to the spot volatility in the limit.

General results on the short-term behavior of the option prices under different process specifications can be found in Carr and Wu (2003a). In the same spirit they investigate the short-term behavior of at-the money and out-of-the-money option prices suggesting a means of testing the presence of jumps. Their approach does not allow detecting the presence of jumps using at-the-money option prices (except for the pure jump case). Proposition 4 suggests that it can be done using at-the-money implied volatilities instead. Indeed, in the pure diffusion model at-the-money implied volatilities converge to σ at the rate of τ. On the contrary, in the jump-diffusion model the rate of convergence is $\sqrt{\tau}$.

3.2.2 General case

In this section we extend the results obtained in the previous section to the general case of Model (1) when the intensity and the jump size distribution depend on the spot volatility.

Proposition 5 *In Model (1) if the implied volatility under the pure diffusion model ($\mu \equiv 0$) satisfies (8), then the implied volatility under the jump-diffusion model has the following asymptotics:*

$$I(\theta, \tau; \sigma) = \sigma + I_1(\theta; \sigma)\sqrt{\tau} + \left(I_2(\theta; \sigma) - \frac{1}{2}\rho b \mu' \right) \tau + O(\tau\sqrt{\tau}), \tag{18}$$

where I_1 and I_2 are the same as in Proposition 3, and μ' stands for the derivative of μ w.r.t. σ.

Proof. see Appendix D. ∎

Proposition 5 shows that the dependence of the parameters of the jump process on spot volatility manifests itself only via the derivative μ' of the expected jump μ. As it might have been expected, it has only a second order effect as it was true for the volatility of volatility b.

3.3 Calibration of the implied volatility smile

An important application of Approximation (18) is the calibration of jump-diffusion stochastic volatility models. Furthermore since the formula is valid for all jump-diffusion models of the type (1) it can be used to test different parametric specifications on market data. This application, however, may be limited for the following reasons. Expression (18) for the implied volatility involves the unobserved spot volatility σ which, by the nature of the model, is changing over time. As a result the formula needs to be calibrated day by day. Such a calibration obviously requires a reasonably large number of short-term options quoted daily. In practice this is not always the case. As an example, S&P500 options are issued only once in a month. To overcome this difficulty we propose a formula that is free of latent variables by replacing the unobserved spot volatility in (18) with an observed implied volatility. This will yield a feasible and simple calibration procedure of arbitrary parametric specifications to option data pooled across different calendar dates.

Let us assume that we observe the at-the-money implied volatility $\widehat{\sigma} = I(0, \tau; \sigma)$ for some τ. Consider the modified moneyness degree $\widehat{\theta}$ obtained by the substitution of the unobserved σ with the observed implied volatility $\widehat{\sigma}$:

$$\widehat{\theta} \equiv \frac{X}{\widehat{\sigma}\sqrt{\tau}}. \tag{19}$$

It is now possible to express implied volatility for the same τ as a function of $\widehat{\theta}$ and $\widehat{\sigma}$. Indeed, from (18) the asymptotic expansion of $\widehat{\sigma} = I(0, \tau; \sigma)$ is:

$$\widehat{\sigma} = \sigma + I_1(0; \sigma)\sqrt{\tau} + \left(I_2(0; \sigma) - \frac{1}{2}\rho b(\sigma)\mu'(\sigma) \right) \tau + O(\tau\sqrt{\tau}). \tag{20}$$

For small τ this function can be inverted to express the spot volatility $\sigma = \sigma\ (\tau, \widehat{\sigma})$. To derive its asymptotics we first write it in a general form:

$$\sigma(\tau, \widehat{\sigma}) = \widehat{\sigma} + \Sigma_1(\widehat{\sigma})\sqrt{\tau} + \Sigma_2(\widehat{\sigma})\tau + O(\tau\sqrt{\tau}). \tag{21}$$

Then we substitute (21) in (20) to obtain the identity:

$$\begin{aligned} \widehat{\sigma} &= \widehat{\sigma} + [I_1(0; \widehat{\sigma}) + \Sigma_1]\sqrt{\tau} \\ &+ \left[\frac{\partial I_1(0; \widehat{\sigma})}{\partial \sigma} \Sigma_1 + I_2(0; \widehat{\sigma}) - \frac{1}{2}\rho b(\widehat{\sigma})\mu'(\widehat{\sigma}) + \Sigma_2 \right] \tau + O(\tau\sqrt{\tau}). \end{aligned}$$

Solving for Σ_1 and Σ_2 we obtain:

$$\begin{aligned} \sigma(\tau, \widehat{\sigma}) &= \widehat{\sigma} - I_1(0; \widehat{\sigma})\sqrt{\tau} \\ &+ \left[I_1(0; \widehat{\sigma}) \frac{\partial I_1(0; \widehat{\sigma})}{\partial \sigma} - I_2(0; \widehat{\sigma}) + \frac{1}{2}\rho b(\widehat{\sigma})\mu'(\widehat{\sigma}) \right] \tau + O(\tau\sqrt{\tau}). \end{aligned} \tag{22}$$

Finally let us replace θ with $\widehat{\sigma}\widehat{\theta}/\sigma$, and σ with (22) in (18) to eliminate σ from the pricing formula.

Proposition 6 *Let $\widehat{\sigma}$ be the at-the-money implied volatility corresponding to time-to-maturity τ, and let $\sigma(\tau, \widehat{\sigma})$ be the inverse function of $\widehat{\sigma} = I(0, \tau; \sigma)$. Then under the assumptions made* *in Proposition 5 the modified implied volatility function:*

$$\widehat{I}(\widehat{\theta},\tau;\widehat{\sigma}) \equiv I\left(\frac{\widehat{\sigma}\widehat{\theta}}{\sigma(\tau,\widehat{\sigma})},\tau;\sigma(\tau,\widehat{\sigma})\right).$$

has the following asymptotics:

$$\begin{aligned}\widehat{I}(\widehat{\theta},\tau;\widehat{\sigma}) &= \widehat{\sigma} + \left[I_1(\widehat{\theta};\widehat{\sigma}) - I_1(0;\widehat{\sigma})\right]\sqrt{\tau} \\ &+ \left[I_1(0;\widehat{\sigma})\left(\frac{\partial I_1(0;\widehat{\sigma})}{\partial\sigma} - \frac{\partial I_1(\widehat{\theta};\widehat{\sigma})}{\partial\sigma} + \frac{\widehat{\theta}}{\widehat{\sigma}}\frac{\partial I_1(\widehat{\theta};\widehat{\sigma})}{\partial\theta}\right)\right. \\ &\left. + I_2(\widehat{\theta};\widehat{\sigma}) - I_2(0;\widehat{\sigma})\right]\tau + O(\tau\sqrt{\tau}). \qquad (23)\end{aligned}$$

where I_1 and I_2 are defined in Proposition 4.

An important remark is in order. Proposition 6 could be easily written in a more general form by letting any implied volatility to serve as an input into the approximation formula and not only the at-the-money one. This would however complicate the expression without adding any practical value to the result. Indeed, in available datasets we always observe prices of options sufficiently close-to-the-money, which can be reasonably considered as being at-the-money.

Formula (23) suggests a means to compute the implied volatility smile using at-the-money implied volatility as an input. The expression avoids the presence of a latent variable. As a consequence, it can be used to calibrate any parametric specification of Model (1) to a set of option prices across calendar dates simultaneously and not day by day. This means that we are able to fit the implied volatility smile and its dynamics in a single step. Of course, this does not come without a sacrifice. Now the volatility drift does not appear in (23) and cannot be inferred from data. However, this seems to be a justified loss since, in any case, the volatility drift cannot be fully identified on the set of short maturity options where our approximation works.

Approximation (23) is likely to be accurate only for near-the-money options. Therefore we can safely use quadratic expansions of the asymptotic terms in (23) around $\theta = 0$ to avoid the possibility of large numerical errors due to highly nonlinear functions $g(\theta)$ and $h(\theta)$. Zhang and Xiang (2006) take a similar approach by considering the quadratic approximation (not expansion) of implied volatilities. They establish an approximate relationship between implied volatilities and the risk-neutral density function, but this does not allow to recover a state variable model specification from option prices. Another important difference is that the method of Zhang and Xiang (2006) cannot be used to capture the information contained in the dynamics of the implied volatility smile since it cannot free itself from a calibration day by day.

Proposition 7 summarizes the final result, which is most relevant for the calibration procedure. For convenience, the approximation formula is now written directly in terms of X rather than θ.

Proposition 7 *Under the assumptions made in Proposition 5:*

$$\begin{aligned} IV \simeq\ & \widehat{\sigma} - \left[\frac{b\rho}{2\widehat{\sigma}} + \frac{\mu}{\widehat{\sigma}}\right] X + \left[\frac{\pi}{4}(-\mu + 2\eta)\left(-\frac{\mu}{\widehat{\sigma}^3} + \frac{2\eta}{\widehat{\sigma}^3} + \frac{\mu'}{\widehat{\sigma}^2} - \frac{2\eta'}{\widehat{\sigma}^2}\right) - \left(\frac{\mu}{2\widehat{\sigma}} + \frac{\lambda}{\widehat{\sigma}}\right)\right. \\ & \left. + \left(-\frac{5\rho^2 b^2}{12\widehat{\sigma}^3} + \frac{b^2}{6\widehat{\sigma}^3} + \frac{\rho^2 b b'}{6\widehat{\sigma}^2} - \frac{\mu b \rho}{2\widehat{\sigma}^3}\right)\right] X^2 \\ & + \frac{\sqrt{2\pi}}{2}\left[(-\mu + 2\eta)\left(\frac{b'\rho}{2\widehat{\sigma}} + \frac{\mu'}{\widehat{\sigma}} - \frac{b\rho}{2\widehat{\sigma}^2} - \frac{\mu}{\widehat{\sigma}^2} + \frac{1}{2}\right) - \lambda + 2\chi\right] X\sqrt{\tau} \\ & + \frac{\sqrt{2\pi}}{4}\frac{-\mu + 2\eta}{\widehat{\sigma}^2\sqrt{\tau}} X^2, \end{aligned} \tag{24}$$

where $\widehat{\sigma}$ *is at-the-money implied volatility corresponding to time-to-maturity* τ, $\eta = \eta(\widehat{\sigma})$, $\chi = \chi(\widehat{\sigma})$, $b = b(\widehat{\sigma})$, $\mu = \mu(\widehat{\sigma})$, $\lambda = \lambda(\widehat{\sigma})$.

