

PRICING AMERICAN OPTIONS UNDER STOCHASTIC VOLATILITY AND STOCHASTIC INTEREST RATES

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Abstract

We introduce a new analytical approach to price American options. Using an explicit and intuitive proxy for the exercise rule, we derive tractable pricing formulas using a short-maturity asymptotic expansion. Depending on model parameters, this method can accurately price options with time-to-maturity up to several years. The main advantage of our approach over existing methods lies in its straightforward extension to models with stochastic volatility and stochastic interest rates. We exploit this advantage by providing an analysis of the impact of volatility mean-reversion, volatility of volatility, and correlations on the American put price.

Key words: American options, stochastic volatility, stochastic interest rates, asymptotic approximation.

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1 Introduction

The valuation of American options is a challenging task, even under the Black-Scholes model (see Detemple (2005) for an extensive review). Several semi-analytical approximations for American option prices have been proposed in the literature (Barone-Adesi and Whaley (1987), Broadie and Detemple (1996), Bunch and Johnson (2001), Ju (1998)). Although these approaches are fast and accurate, they cannot easily be extended beyond the Black-Scholes model.

It has been firmly established that the Black-Scholes model is not consistent with quoted option prices. The literature advocates the introduction of stochastic volatility and/or jumps to reproduce the implied volatility smile observed in the market. The introduction of an additional stochastic volatility factor enormously complicates the pricing of American options. Presently, this can only be done by means of numerical schemes, which involve solving integral equations (Kim (1990), Huang, Subrahmanyam and Yu (1996), Sullivan (2000), Detemple and Tian (2002)), performing Monte Carlo simulations (Broadie and Glasserman (1997), Longstaff and Schwartz (2001), Rogers (2002), Haugh and Kogan (2004)), or discretizing the partial differential equation (Brennan and Schwartz (1977), Clarke and Parrott (1999), Ikonen and Toivanen (2007)).

The early exercise premium of the American put option depends on the cost of carry determined by interest rates. Consequently, the volatility of interest rates does affect the decision to exercise this option at any point in time. This fact is recognized in the literature dealing with models with stochastic interest rates (Ho, Stapleton and Subrahmanyam (1997), Menkveld and Vorst (2001), Detemple and Tian (2002)). This literature, however, considers only two-factor extensions of the Black-Scholes model assuming that the volatility of the underlying asset is constant.

Numerical approaches are complicated to implement when both volatility and interest rates are stochastic. Presently, there exist two ways of computing the option price in this case: a discretization of the partial differential equation (PDE), and the simulation-based method of Longstaff and Schwartz (2001). From the analogy with standard tree approaches, a PDE solver amounts to reconnecting three trees: a tree for the price of the underlying asset, a tree for the mean-reverting stochastic volatility and a tree for the stochastic interest rates. As a result, the implementation of a PDE solver is case specific. Such an implementation is feasible in the case of a three-factor affine specification, but next to impossible in the general setting. Further it may raise stability issues (especially when computing option Greeks). Being easier to implement, the method of Longstaff and Schwartz (2001) is extremely time-consuming, making it not realistic from a practical point of view.

In this paper we propose a new analytical approach that is both computationally tractable and general enough to be successfully applied to a three-factor diffusion model without jumps. Here we limit our analysis of American payoffs to diffusion processes. For European payoffs there is a large recent literature on models based

on Levy processes (see e.g. Carr and Wu (2004), Bakshi, Carr, and Wu (2008), and references therein). The characterization of American put prices under Levy processes raises complex issues related to the possibility of discontinuities of the underlying process at the exercise boundary (see e.g. Chesney and Jeanblanc (2004), Alili and Kyprianou (2005), and Lamberton and Mikou (2008) for recent discussions), and extending our method is not straightforward in such a context.

Our approach is based on the idea of substituting the optimal exercise rule with a simple (suboptimal) exercise rule for which an approximate solution is easy to find and fast to compute. Similar ideas have already been explored in the literature in the context of the Black-Scholes model (Broadie and Detemple (1996), Carr (1998), Ju (1998)). Our proxy rule is to exercise the option as soon as its moneyness measured in standard deviations reaches some specified level. The rationale is that an option should be exercised when it can be considered sufficiently in-the-money (large moneyness). The option price under this rule appears to have a regular asymptotic behavior near maturity with an asymptotic expansion available in a closed-form for a broad class of models. The American option price is then approximated by the maximum over these option prices. We provide several numerical experiments and comparisons showing that our method performs well with respect to computational time and accuracy.