Let us conclude this section by making several observations. Formula (24) is a quadratic approximation of the implied volatility smile with a correction for time-to-maturity (the last two lines in (24)). The structure of the approximation formula suggests that its accuracy should strongly depend on the extent to which the shape of the implied volatility smile is dependent on time-to-maturity. Note also that the correction term appears only when there are jumps in returns. This fact does not mean that the shape of implied volatility is independent of time-to-maturity in the absence of jumps. Indeed, it depends on τ indirectly via the at-the-money implied volatility $\widehat{\sigma}$. Finally observe that $-\mu + 2\eta = \lambda E\,|\Delta J| > 0$ meaning that the curvature of the implied volatility smile diminishes as time-to-maturity increases. This is consistent with empirical facts.

4 Accuracy of the calibration formula

To determine the domain of application of our approximation we will study its performance in pricing options based on realistic model parameters. For the first numerical exercise we consider an affine jump-diffusion model with parameter values borrowed from Pan (2002). Model (1) becomes using $v_t = \sigma_t^2$:

$$\begin{aligned} dS_t &= (r - \delta - \mu(\sqrt{v_t}))S_t dt + \sqrt{v_t} S_t dW_t^{(1)} + S_t dJ_t, \\ dv_t &= \kappa(\overline{v} - v_t)dt + \sigma_v \sqrt{v_t} \left(\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)} \right), \end{aligned}$$

where $\lambda(v_t) = \lambda_0 + \lambda_1 v_t$.

Assuming a normal distribution of the jump size with mean μ_J and standard deviation σ_J Pan (2002) obtains the following parameter estimates: $\lambda_0 = 0$, $\lambda_1 = 12.3$, $\kappa = 6.4$, $\overline{v} = .015$, $\sigma_v = .3$, $\mu_J = -.21$, $\sigma_J = .04$, $\rho = -0.53$.

We have used both Approximations (23) and (24). As expected the quadratic approximation is not significantly different from (23) whenever the latter is accurate. Moreover it performs much better in other cases where (23) produces large numerical errors due to exponential terms. Taking this into account we have decided to continue with using Formula (24) both in the numerical experiments and in the empirical calibration[2].

Figure 1 plots the implied volatilities and their approximations based on (24). Three levels of the spot volatility are considered: 8% (low), 16%(medium) and 24%(high) on an annual basis, which approximately correspond to daily levels of 0.5%, 1% and 1.5%. In accordance with the typical definition of short-term options (Bakshi et. al. (1997), Chernov and Ghysels (2000), Pan (2002) e.g.) we consider options with a time-to-maturity of maximum 60 days. We also exclude very short-term options since, in practice, their prices are very noisy. For the same reason empirical studies also tend to exclude these options from the analysis since their unreliability might affect the results significantly (Jones (2003), Pan (2002)). Here we follow Pan (2002) by considering only options expiring at least after 15 days.

Let us observe from Figure 1 that for moneyness $X \in [-0.05, 0.05]$ the approximation is reasonably accurate for different levels of volatility and time-to-maturity. Figure 1 also suggests that the accuracy slightly diminishes at low levels of volatility. To explain this

[2] We would like to thank the anonymous referee for suggesting us to use the quadratic approximation.

fact recall the discussion at the end of the previous section. The lower the volatility the stronger the dependence of the shape of implied volatility on time-to-maturity. As a result, our approximation becomes less accurate with lower levels of volatility.

To make the numerical analysis more complete let us perform another exercise using a model with a CEV-type specification. For this purpose we select the CEV model without jumps estimated by Aït-Sahalia and Kimmel (2004):

$$\begin{aligned} S_t &= (r-\delta)S_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \\ dv_t &= \kappa(\overline{v} - v_t)dt + \sigma_v v_t^{\phi} \left(\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)}\right), \end{aligned}$$

with $\kappa = 3$, $\overline{v} = .052$, $\sigma_v = 1.37$, $\phi = .85, \rho = -.77$.

To compute option prices we use a Monte-Carlo simulation engine employing two variance reduction techniques: antithetic variables and conditional simulation (see Jones (2003) for details). Figure 2 plots the implied volatilities and their approximations for the same different levels of volatility and time-to-maturity. We observe that the picture is largely the same, and that the remark on the effect of low levels of volatility remains valid.

To conclude the section let us summarize our findings. Approximation (24) seems to be reasonably accurate for near-the-money options ($|X| \leq 0.05$) with short time-to-maturity (maximum 60 days to expiration). Therefore the formula can be safely calibrated to a set of option prices satisfying these restrictions pooled across different calendar dates to extract the information contained in the dynamics of the implied volatility smile.

5 Empirical calibration to S&P500 option prices

In this section we take the Approximation (24) to S&P500 option price data. We use data on implied volatilities of S&P500 options constructed in Aït-Sahalia and Lo (1998) and covering a period of one year. This database contains implied volatilities and forward prices so that moneyness X can be easily computed. The formula is calibrated to implied volatilities by minimizing the sum of squared errors (nonlinear least squares). Implied volatilities are calibrated by means of (24) with $\widehat{\sigma}$ corresponding to the implied volatility of the option closest-to-the-money.

Following the analysis of the accuracy of our approximation we deal only with options with time-to-maturity between 15 and 60 days and strike price within 5% of the forward price. Recall that our approximation becomes less accurate when the spot volatility is too low. To make sure that the analysis is reliable we drop the observations of implied volatility smile with the closest-to-the-money implied volatility $\widehat{\sigma} < 8\%$. The sample of option prices satisfying all the restrictions is fairly large consisting of 2537 observations. Note that here we do not count those observations that are used as inputs in the approximation formula.

The dataset represents a collection of implied volatility smiles observed at different calendar dates and time-to-maturity. In all the cases the moneyness of the closest-to-the-money option does not exceed 0.006 in absolute value, which seems to be sufficiently close to zero. On average, we have 9 points per implied volatility smile with moneyness X almost uniformly distributed in the range $[-0.05, 0.05]$. Pooling data across calendar dates allows observations of implied volatility smiles with near time-to-maturity and near at-the-money implied volatility complement each other (see Figure 3c).

We calibrate the following general CEV-type model:

$$\begin{aligned} dS_t &= (r - \delta - \mu(\sigma_t)) S_t dt + \sigma_t S_t dW_t^{(1)} + S_t dJ_t, \\ d\sigma_t &= (...)dt + \beta \sigma_t^{\phi} \left(\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)} \right), \end{aligned} \tag{25}$$

with the expected jump being a power function of the volatility $\mu(\sigma) \equiv \lambda(\sigma) E(\Delta J) = \lambda_0 E(\Delta J) \sigma^{\psi}$. Here the parameter ψ measures the elasticity of the jump intensity λ with respect to σ. For the sake of parsimony we assume $\eta = \chi = 0$ meaning that the risk neutral probability of a positive jump is negligible. This assumption is reasonable for the model parameters considered in the previous section and encountered in the available empirical literature. Note also that positive jumps under the risk-neutral measure are typically much less likely than under the historical measure due to a significant risk premium that investors demand for the risk of a market crash.

Six model parameters are calibrated: $\beta\rho, \phi, \lambda_0 E(\Delta J), \psi, \rho, E(\Delta J)$. The first four parameters determine the leading term I_1 in Expansion (18); so we expect them to be inferred accurately. Table 1 reports the estimates of the six parameters for the CEV-type model (25) and three particular cases: the affine specification ($\phi = 0, \psi = 2$), the CEV-type model without jumps ($\lambda_0 = 0$), and the Heston model ($\lambda_0 = 0$, $\phi = 0$). The calibration results are largely consistent with those obtained in the empirical literature. Indeed, the volatility of variance in the Heston model $\sigma_v = 2\beta \simeq .37$ (see the first column in Table 1) is close to .36 obtained in Jones (2003) and .43 found in Aït-Sahalia and Kimmel (2004)). The estimate of the expected jump size $-.14$ in the affine model agrees with the value obtained in Pan (2002) ($-.19$). Observe that the estimate of the correlation ρ in the affine model reaches the lower boundary -1 of the constrained parameter space $[-1, 1]$, and that usual inference on that parameter does not apply.

Based on the standard errors given in parenthesis we conclude that jumps in index return are significant ($\lambda_0 E(\Delta J) < 0$). This fact has long been firmly established in the empirical literature using parametric models (see Bakshi et al. (1997), Bates (2000a) e.g.) and model free approaches (Carr and Wu (2003a)). The estimate of the elasticity of the jump intensity with respect to volatility suggests that the affine specification ($\psi = 2$) is statistically rejected. However we cannot reject the hypothesis that the actual specification is close to the affine one.

On the contrary, the affine specification for the volatility of volatility is firmly rejected ($\phi >> 0$). This result remains valid also if we assume a pure diffusion model, where an affine specification of the volatility of volatility corresponds to the Heston model. This result is consistent with the recent literature which tends to reject the Heston model when testing against a CEV-type one (Jones (2003), Aït-Sahalia and Kimmel (2004)). We contribute to this literature by observing that the affine specification of the volatility of volatility is also rejected in the model with jumps in returns.

Let us examine the effect of the model restrictions on the pricing errors. For this purpose we compute the goodness-of-fit criterion in terms of relative errors denoted by $\sqrt{\text{ASE}}$. The $\sqrt{\text{ASE}}$ is defined as the square root of the Average of Squared Errors in implied volatilities. The magnitude of the fitted errors indicates that the affine restrictions do not impact the accuracy of the model fit dramatically. Indeed, the average error in implied volatility increases only by 0.02% from 0.35% to 0.37% for jump-diffusion models and by 0.03% from 0.43% to 0.46% for pure diffusion models. On the other hand, the introduction of jumps allows reducing the error by 0.09% for affine models and 0.08% for CEV models.

We conclude this section by verifying the robustness of our estimates. Recall that in the calibration we use implied volatilities with moneyness restricted to $-0.05 \leq X \leq 0.05$ to guarantee that there is no impact of the approximation error. To make sure that our results are robust we compare our estimates with those obtained by imposing stricter restrictions on the moneyness. In Figure 5 we plot the inferred shapes of expected jump $\mu(\sigma)$ and the volatility of volatility $b(\sigma)\rho$ based on three samples of option data corresponding to moneyness restrictions $-0.05 \leq X \leq 0.05$, $-0.04 \leq X \leq 0.04$, and $-0.03 \leq X \leq 0.03$ of sizes 2537, 2012, and 1439, respectively. We can see that results are not much different across the three samples. This provides indeed good evidence of the robustness of our results since the number of observations in the smallest sample is less then half the number of observations initially used in the model calibration.

6 Concluding remarks

In this paper we have derived an asymptotic formula for implied volatilities of European options under a two-factor jump-diffusion stochastic volatility model. The asymptotics correspond to the short-term behavior of implied volatilities under parameterization widely used in practice. The calibration formula appears to be simple and useful for practical purposes. It allows testing parametric model specifications using option data pooled across calendar dates, moneyness and time-to-maturity. An interesting empirical result is the rejection of an affine specification of the volatility of volatility in favor of a CEV-type specification even if jumps are taken into account. The other interesting empirical result is that an affine specification of the jump intensity seems to be adequate.

Possible areas of future applications include the use of the asymptotic formula in "speedy" pricing of very short-term options typically traded in foreign exchange markets. We also believe that the asymptotic analysis can be further extended to American options where analytical approximations of this type are not yet available. This is on our research agenda.

Appendix A. Proof of Proposition 1.

The steps of the proof are as follows. First we start with deriving a PDE for the implied volatility and then identify terms I_0, I_1, I_2 in the generic asymptotic representation:

$$I(\theta,\tau;\sigma) = I_0(\theta;\sigma) + I_1(\theta;\sigma)\sqrt{\tau} + I_2(\theta;\sigma)\tau + O(\tau\sqrt{\tau}). \tag{26}$$

The terms I_1 and I_2 will be characterized through two second order ODEs, whose solutions are taken in the class of polynomials. The call option price under Model (1) is determined by the strike price K, maturity date T, two stochastic factors S and σ, and time t. It is convenient to pass from T to time-to-maturity $\tau = T - t$, and from S to moneyness $X_t = \log\left(S_t e^{(r-\delta)\tau}/K\right)$. Let us denote the call option price by $C(X, K, \tau, \sigma)$. Its associated PDE under Model (1) with no jumps can be written as:

$$-C_\tau + \frac{1}{2}\sigma^2\left(C_{XX} - C_X\right) + a(\sigma)C_\sigma + \frac{1}{2}b^2(\sigma)C_{\sigma\sigma} + \sigma b(\sigma)\rho C_{\sigma X} = 0, \tag{27}$$

where subscripts refer to differentiation.

By the definition of implied volatility, we have $C = C^{BS}(X, K, \tau, \mathbf{I}(X, \tau, \sigma_t))$, where C^{BS} denotes the Black-Scholes formula (see (2)). After plugging this expression into Equation (27) the PDE for the implied volatility can be obtained after some simplifications. We skip these technical details since such a derivation is in Ledoit et al. (2002). Our model (without jumps in the price) is a particular case of their setting, and the PDE of the implied volatility is here:

$$\begin{aligned}
&\frac{\rho b\sigma}{\mathbf{I}\sqrt{\tau}}d_2\mathbf{I}_\sigma - \frac{1}{2}\frac{b^2}{\mathbf{I}}d_1 d_2\left(\mathbf{I}_\sigma\right)^2 - b\rho\sigma\mathbf{I}_{X\sigma} + \frac{\mathbf{I}^2 - \sigma^2}{2I\tau} + \mathbf{I}_\tau \\
&+\frac{\sigma^2}{\mathbf{I}\sqrt{\tau}}d_2\mathbf{I}_X - \frac{\rho b\sigma}{\mathbf{I}}d_1 d_2\mathbf{I}_X\mathbf{I}_\sigma - \frac{1}{2}\frac{\sigma^2}{\mathbf{I}}d_1 d_2\left(\mathbf{I}_X\right)^2 \\
&-\frac{1}{2}\sigma^2\left(\mathbf{I}_{XX} - \mathbf{I}_X\right) - a\mathbf{I}_\sigma - \frac{1}{2}b^2\mathbf{I}_{\sigma\sigma} = 0,
\end{aligned} \tag{28}$$

where $d_1 = \dfrac{X}{\mathbf{I}\sqrt{\tau}} + \dfrac{1}{2}\mathbf{I}\sqrt{\tau}$, and $d_2 = \dfrac{X}{\mathbf{I}\sqrt{\tau}} - \dfrac{1}{2}\mathbf{I}\sqrt{\tau}$.
Ledoit et al. (2002) also show that $\mathbf{I}(0, 0;\sigma) = I_0(0;\sigma) = \sigma$.

Let us now pass from X to $\theta = X/(\sigma\sqrt{\tau})$ and from $\mathbf{I}(X, \tau;\sigma)$ to $I(\theta, \tau;\sigma)$. We have to make the following replacements for the function $\mathbf{I}$ in (28):

$$
\begin{aligned}
\mathbf{I} &\rightarrow \sigma + I_1\sqrt{\tau} + I_2\tau + O\left(\tau\sqrt{\tau}\right), \\
\mathbf{I}^2 &\rightarrow \sigma^2 + 2\sigma I_1\sqrt{\tau} + (I_1^2 + 2\sigma I_2)\tau + O\left(\tau\sqrt{\tau}\right), \\
\mathbf{I}^3 &\rightarrow \sigma^3 + 3\sigma^2 I_1\sqrt{\tau} + O\left(\tau\right),
\end{aligned}
$$

and its derivatives:

$$
\begin{aligned}
\mathbf{I}_X &\rightarrow \frac{1}{\sigma}I_{1\theta} + \frac{1}{\sigma}I_{2\theta}\sqrt{\tau} + O\left(\tau\right), \\
\mathbf{I}_{XX} &\rightarrow \frac{1}{\sigma^2}I_{1\theta\theta}\frac{1}{\sqrt{\tau}} + \frac{1}{\sigma^2}I_{2\theta\theta} + O\left(\sqrt{\tau}\right), \\
\mathbf{I}_\sigma &\rightarrow 1 + \left(I_{1\sigma} - \frac{1}{\sigma}\theta I_{1\theta}\right)\sqrt{\tau} + O\left(\sqrt{\tau}\right), \\
\mathbf{I}_{\sigma\sigma} &\rightarrow 0 + O\left(\sqrt{\tau}\right) \\
\mathbf{I}_{X\sigma} &\rightarrow \frac{1}{\sigma}I_{1\sigma\theta} - \frac{1}{\sigma^2}I_{1\theta} - \frac{1}{\sigma^2}\theta I_{1\theta\theta} + O\left(\sqrt{\tau}\right), \\
\mathbf{I}_\tau &\rightarrow \left(\frac{1}{2}I_1 - \frac{1}{2}\theta I_{1\theta}\right)\frac{1}{\sqrt{\tau}} + I_2 - \frac{1}{2}\theta I_{2\theta} + O\left(\sqrt{\tau}\right),
\end{aligned}
$$

also setting $d_1 = \dfrac{\theta\sigma}{I} + \dfrac{1}{2}I\sqrt{\tau}$, and $d_2 = \dfrac{\theta\sigma}{I} - \dfrac{1}{2}I\sqrt{\tau}$.