Taking advantage of our computationally efficient method, we study the effect of introducing stochastic volatility and stochastic interest rates on the American put price and its components: the European put price and the early exercise premium. Using a simple economic argument we guess that the effect on the American put price should be equal in sign but smaller in magnitude than the impact on the European put price. We confirm this conjecture by providing a detailed analysis of the impact of different generalizations of the Black-Scholes model using a range of plausible model parameter values borrowed from empirical studies of stock index and currency options. Our analysis does not extend easily to the case of an underlying asset paying discrete dividends before the option maturity.

The paper is organized as follows. In Section 2 we describe our approach in the context of the Black-Scholes model. We provide a motivation for our approach, discuss intuitively its main features, and compare it with other available methods. We focus our discussion on the American put option. We can use the put-call symmetry to obtain prices, derivatives of prices with respect to parameters, and early exercise boundaries for American call options from the properties of the corresponding put options (see e.g. Schroder (1999) and Section 3.6 in the survey article of Broadie and Detemple (2004)). In Section 3 we generalize our approach to incorporate multifactor models with stochastic volatility and stochastic interest rates. We run several numerical experiments to show that our approach is accurate for reasonable model parameters, and investigate the impact of stochastic volatility and stochastic interest rates on the components of the American put price. Section 4 concludes the

paper. Technical proofs and results are gathered in Appendices. All the Matlab codes used in this paper are available from the authors on request.

2 Black-Scholes model

In this section we consider the Black-Scholes model where the price S of the underlying asset follows a log-normal diffusion process $dS_t = (r - \delta)S_t dt + \sigma S_t dW_t^{(1)}$, with constant interest rate r , dividend yield δ , and volatility σ . This section is aimed at presenting intuitively our approach and developing an analytical approximation for the American put in a simple setting. In the next section we look at richer settings where two additional Brownian motions $W_t^{(2)}$ and $W_t^{(3)}$ are introduced in a three-factor model.

2.1 Short-maturity asymptotics for American option prices

An American put option with strike price K and maturity date T is a derivative that gives its owner the right to receive $\max(K - S_{t'}, 0)$ at any point in time $t \leq t' \leq T$. Under the Black-Scholes model the price $\mathbf{P}(S, t)$ of this option satisfies the partial differential equation (PDE):

$$\mathbf{P}_t + (r - \delta)S\mathbf{P}_S + \frac{1}{2}\sigma^2 S^2 \mathbf{P}_{SS} - r\mathbf{P} = 0, \quad (1)$$

with boundary conditions:

$$\begin{aligned} \mathbf{P}(\infty, t) &= 0, & \mathbf{P}(S, T) &= \max(K - S, 0), \\ \mathbf{P}(\bar{S}(T - t), t) &= \max(K - \bar{S}(T - t), 0), & \mathbf{P}_S(\bar{S}(T - t), t) &= -1. \end{aligned} \quad (2)$$

Here subscripts denote differentiation with respect to time t and price S ; $\bar{S}(\tau)$ is the early exercise price, which depends on the option time-to-maturity $\tau = T - t$. The last boundary condition in (2) is the so-called "smooth pasting" condition. Note that the European put also satisfies (1) plus the four boundary conditions with $\bar{S}(\tau) = 0$. The unique solution for the American option price is then determined by requiring that $\mathbf{P}(S, t) \geq \max(K - S, 0)$.

Solving PDE (1) given the five conditions above is a non-trivial task. The only known analytical solution to this problem ² is found by Zhu (2006) in the form of a Taylor series expansion. While the emphasis of that paper is to show the existence of an exact analytical solution, such a solution does not have a clear advantage over some fast numerical schemes from a computational point of view. The convergence is rather slow, and the expansion terms are given recursively by a complicated analytical formula.

A number of approximation methods exist in the literature. In particular, the behavior of $\bar{S}(\tau)$ near maturity

²Except, of course, the trivial case when there is no early exercise premium.

(small τ) has attracted lots of attention as a promising way to derive an analytical formula (Alobaidi and Mallier (2001), Barles et al. (1995), Chevalier (2005), Dewynne et al. (1993), Evans, Keller and Kuske (2002), Goodman and Ostrov (2002), Lamberton and Villeneuve (2003); see also Levendorski (2007) and references therein for further general results). However, this approach has not produced a sufficiently accurate approximation under realistic model parameters (see the numerical examples in Mallier (2002)). Although a high-order short-maturity asymptotic approximation of $\bar{S}(\tau)$ is available in an analytical form (see Alobaidi and Mallier (2004)), accuracy remains an issue. In a related paper Chen and Chadam (2007) use short-maturity asymptotics to derive an implicit approximation that appears to be accurate for time-to-maturity less than several months. However, instead of using a truncated asymptotic expansion, they choose an ad hoc functional form with correct asymptotic behavior near maturity.