Using these expressions and (28) we get after some algebra:

$$
A\frac{1}{\sqrt{\tau}} + B + O(\sqrt{\tau}) = 0,
$$

where $\quad A = \rho b\theta + \dfrac{3}{2}I_1 + \dfrac{1}{2}\theta I_{1\theta} - \dfrac{1}{2}I_{1\theta\theta},$

$$
\begin{aligned}
B &= \rho b\theta I_{1\sigma} - \frac{\rho b\theta^2}{\sigma}I_{1\theta} - \frac{2\rho b\theta}{\sigma}I_1 - \frac{\rho b\sigma}{2} - \frac{b^2\theta^2}{2\sigma} - \rho b I_{1\sigma\vartheta} + \frac{\rho b}{\sigma}I_{1\theta} \\
&\quad + \frac{\rho b\theta}{\sigma}I_{1\theta\theta} - \frac{I_1^2}{2\sigma} - \frac{2\theta I_1}{\sigma}I_{1\theta} - \frac{\rho b\theta^2}{\sigma}I_{1\theta} - \frac{\theta^2}{2\sigma}\left(I_{1\theta}\right)^2 - a \\
&\quad + 2I_2 + \frac{1}{2}\theta I_{2\theta} - \frac{1}{2}I_{2\theta\theta}.
\end{aligned} \tag{29}
$$

After putting A equal to zero, we arrive at the following ODE for I_1:

$$-\frac{3}{2}I_1 - \frac{1}{2}\theta I_{1\theta} + \frac{1}{2}I_{1\theta\theta} = \rho b\theta.$$

Taking into account the assumption made in the proposition and (7), we select the linear solution:

$$I_1(\theta,\sigma) = -\frac{\rho b\theta}{2}. \tag{30}$$

Now setting the second asymptotic term B equal to zero and taking into account (30), we get the ODE for I_2:

$$\begin{aligned} -2I_2 - \frac{1}{2}\theta I_{2\theta} + \frac{1}{2}I_{2\theta\theta} &= \left(\frac{5}{4}\frac{\rho^2 b^2}{\sigma} - \frac{1}{2}\frac{b^2}{\sigma} - \frac{1}{2}\rho^2 bb'\right)\theta^2 \\ &\quad -a - \frac{\rho b\sigma}{2} - \frac{\rho^2 b^2}{2\sigma} + \frac{1}{2}\rho^2 bb', \end{aligned}$$

where b' denotes the derivative of b w.r.t. σ. Using again (7), we select the quadratic solution to this ODE:

$$\begin{aligned} I_2 &= \left(-\frac{5}{12}\frac{\rho^2 b^2}{\sigma} + \frac{1}{6}\frac{b^2}{\sigma} + \frac{1}{6}\rho^2 bb'\right)\theta^2 \\ &\quad + \frac{a}{2} + \frac{\rho b\sigma}{4} + \frac{1}{24}\frac{\rho^2 b^2}{\sigma} + \frac{1}{12}\frac{b^2}{\sigma} - \frac{1}{6}\rho^2 bb'. \end{aligned}$$

Appendix B. Proof of Proposition 2

Let us use $N = N(z)$ and $n = n(z)$ to abbreviate the Gaussian cdf and pdf evaluated at z. The proof of this proposition uses the definition of implied volatility underlying Expression (2), which can be written as:

$$C \equiv Ke^{-r\tau}(e^{\theta\sigma\sqrt{\tau}}N_1 - N_2), \tag{31}$$

with $N_1 = N\left(\frac{\sigma\theta}{I} + \frac{1}{2}I\sqrt{\tau}\right)$, $N_2 = N\left(\frac{\sigma\theta}{I} - \frac{1}{2}I\sqrt{\tau}\right)$, and I a shorthand for $I(\theta, \tau; \sigma)$.

Given the asymptotic expansion (12), we can write:

$$\begin{aligned} \frac{\sigma\theta}{I} + \frac{I}{2}\sqrt{\tau} &= \theta + \left(-\frac{\theta I_1}{\sigma} + \frac{\sigma}{2}\right)\sqrt{\tau} + \left(-\frac{\theta I_2}{\sigma} + \frac{\theta I_1^2}{\sigma^2} + \frac{I_1}{2}\right)\tau \\ &+ \left(-\frac{\theta I_3}{\sigma} + \frac{2\theta I_1 I_2}{\sigma^2} - \frac{\theta I_1^3}{\sigma^3} + \frac{I_2}{2}\right)\tau\sqrt{\tau} + O(\tau^2), \end{aligned}$$

and

$$\begin{aligned} \frac{\sigma\theta}{I} - \frac{I}{2}\sqrt{\tau} &= \theta + \left(-\frac{\theta I_1}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\tau} + \left(-\frac{\theta I_2}{\sigma} + \frac{\theta I_1^2}{\sigma^2} - \frac{I_1}{2}\right)\tau \\ &+ \left(-\frac{\theta I_3}{\sigma} + \frac{2\theta I_1 I_2}{\sigma^2} - \frac{\theta I_1^3}{\sigma^3} - \frac{I_2}{2}\right)\tau\sqrt{\tau} + O(\tau^2). \end{aligned}$$

Substituting these expressions in (37) and using the Taylor series expansion:

$$N(y + \Delta y) = N(y) + n(y)\Delta y - \frac{1}{2}yn(y)(\Delta y)^2 + \frac{1}{6}(y^2 - 1)n(y)(\Delta y)^3 + O((\Delta y)^4),$$

allow us to compute the expansion of the call option price. After some simple but tedious algebra we finally obtain:

$$CK^{-1} = e^{-r\tau}(e^{\theta\sigma\sqrt{\tau}}N_1 - N_2) = \Lambda_1\sqrt{\tau} + \Lambda_2\tau + \Lambda_3\tau\sqrt{\tau} + O(\tau^2),$$

where

$$\Lambda_1 = \sigma(n + \theta N),$$

$$\Lambda_2 = \frac{\theta^2 N}{2}\sigma^2 + \frac{\theta n}{2}\sigma^2 + nI_1,$$

$$\Lambda_3 = \frac{\theta^3 N}{6}\sigma^3 + \frac{\theta^2 n}{6}\sigma^3 - \frac{n}{24}\sigma^3 + \frac{n\theta^2 I_1^2}{2\sigma} + \frac{\sigma}{2}\theta n I_1 + nI_2 - \sigma r(n + \theta N),$$

and $n = n(\theta)$, $N = N(\theta)$ are the pdf and cdf of the standard normal distribution evaluated at θ.

Appendix C. Proof of Proposition 3

Using the representation of the option price as an expectation of its discounted future payoff, we can write:

$$C_t = e^{-r\tau} E_t \left(S_T - K\right)_+ = e^{-r\tau} \sum_{i=0}^{\infty} \Pr(i \text{ jumps}) E_t \left\{ \left(S_T - K\right)_+ \mid i \text{ jumps} \right\}.$$

From the properties of the Poisson process we have:

$$\begin{aligned} \Pr(\text{no jumps}) &= 1 - \lambda\tau + O(\tau^2), \\ \Pr(\text{one jump}) &= \lambda\tau + O(\tau^2), \\ \Pr(i \text{ jumps}) &= O(\tau^2) \qquad \text{for } i \geq 2. \end{aligned}$$

This means that we may ignore the possibility of multiple jumps during the lifetime of the option since we are looking for an asymptotic expansion of option prices up to $O(\tau\sqrt{\tau})$. Using this we can write:

$$\begin{aligned} C_t &= e^{-r\tau} E_t \left(S_T - K\right)_+ \\ &= \lambda\tau e^{-r\tau} E_t \left\{ \left(S_T - K\right)_+ |\text{jump} \right\} + (1 - \lambda\tau) e^{-r\tau} E_t \left\{ \left(S_T - K\right)_+ |\text{no jump} \right\} + O(\tau^2) \\ &= \lambda\tau E_t \left\{ \left(S_T - K\right)_+ |\text{jump} \right\} \qquad\qquad (32) \\ &\qquad + (1 - (\lambda + \mu)\tau) e^{-(r-\mu)\tau} E_t \left\{ \left(S_T - K\right)_+ |\text{no jump} \right\} + O(\tau^2). \end{aligned}$$

Let us first evaluate the conditional expectation $E_t \left\{ \left(S_T - K\right)_+ |\text{jump} \right\}$ of the option payoff up to the term of order $\sqrt{\tau}$ given that a jump occurs. From (1), the log of the ratio of the price to the strike given that a jump occurs is equal to:

$$\begin{aligned} \log\left(S_T/K\right) &= \log(S_t/K) + \left(r - \delta - \mu - \frac{\sigma^2}{2}\right)\tau + \int_t^T \sigma_s dW_s + \log(1 + \Delta J) \\ &= \sigma_t \theta \sqrt{\tau} + \sigma_t (W_T - W_t) + \log(1 + \Delta J) + O(\tau). \end{aligned}$$

Note that $(W_T - W_t)$ is of order $\sqrt{\tau}$. Hence:

$$\begin{aligned}
E_t\left\{(S_T-K)_+|\text{jump}\right\} &= KE_t\left\{\left(e^{\log S_T/K}-1\right)_+|\text{jump}\right\} \\
&= KE_t\{[(1+\Delta J) \\
&\quad \times\left(1+\sigma_t\theta\sqrt{\tau}+\sigma_t(W_T-W_t)\right)-1]_+|\text{jump}\Big\}+O(\tau) \\
&= KE_t\{[\Delta J+(1+\Delta J) \\
&\quad \times\left(\sigma_t\theta\sqrt{\tau}+\sigma_t(W_T-W_t)\right)]_+|\text{jump}\Big\}+O(\tau) \\
&= KE\,(\Delta J)_+ \\
&\quad +K\sigma_t\theta(\Pr(\Delta J>0)+E\,(\Delta J)_+)\sqrt{\tau}+O(\tau). \qquad (33)
\end{aligned}$$

The last equality is easy to understand intuitively: for small τ, the event

$$\{[\Delta J+(1+\Delta J)(\sigma_t\theta\sqrt{\tau}+\sigma_t(W_T-W_t))]>0\}$$

happens "approximately" if and only if the event $\{\Delta J>0\}$ happens [3].

The rigorous argument is the following. We have

$$\begin{aligned}
E_t\left\{(S_T-K)_+|\text{jump}\right\} &= KE_t\left\{\left(e^{\log S_T/K}-1\right)_+|\text{jump}\right\} \\
&= KE_t\{(\Delta J+(1+\Delta J)\xi) \\
&\quad \times\mathbf{1}_{\{(\Delta J+(1+\Delta J)\xi)>0\}}|\text{jump}\Big\}+O(\tau).
\end{aligned}$$

where $\xi\sim N(\sigma\theta\sqrt{\tau},\sigma\sqrt{\tau})$ - independent of the jump. Now let us write the expectation

[3] Note, that in the model with contemporaneous jump in returns and volatility there appears an additional term $\Delta\sigma(W_T-W_{t'})$, where t' is the time of the jump and $\Delta\sigma$ - the size of the jump in volatility. This term is also of order $\sqrt{\tau}$ but has zero expectation (conditional on the jump occurence). Hence, using the same logic, we conclude that the introduction of a contemporaneous jump in volatility does not affect the analytical result.

explicitly (for ease of notation we omit conditioning):

$$
\begin{aligned}
& E\left\{(\Delta J+(1+\Delta J)\xi)\mathbf{1}_{\{(\Delta J+(1+\Delta J)\xi)>0\}}\right\} \\
= & \int_{-1}^{\infty} f(x)\frac{1}{\sqrt{2\pi}(\sigma\sqrt{\tau})}\int_{-x/(1+x)}^{\infty}(x+(1+x)y)e^{-\frac{1}{2}\frac{(y-\sigma\theta\sqrt{\tau})^2}{(\sigma\sqrt{\tau})^2}}dydx \\
= & \int_{-1}^{\infty} xf(x)N(\theta+\frac{x}{\sigma(1+x)\sqrt{\tau}})dx \qquad (34) \\
& +\sqrt{\tau}\int_{-1}^{\infty}(1+x)f(x)[\sigma n(\theta+\frac{x}{\sigma(1+x)\sqrt{\tau}})+\theta\sigma N(\theta+\frac{x}{\sigma(1+x)\sqrt{\tau}})]dx,
\end{aligned}
$$

where f is the density of the jump size distribution defined on the domain of possible jump values between -1 and $+\infty$ [4].

The first integral on the right hand side of (34) can be transformed using standard integration by parts with subsequent change of variable $y = x/(\sigma\sqrt{\tau})$. Denoting $G(x) = \int_0^x sf(s)ds$ we obtain:

$$
\begin{aligned}
\int_{-1}^{\infty} xf(x)N(\theta+\frac{x}{\sigma(1+x)\sqrt{\tau}})dx &= G(\infty)N\left(\theta+\frac{1}{\sigma\sqrt{\tau}}\right) \\
& -\frac{1}{\sigma\sqrt{\tau}}\int_{-1}^{\infty}\frac{G(x)}{(1+x)^2}n(\theta+\frac{x}{\sigma(1+x)\sqrt{\tau}})dx \\
&= G(\infty)N\left(\theta+\frac{1}{\sigma\sqrt{\tau}}\right) \\
& -\int_{-1/(\sigma\sqrt{\tau})}^{\infty}\frac{G(y\sigma\sqrt{\tau})}{(1+y\sigma\sqrt{\tau})^2}n(\theta+\frac{y}{1+y\sigma\sqrt{\tau}})dy \\
&= E\left(\Delta J\right)_+ + O(\tau).
\end{aligned}
$$

[4] Recall, that we deal with jumps in percentage points. Since stock price is always non-negative, jump cannot take value less than -1.

The last equality follows from the fact that $N\left(\theta+\frac{1}{\sigma\sqrt{\tau}}\right) = 1+o(\tau^k)$ for any $k>0$, $G(0)=G'(0)=0$ and $G(\infty)=E\left(\Delta J\right)_+$. In a similar way we approximate the other two integrals on the right hand side of (34):

$$\sigma\sqrt{\tau}\int_{-1}^{\infty}(1+x)f(x)n(\theta+\frac{x}{\sigma(1+x)\sqrt{\tau}})dx = O(\tau),$$

$$\theta\sigma\sqrt{\tau}\int_{-1}^{\infty}(1+x)f(x)N(\theta+\frac{x}{\sigma(1+x)\sqrt{\tau}})dx = \theta\sigma\sqrt{\tau}\left(\Pr\left(\Delta J>0\right)+E\left(\Delta J\right)_+\right)+O(\tau).$$

The other term $e^{-(r-\mu)\tau}E_t\left\{(S_T-K)_+\,|\text{no jump}\right\}$ on the right hand side of (32) can be evaluated using the asymptotics of the call option price obtained for the pure diffusion case. Indeed, conditional on no jump, we have a joint dynamics of price and volatility analogous to the diffusion case except that r should be replaced by $r-\mu$. We can use Propositions 1 and 2 to obtain asymptotics corresponding to a slightly different definition of moneyness degree:

$$\widetilde{\theta} = \frac{\log(S_t e^{(r-\mu-\delta)\tau}/K)}{\sigma\sqrt{\tau}}.$$

That is, we have:

$$\begin{aligned} & e^{-(r-\mu)\tau}E_t\left\{(S_T-K)_+\,|\text{no jump}\right\} \\ = \;& K\sigma\left[\widetilde{n}+\widetilde{\theta}\widetilde{N}\right]\sqrt{\tau}+K\Lambda_2\tau+K\Lambda_3\tau\sqrt{\tau}+O(\tau^2), \end{aligned} \tag{35}$$

where $\Lambda_2 = \frac{\widetilde{\theta}^2\widetilde{N}}{2}\sigma^2+\frac{\widetilde{\theta}\widetilde{n}}{2}\sigma^2+\widetilde{n}\widetilde{I}_1$,

$$\Lambda_3 = \frac{\widetilde{\theta}^3\widetilde{N}}{6}\sigma^3+\frac{\widetilde{\theta}^2\widetilde{n}}{6}\sigma^3-\frac{\sigma^3\widetilde{n}}{24}+\frac{\widetilde{n}\widetilde{\theta}^2\widetilde{I}_1^2}{2\sigma}+\frac{\sigma}{2}\widetilde{\theta}\widetilde{n}\widetilde{I}_1+\widetilde{n}\widetilde{I}_2-\sigma(r-\mu)(\widetilde{n}+\widetilde{\theta}\widetilde{N}),$$

and $\widetilde{I}_1 = I_1(\widetilde{\theta};\sigma)$, $\widetilde{I}_2 = I_2(\widetilde{\theta};\sigma)$ are the same as in Proposition 1, $\widetilde{n}=n(\widetilde{\theta})$, $\widetilde{N}=N(\widetilde{\theta})$. To obtain the asymptotics corresponding to θ we use the following relationship:

$$\widetilde{\theta} = \theta-\frac{\mu}{\sigma}\sqrt{\tau},$$

which should be plugged in (35). Using a Taylor series expansion, we can write:

$$n(\widetilde{\theta}) + \widetilde{\theta}N(\widetilde{\theta}) = n(\theta) + \theta N(\theta) - N(\theta)\frac{\mu}{\sigma}\sqrt{\tau} + \frac{1}{2}n(\theta)\left(\frac{\mu}{\sigma}\right)^2\tau + O(\tau\sqrt{\tau}),$$

$$N(\widetilde{\theta}) = N(\theta) - n(\theta)\frac{\mu}{\sigma}\sqrt{\tau} + O(\tau), \quad n(\widetilde{\theta}) = n(\theta) + \theta n(\theta)\frac{\mu}{\sigma}\sqrt{\tau} + O(\tau).$$

After substitution of these expressions in (35) and collecting terms of the same order, we arrive at:

$$e^{-(r-\mu)\tau}E_t\left\{(S_T - K)_+ \,|\text{no jump}\right\} = \sigma\left(n + \theta N\right)\sqrt{\tau} + K\Lambda_2^*\tau + K\Lambda_3^*\tau\sqrt{\tau} + O(\tau^2), \qquad (36)$$

where $\quad \eta = \lambda E\left(\Delta J\right)_+, \qquad \Lambda_2^* = \dfrac{\theta^2 N}{2}\sigma^2 + \dfrac{\theta n}{2}\sigma^2 - \mu N + nI_1,$

$$\begin{aligned}\Lambda_3^* &= \frac{\theta^3 N}{6}\sigma^3 + \frac{\theta^2 n}{6}\sigma^3 - \frac{\sigma^3 n}{24} + \frac{\mu^2 n}{2\sigma} + \frac{\sigma\mu n}{2} + \frac{n\theta^2 I_1^2}{2\sigma} + \frac{\mu b\rho n}{2\sigma} \\ &\quad + \left(\frac{\sigma}{2} + \frac{\mu}{\sigma}\right)\theta n I_1 + nI_2 - \sigma r(n + \theta N),\end{aligned}$$

and $I_1 = I_1(\theta;\sigma)$, $I_2 = I_2(\theta;\sigma)$, $n = n(\theta)$, $N = N(\theta)$.

Let us now substitute (33) and (36) in (32), which yields (14) with:

$$\Gamma_1 = \sigma\left(n + \theta N\right),$$

$$\Gamma_2 = \frac{\theta^2 N}{2}\sigma^2 + \frac{\theta n}{2}\sigma^2 - \mu N + nI_1 + \eta,$$

$$\begin{aligned}\Gamma_3 &= \frac{\theta^3 N}{6}\sigma^3 + \frac{\theta^2 n}{6}\sigma^3 - \frac{\sigma^3 n}{24} + \frac{\mu^2 n}{2\sigma} + \frac{\sigma\mu n}{2} + \frac{\mu b\rho n}{2\sigma} + \frac{n\theta^2 I_1^2}{2\sigma} \\ &\quad + \left(\frac{\sigma}{2} + \frac{\mu}{\sigma}\right)\theta n I_1 + \sigma\theta(\chi + \eta) + nI_2 - \sigma(\lambda + \mu + r)\left(n + \theta N\right),\end{aligned}$$

where we have denoted the expected positive jump by $\eta = \lambda E(\Delta J)_+$, and the probability of positive jump by $\chi = \lambda\ \Pr\left(\Delta J > 0\right)$. Functions $I_1(\theta;\sigma)$ and $I_2(\theta;\sigma)$ are the same as in Proposition 1, while $n = n(\theta)$, $N = N(\theta)$ are the pdf and cdf of the standard normal distribution evaluated at θ.