We now explain why a direct short-maturity analysis does not yield an applicable formula for American options. This will motivate our approach to option pricing described in the rest of the paper. Let us take $\delta = 0$, and introduce a convenient parameterization, denoting:

$$\theta = \frac{\ln(K/S)}{\sigma\sqrt{\tau}}. \quad (3)$$

This ratio is called normalized moneyness, and is frequently used in the literature (see e.g. Bates (2000), Carr and Wu (2003)). It measures the distance between the logarithm of the price of the underlying asset and the logarithm of the strike price in terms of standard deviations. Strictly speaking, we should take the ratio of the strike price to the forward price to take into account the drift in $\ln S$. The two definitions, however, are equivalent when time-to-maturity is small.

As an example, consider now the choice to exercise a put option now or wait till maturity. If the option is exercised now then the option holder receives $\mathbf{P} = K - S_t$ (provided that $K > S_t$). Since the expected discounted payoff of the option at maturity is equal to the European put price \mathbf{P}^E , put-call parity says that:

$$\mathbf{P}^E = Ke^{-r\tau} - S_t + \mathbf{C}^E. \quad (4)$$

This equality combined with $\mathbf{P} = K - S_t$ implies that the early exercise premium $\mathbf{P} - \mathbf{P}^E$ is given by:

$$K(1 - e^{-r\tau}) - \mathbf{C}^E. \quad (5)$$

The early exercise decision may be interpreted as equivalent to giving up a call option while putting money in a bank account. If K is sufficiently greater than S_t (i.e., the option is said to be deep in-the-money) then the

European call option price C^E is small. In this case the interest rate income $K(1 - e^{-r\tau}) > 0$ exceeds the call price meaning that the early exercise has a positive premium.

Let us proceed by noting that the early exercise level of the normalized moneyness $\bar{\theta}(\tau)$ has the following asymptotic behavior under zero dividends (see e.g. Barles et al. (1995)³):

$$\bar{\theta}(\tau) = \frac{\ln(K/\bar{S}(\tau))}{\sigma\sqrt{\tau}} \sim \sqrt{\ln(1/\tau)}. \quad (6)$$

It follows that when time-to-maturity decreases, no matter how deep in-the-money (as measured by θ) the put option is, it still is suboptimal to exercise before expiry. This result seems to contradict the rationale behind early exercise. Intuitively, we expect that when θ is sufficiently large, say 2 (this corresponds to the well-known 2-sigma rule for tail events under normality), the call option in (5) becomes negligible and it is optimal to exercise the American put right away.

This reasoning appears to be fundamentally wrong when time-to-maturity is very small. Medvedev and Scaillet (2007) show that the call option price satisfies $C^E \sim \sqrt{\tau}$ for given fixed θ (see also (18)). Consequently, when time-to-maturity is very small, the interest income $K(1 - e^{-r\tau}) \sim \tau$ is only second order relevant, and the American put converges to a European put. This formal explanation suggests that the exact asymptotics (6) is most likely to be accurate only in a region where the American put loses its early exercise advantage. As a result, a direct short-maturity asymptotic analysis is unable to deliver a good approximation under realistic model parameters.

In this paper we show that it is still possible to rely on a short-maturity asymptotic analysis if we modify the initial problem. Inspired by the "wrong" intuition behind the early exercise, we introduce an explicit exercise rule based on the normalized moneyness.

2.2 Modified problem

Let us consider a modified version of problem (1), with the smooth pasting condition in (2) replaced by an explicit exercise rule. The new problem is defined by the same PDE

$$\mathbf{P}_t + (r - \delta)S\mathbf{P}_S + \frac{1}{2}\sigma^2 S^2 \mathbf{P}_{SS} - rP = 0, \quad (7)$$

with boundary conditions:

$$\mathbf{P}(\infty, t) = 0, \quad (8)$$

³When $\delta = 0$, $r > 0$, we get $\delta < r$. This yields the case of less well-behaved asymptotics near expiry where the expansion of $\bar{\theta}(\tau)$ involves logarithms.

$$\mathbf{P}(S, T) = \max(K - S, 0), \quad (9)$$

$$\mathbf{P}(\bar{S}(T - t), t) = \max(K - \bar{S}(T - t), 0), \quad (10)$$

where $\bar{S}(T - t)$ satisfies $\bar{S}(T - t) = Ke^{-\sigma y \sqrt{T-t}}$.