Appendix D. Proof of Proposition 5

Let us use $N = N(z)$ and $n = n(z)$ to abbreviate the Gaussian cdf and pdf evaluated at z. We can safely assume $r = \delta = 0$ since the implied volatility as a function of moneyness, time-to-maturity and volatility is independent of the risk-free rate and the dividend yield. The fundamental PDE for option price under the general setting of Model (1) is:

$$\begin{aligned} -C_\tau + \frac{1}{2}\sigma^2 \left(C_{XX} - C_X\right) + a(\sigma)C_\sigma + \frac{1}{2}b^2(\sigma)C_{\sigma\sigma} + \sigma b(\sigma)\rho C_{\sigma X} \\ +\lambda E\left[C(X + \log(1+\Delta J)) - C(X)\right] - \mu C_X = 0. \end{aligned} \tag{37}$$

Equation (37) differs from (27) by the last two terms on the left hand side. Similarly the PDE for the implied volatility in the general case differs from (28) in the following term on the left hand side:

$$D = \frac{-\lambda E\left[C(X + \log(1+\Delta J)) - C(X)\right] + \mu C_X}{C_\sigma^{BS}}, \tag{38}$$

where $C_\sigma^{BS} = C_\sigma^{BS}(X, K, \tau, \mathbf{I}(X, \tau, \sigma_t))$ is the derivative of the Black-Scholes formula with respect to volatility evaluated at the corresponding implied volatility.

Let us now derive asymptotics of the additional term (38). First, we find that:

$$\begin{aligned} C(X + \log(1+\Delta J)) &= C^{BS}(X + \log(1+\Delta J), K, \tau, \mathbf{I}^+) \\ &= S(1+\Delta J)N_1 - KN_2 = S\left[(1+\Delta J)N_1 - e^{-\theta\sigma\sqrt{\tau}}N_2\right], \end{aligned} \tag{39}$$

where $N_1 = N\left(\frac{\sigma\theta}{I^+} + \frac{\log(1+\Delta J)}{I^+\sqrt{\tau}} + \frac{1}{2}I^+\sqrt{\tau}\right)$, $N_2 = N\left(\frac{\sigma\theta}{I^+} + \frac{\log(1+\Delta J)}{I^+\sqrt{\tau}} - \frac{1}{2}I^+\sqrt{\tau}\right)$, and $\mathbf{I}^+$ and I^+ denote the implied volatility right after the jump:

$$I^+ = I\left(\theta + \frac{\log(1+\Delta J)}{\sigma\sqrt{\tau}}, \tau; \sigma\right).$$

If $\Delta J < 0$, respectively $\Delta J > 0$, then both N_1 and N_2 converge exponentially to zero, respectively to one. So, intuitively, when looking for asymptotics of the expectation, we may set N_1 and N_2 equal to their limits and, using (39), write:

$$\begin{aligned}\lambda E\left\{C(X+\log(1+\Delta J))\right\} &= \lambda S\left[E(\Delta J)_+ + \Pr(\Delta J>0)\theta\sigma\sqrt{\tau}+O(\tau)\right]\\ &= S\left[\eta+\chi\theta\sigma\sqrt{\tau}+O(\tau)\right]. \qquad (40)\end{aligned}$$

This intuition is not entirely correct. However, it yields correct expression for the first order asymptotics (40). The rigorous argument is the following. Using integration by parts, we have

$$\begin{aligned}E(1+\Delta J)N_1 &= \int_{-1}^{\infty}(1+x)N\left(\frac{\sigma\theta}{I^+}+\frac{\log(1+x)}{I^+\sqrt{\tau}}+\frac{I^+\sqrt{\tau}}{2}\right)f(x)dx\\ &= G(\infty)-\int_{-1}^{\infty}G(x)n\left(\frac{\sigma\theta}{I^+}+\frac{\log(1+x)}{I^+\sqrt{\tau}}+\frac{I^+\sqrt{\tau}}{2}\right)\\ &\quad\times\left(-\frac{\sigma\theta}{I^{+2}}I^+_x-\frac{\log(1+x)}{I^{+2}\sqrt{\tau}}I^+_x+\frac{1}{(1+x)I^+\sqrt{\tau}}+I^+_x\frac{\sqrt{\tau}}{2}\right)dx,\end{aligned}$$

where $G(x)=\int_0^x(1+s)f(s)ds$, $I^+=I\left(\theta+\frac{\log(1+x)}{\sigma\sqrt{\tau}},\tau;\sigma\right)$, and $f(x)$ denotes the pdf of the jump-size distribution. Here we have also used $\lim_{x\to\infty}N\left(\frac{\sigma\theta}{I^+}+\frac{\log(1+x)}{I^+\sqrt{\tau}}+\frac{I^+\sqrt{\tau}}{2}\right)=1$, which follows from the fact that the call option will be exercised for sure if the jump is very large (see (39)).

After the change of variable $y=\log(1+x)/(\sigma\sqrt{\tau})$ we get:

$$\begin{aligned}E(1+\Delta J)N_1 &= G(\infty)-\int_{-1}^{\infty}G(e^{\sigma y\sqrt{\tau}}-1)n\left(\frac{\sigma\theta}{I^+}+\frac{\sigma y}{I^+}+\frac{I^+\sqrt{\tau}}{2}\right)\\ &\quad\times\left(-\frac{\sigma\theta}{I^{+2}}I^+_y-\frac{\sigma y}{I^{+2}}I^+_y+\frac{\sigma}{I^+}+\frac{\sqrt{\tau}}{2}I^+_y\right)dy \qquad (41)\\ &= E(\Delta J)_+ + P(\Delta J>0)-\sqrt{\tau}f(0)\sigma\int_{-1}^{\infty}yn\left(\theta+y\right)dy+O(\tau),\end{aligned}$$

where we have used $G(0)=0$, $G'(0)=f(0)$, $I^+=\sigma+O(\sqrt{\tau})$, and $I^+_y=I_\theta=O(\sqrt{\tau})$.

In a similar way, we obtain:

$$EN_2 = P(\Delta J > 0) - \sqrt{\tau} f(0)\sigma \int_{-1}^{\infty} yn\,(\theta + y)\,dy + O(\tau). \tag{42}$$

Now using (39) and Expressions (41) and (42), we arrive at (40).

Proposition 2 suggests that:

$$C(S) = K\sigma\,[n + \theta N]\,\sqrt{\tau} + O(\tau) = S\sigma\,[n(\theta) + \theta N(\theta)]\,\sqrt{\tau} + O(\tau). \tag{43}$$

Let us further denote by $\widehat{I}_1$ and $\widehat{I}_2$ the new asymptotic terms in the expansion of the implied volatility function $I(\theta, \tau; \sigma)$. It is left to find the relationship between these terms, and I_1, I_2 obtained in Proposition 4. The partial derivative of the Black-Scholes formula with respect to the volatility can be written as:

$$\begin{aligned} C_\sigma^{BS}(I) &= S\sqrt{\tau}n\left(\frac{\sigma\theta}{I} + \frac{I}{2}\sqrt{\tau}\right) \\ &= S\sqrt{\tau}n(\theta)\left[1 + \theta\left(\frac{\theta\widehat{I}_1}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\tau} + O(\tau)\right]. \end{aligned} \tag{44}$$

Now putting (43), (44) and (40) together, we obtain:

$$\begin{aligned} & -\frac{\lambda E\left[C(X + \log(1 + \Delta J)) - C(X)\right]}{C_\sigma^{BS}} \\ = \quad & -\frac{\eta}{n}\frac{1}{\sqrt{\tau}} + \frac{1}{n}\left[\chi\theta\sigma - \theta\eta\left(\frac{\theta\widehat{I}_1}{\sigma} - \frac{\sigma}{2}\right) - \sigma\,[n + \theta N]\right] + O(\sqrt{\tau}), \end{aligned} \tag{45}$$

where $n = n(\theta)$, $N = N(\theta)$.

Similarly we obtain:

$$\frac{\mu C_X}{C_\sigma^{BS}} = \mu\frac{N}{n}\frac{1}{\sqrt{\tau}} - \mu\frac{1}{n}\left(\frac{\theta\widehat{I}_1}{\sigma} - \frac{\sigma}{2}\right)(n + \theta N) + O(\sqrt{\tau}). \tag{46}$$

The substitution of (45) and (46) in (38) yields:

$$
\begin{aligned}
D &= \left[\mu\frac{N}{n} - \frac{\eta}{n}\right]\frac{1}{\sqrt{\tau}} + \frac{1}{n}\left[\chi\theta\sigma - \theta\eta\left(\frac{\theta\widehat{I}_1}{\sigma} - \frac{\sigma}{2}\right) - \sigma\left[n + \theta N\right]\right. \\
&\quad \left. - \mu\left(\frac{\theta\widehat{I}_1}{\sigma} - \frac{\sigma}{2}\right)(n + \theta N)\right] + O(\sqrt{\tau}).
\end{aligned}
$$

After adding this term to (28) we deduce the following asymptotics of the PDE for the implied volatility:

$$
\widehat{A}\frac{1}{\sqrt{\tau}} + \widehat{B} + O(\sqrt{\tau}) = 0,
$$

with $\quad \widehat{A} = \mu\dfrac{N}{n} - \dfrac{\eta}{n} + A,$

$$
\widehat{B} = \frac{1}{n}\left[(\chi + \eta/2)\theta\sigma - \eta\frac{\theta^2\widehat{I}_1}{\sigma} - \sigma\left[n + \theta N\right] - \mu\left(\frac{\theta\widehat{I}_1}{\sigma} - \frac{\sigma}{2}\right)(n + \theta N)\right] + B,
$$

and where A, B are the same as in the proof of Proposition 1 except for $\widehat{I}_1$ and $\widehat{I}_2$ replacing I_1 and I_2.

By proceeding as in the proof of Proposition 1, we derive the ODE for $\widehat{I}_1$:

$$
-\frac{3}{2}\widehat{I}_1 - \frac{1}{2}\theta\widehat{I}_{1\theta} + \frac{1}{2}\widehat{I}_{1\theta\theta} = \rho b\theta + \mu\frac{N(\theta)}{n(\theta)} - \frac{\eta}{n(\theta)}. \tag{47}
$$

We can easily verify that

$$
\widehat{I}_1 = I_1 = -\frac{\rho b\theta}{2} - \mu\frac{N(\theta)}{n(\theta)} + \frac{\eta}{n(\theta)} \tag{48}
$$

is the solution of (47) (see Proposition 4).

The ODE for $\widehat{I}_2$ has the same homogeneous part as in the pure diffusion case since $\widehat{I}_2$ does not enter D. However, the non-homogeneous part of this ODE in the jump-diffusion case is, of course, different. Let us denote it as $Q(\theta, \sigma)$ in the particular case of a jump process with time invariant parameters, the case considered in Proposition 4. The non-homogeneous part in the general case differs from Q due to terms with partial derivative of $\widehat{I}_1$ with respect to σ in B. Indeed, μ and η depend on σ, so the partial derivative $\dfrac{\partial\widehat{I}_1}{\partial\sigma}$ will include $\mu' = \dfrac{\partial\mu}{\partial\sigma}$ and

$\eta' = \dfrac{\partial \eta}{\partial \sigma}$.

Using the expression for $\widehat{I}_1$ (48) and the expression for B (29), we obtain the ODE:

$$-2\widehat{I}_2 - \frac{1}{2}\theta \widehat{I}_{2\theta} + \frac{1}{2}\widehat{I}_{2\theta\theta} = Q(\theta, \sigma) + \rho b \mu'. \tag{49}$$

From Proposition 4 we know that $-2I_2 - \dfrac{1}{2}\theta I_{2\theta} + \dfrac{1}{2}\sigma^2 I_{2\theta\theta} = Q(\theta, \sigma)$.

So the natural candidate for the solution to (49) is:

$$\widehat{I}_2 = I_2 - \frac{1}{2}\rho b \mu',$$

which gives the stated result.

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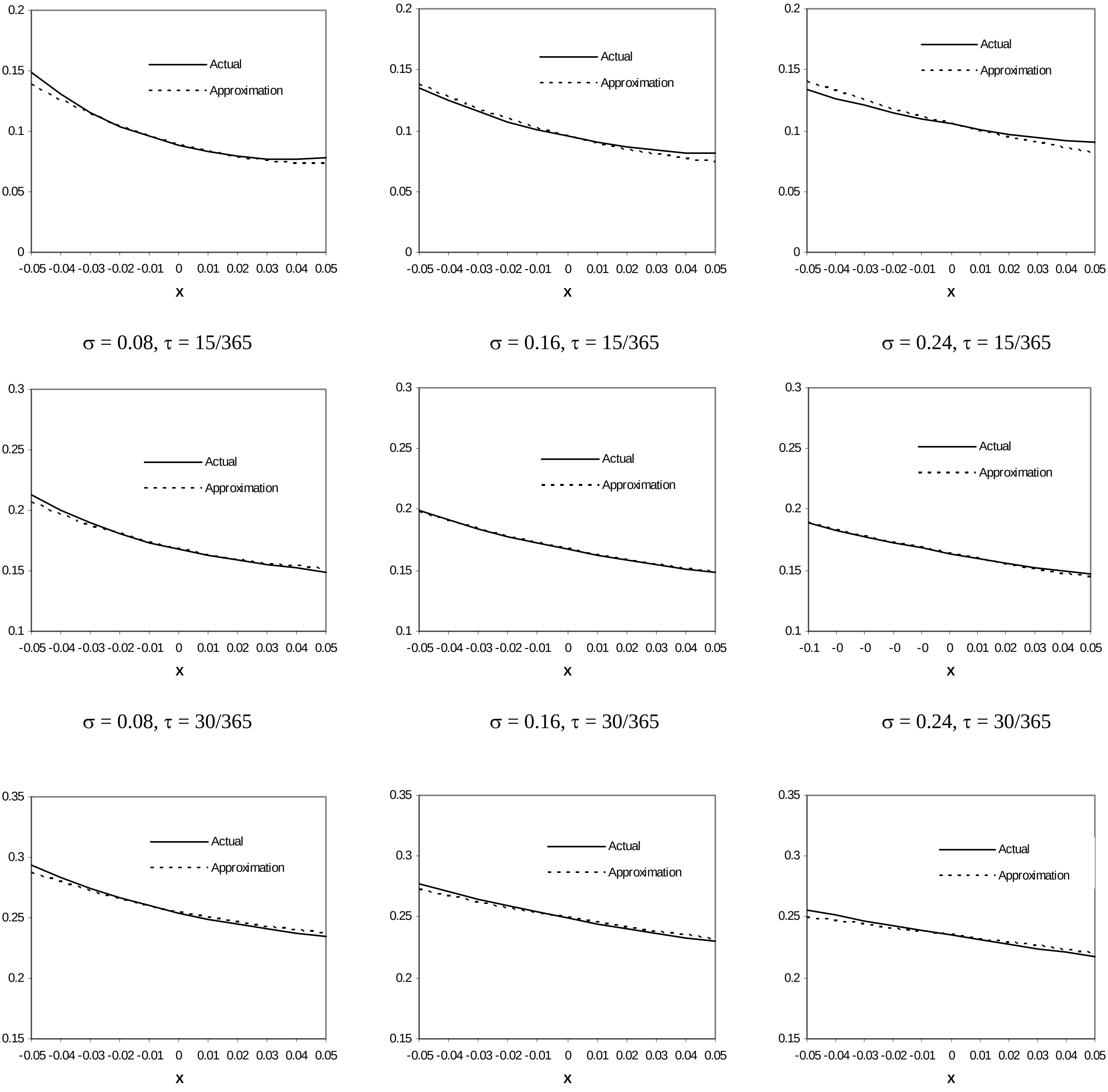
	Heston	CEV SV	Affine SVJ	CEV SVJ
$\beta\rho$	-.09 (.00)	-1.94 (.07)	-.03 (.00)	-33.3 (10.6)
ρ	-.46 (.00)	-.54 (.00)	-1.0*	-0.81 (.08)
ϕ	0	1.30 (.13)	0	2.61 (.13)
$\lambda_0\, E(\Delta J)$	0	0	-4.98 (.11)	-.92 (.39)
$E(\Delta J)$	…	…	-.14 (.00)	-.09 (.01)
ψ	…	…	2	2.32 (.16)
$\sqrt{ASE}$	.0046	.0043	.0037	.0035

* *the lower boundary for the correlation parameter is reached.*

Table 1. The results of the calibration to quoted S&P500 option prices.

Four models are calibrated on 2537 observations: the Heston (1993) model, the CEV-type stochastic volatility model without jumps, the affine jump-diffusion stochastic volatility (SVJ) model, and the CEV-type jump-diffusion stochastic volatility model. $\sqrt{ASE}$ denotes the square root of the average of squared errors in implied volatilities. Standard errors are given in parentheses.



σ = 0.08, τ = 15/365 | σ = 0.16, τ = 15/365 | σ = 0.24, τ = 15/365

σ = 0.08, τ = 30/365 | σ = 0.16, τ = 30/365 | σ = 0.24, τ = 30/365

σ = 0.08, τ = 60/365 | σ = 0.16, τ = 60/365 | σ = 0.24, τ = 60/365

Figure 1. Actual implied volatilities and their approximations for the jump-diffusion model.
The model parameters are borrowed from from Pan (2002):

$$dS_t = \left(r - \delta - \lambda_t E(\Delta J)\right) S_t dt + \sqrt{v_t} S_t dW_t^{(1)} + S_t dJ_t,$$
$$dv_t = \kappa(\bar{v} - v_t)dt + \sigma_v \sqrt{v_t}\left(\rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(2)}\right)$$
$$\lambda_t = \lambda_0 + \lambda_1 v_t,\ \ v_t = \sigma_t^2,$$

with $\kappa = 6.4,\ \ \bar{v} = .015,\ \ \rho = -.53,\ \ \lambda_0 = 0,\ \ \lambda_1 = 12.3,\ \ E(\Delta J) = -.19,\ \ \sigma_v = .3.$