The unique solution to this problem is the price of a barrier put option that is exercised as soon as the normalized moneyness reaches the barrier level y . There are few examples of boundaries for which the distribution of the first passage time of a Brownian motion is known in a closed-form and helps getting an explicit pricing formula, but they do not suit our setting. As we have already noted, the proxy for the optimal exercise rule is intuitively appealing since it is based on a normalized measure of moneyness. Hence, if the barrier level y is chosen around 2 to approximate the exercise boundary of the American option we expect the solution to the modified problem to be close to the true American option price.

To derive proper asymptotic expansions, we rewrite PDE (7) in terms of (θ, τ) instead of (S, t) . Using the definition of θ in (3), and setting $P(\theta, \tau) = \mathbf{P}(Ke^{-\sigma\theta\sqrt{\tau}}, T - \tau)$, we make the following substitutions in (7): $\mathbf{P}_t = -P_\tau + \frac{\theta}{2\tau}P_\theta$, $\mathbf{P}_S = -\frac{1}{\sigma S\sqrt{\tau}}P_\theta$, and $\mathbf{P}_{SS} = \frac{1}{\sigma^2 S^2 \tau}P_{\theta\theta} + \frac{1}{\sigma S^2 \sqrt{\tau}}P_\theta$. Simplifying, we obtain:

$$\theta P_\theta + P_{\theta\theta} + \frac{1}{\sigma} [\sigma^2 + 2(\delta - r)] P_\theta \sqrt{\tau} - 2(P_\tau + rP)\tau = 0. \quad (11)$$

As we shall see in the next section, there is a unique solution to (11) satisfying boundary conditions (8) and (10) in the form:

$$P(-\infty, \tau) = 0, \quad (12)$$

$$P(y, \tau) = K \max(1 - e^{-\sigma y \sqrt{\tau}}, 0) = K(1 - e^{-\sigma y \sqrt{\tau}}), \quad (13)$$

and which has regular asymptotics near maturity of the form:

$$P(\theta, \tau) = \sum_{n=1}^{\infty} P_n(\theta) \tau^{\frac{n}{2}}, \quad (14)$$

where $P_n(\theta)$, $n = 1, 2, \dots$, are the coefficients of the short-maturity asymptotic expansion in τ . The characterization of $P_n(\theta)$ is given in Proposition 1 below. Note that condition (9) is implicit in (14). Indeed, when $\tau = 0$ and θ is held fixed, we have $S = K$, and $\max(K - S, 0) = 0$. Thus, $P(\theta, 0) = 0$, a condition implied by (14).

Let us denote the solution to (11) with conditions (12), (13), and (14) by $P(\theta, \tau; y)$. The American put price $P(\theta, \tau)$ can be approximated from below by:

$$P(\theta, \tau) \simeq \max_{y \geq \theta} P(\theta, \tau; y) = P(\theta, \tau; \tilde{y}(\theta, \tau)). \quad (15)$$

2.3 Asymptotic expansion

In the following proposition we describe the series representation of the general solution to (11) without boundary condition (13). Then we show how a unique solution is determined by requiring (13). To make the presentation more compact, let us introduce some notation. The set of polynomials in θ of the form:

$$a_n\theta^n + a_{n-2}\theta^{n-2} + a_{n-4}\theta^{n-4} + \dots + a_m\theta^m, \quad m = \text{mod}(n, 2),$$

is denoted by $\Pi^1(n, \theta)$, and the subset for which $a_n = 1$ is denoted by $\Pi^0(n, \theta)$. Here, $m = 1$ if n is an odd number, and 0 if an even number. In addition, let us set: $\Phi(\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\theta} e^{-\frac{s^2}{2}} ds$, $\phi(\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}}$.

Proposition 1 *Consider partial differential equation (11) with boundary condition (12) and the regular asymptotic expansion (14) in the vicinity of $(0, 0)$. For any solution to this problem there exist constants C_1, C_2, \dots such that for each n : $P_n(\theta) = C_n [p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta)] + p_n^1(\theta)\Phi(\theta) + q_n^1(\theta)\phi(\theta)$, where $p_n^0 \in \Pi^0(n, \theta)$, $p_n^1 \in \Pi^1(n-2, \theta)$, $q_n^0 \in \Pi^0(n-1, \theta)$, and $q_n^1 \in \Pi^1(n-3, \theta)$ with coefficients depending on model parameters and C_1, C_2, \dots, C_{n-1} .*