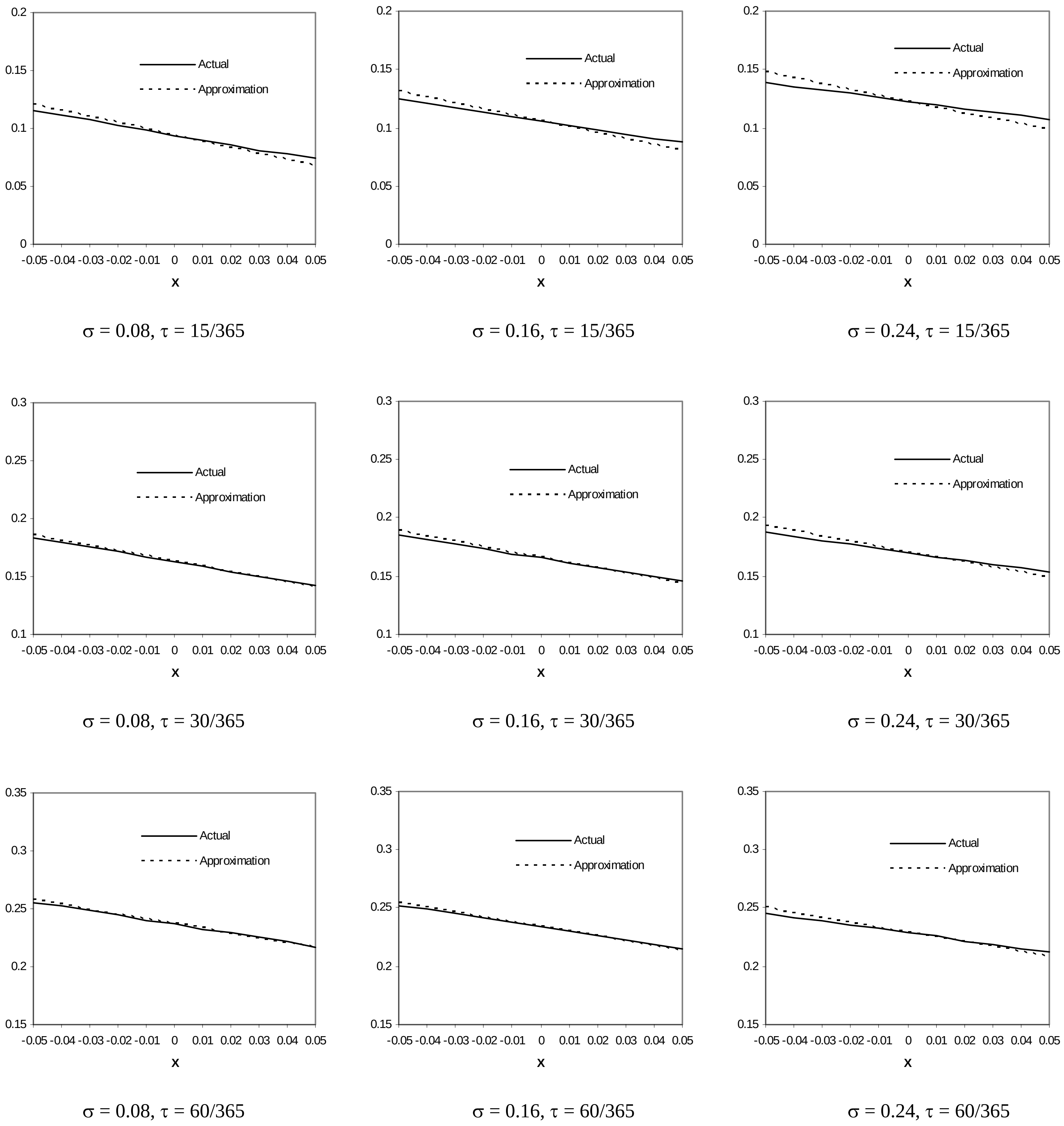
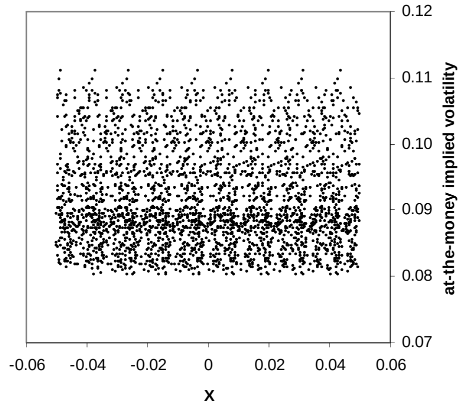


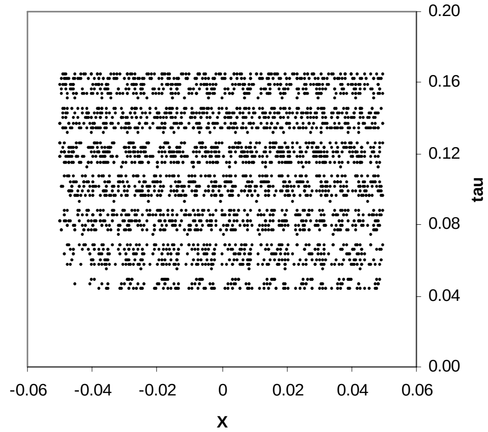
Figure 2. Actual implied volatilities and their approximations for the CEV diffusion model.
The model parameters are borrowed from Aït-Sahalia and Kimmel (2004):

$$dS_t = (r-\delta)S_t dt + \sqrt{v_t}\, S_t\, dW_t^{(1)},$$
$$dv_t = \kappa(\bar{v} - v_t)dt + \sigma_v v_t^{\gamma}\left(\rho W_t^{(1)} + \sqrt{1-\rho^2}\, W_t^{(2)}\right)$$
$$v_t = \sigma_t^2,$$

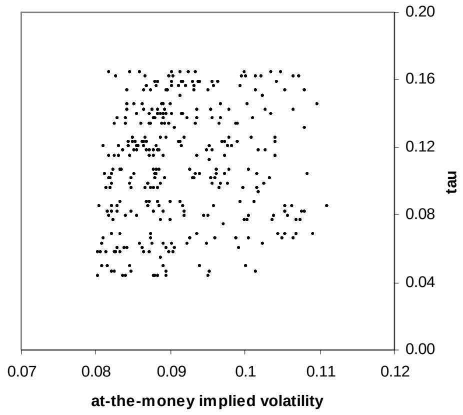
with $\kappa = 3,\ \bar{v} = .052,\ \rho = -.77,\ \sigma_v = 1.37,\ \gamma = .85.$



(a)



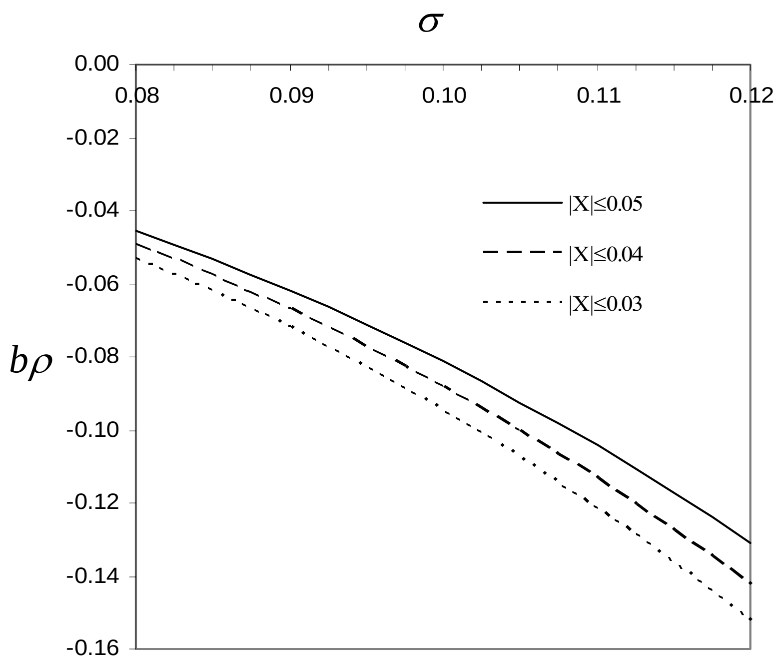
(b)



(c)

Figure 3. The distribution of observations in the dimensions of moneyness, time-to-maturity and at-the-money implied volatility.

The first two scatter plots are built for 2537 observations showing how dense the observations of implied volatilities are across moneyness. The third scatter plot is based on 324 observations of implied volatility smile. The at-the-money implied volatility is the implied volatility which is the closest-to-the-money.



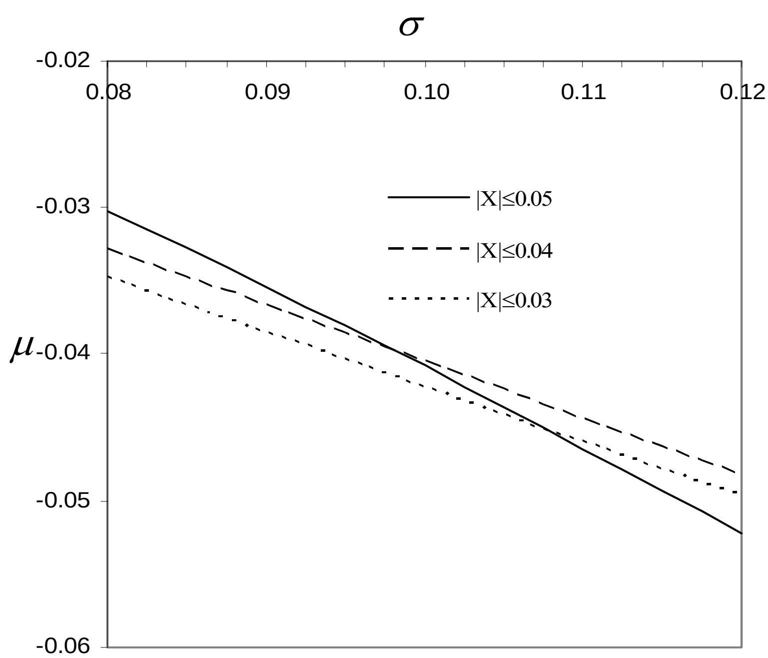


Figure 4. Robustness of model parameter estimates

The two graphs show the inferred shapes of the volatility of volatility and the expected jump size as functions of volatility based on datasets corresponding to three restrictions on the option moneyness $-0.05 \le X \le 0.05$, $-0.04 \le X \le 0.04$ and $-0.03 \le X \le 0.03$.