Proof. See Appendix A. ■

Proposition 1 describes the form of the asymptotic expansion of the general solution (14) with appropriate asymptotics given by (12). To obtain a unique N th order expansion, we need to determine N constants C_n , $n = 1, \dots, N$. Let us show how to do this using a 2nd order expansion of equation (14) as an illustration. Using Proposition 1 we find by substitution:

$$P(\theta, \tau; y) = C_1 [\theta\Phi(\theta) + \phi(\theta)] \sqrt{\tau} + \left[C_2 ((\theta^2 + 1) \Phi(\theta) + \theta\phi(\theta)) + \frac{\sigma C_1}{2} \Phi(\theta) \right] \tau + O(\tau\sqrt{\tau}). \quad (16)$$

The coefficients C_1 and C_2 are uniquely determined by imposing the early exercise condition (13). Indeed, the short-maturity expansion of the payoff function is:

$$P(y, \tau; y) = K [1 - \exp(-\sigma y \sqrt{\tau})] = \sigma y K \sqrt{\tau} - \frac{\sigma^2 y^2 K}{2} \tau + O(\tau\sqrt{\tau}). \quad (17)$$

Equating expansion (16) at $\theta = y$ to expansion (17) allows us to identify the missing coefficients. The expressions for these coefficients can be found in Appendix B, where we present the short-maturity expansion of $P(\theta, \tau; y)$ up to the 4th order.

As a consistency check, recall that the European put price corresponds to $y = \infty$. So, as $y \rightarrow \infty$, we have $C_1(\infty) = \sigma K$, $C_2(\infty) = -\frac{K\sigma^2}{2}$, and

$$P(\theta, \tau; \infty) = \sigma K(\theta\Phi(\theta) + \phi(\theta))\sqrt{\tau} - K \left[\frac{\sigma^2}{2} (\theta^2\Phi(\theta) + \theta\phi(\theta)) + r\Phi(\theta) \right] \tau + O(\tau). \quad (18)$$

This is exactly the asymptotic behavior of the European put implied by the put-call parity and results in Medvedev and Scaillet (2007) for the European call. It follows that the European put (call) converges to zero at the rate of $\sqrt{\tau}$ when τ goes to zero for given fixed θ . Observe also that the leading order in expansion (18) coincides with the put price under an arithmetic Brownian motion specification for the stock price (Bachelier formula). Both types of models are equivalent near maturity.

2.4 Early exercise price

So far we have dealt with the pricing of the American put. However, we did not address the issue of how to decide on the early exercise of the option. An approximation $\tilde{\theta}$ for the true early exercise level of moneyness $\bar{\theta} = \frac{\ln(K/\bar{S})}{\sigma\sqrt{\tau}}$ can be defined in the following way:

$$\tilde{\theta}(\tau) = \arg \min_{\theta} \{ \tilde{y}(\theta, \tau) = \theta \}, \quad (19)$$

where \tilde{y} is implicitly defined in (15). That is, the minimum level of moneyness such that the American put is best approximated by its payoff and, therefore, should be exercised immediately. When $\theta = \bar{\theta}$, the payoff also delivers the best approximation; consequently from (19) we necessarily have $\tilde{\theta} \leq \bar{\theta}$.

We use barrier options to approximate the American put, and this may raise concerns about the quality of the approximation of the early exercise boundary especially near maturity where, under zero dividends, $\bar{\theta}(\tau) \sim \sqrt{\ln(1/\tau)}$ (see (6)). Our numerical experiments, however, suggest that the approximation appears to be indistinguishable from the true boundary. Proposition 2 provides a formal justification for this. It shows that the "smooth-pasting" condition is satisfied at the early exercise boundary, and $\tilde{\theta}(\tau)$ has the correct short-maturity asymptotic behavior.

Proposition 2 *Let $\tilde{\theta}(\tau)$ be the approximation of the early exercise boundary defined by (19) then*

- 1) *the barrier put option $P(\theta, \tau; \tilde{\theta}(\tau))$ is tangent to its payoff at $\theta = \tilde{\theta}(\tau)$.*
- 2) *under zero dividends, $\tilde{\theta}(\tau) \sim \sqrt{\ln(1/\tau)}$ when τ goes to 0.*

Proof. See Appendix C. ■

We conclude this section by noting that, in practice, it is not necessary to solve problem (19). The decision

to exercise the option should be based on a comparison between θ and the value-maximizing boundary $\tilde{y}(\theta, \tau)$. If $\theta = \tilde{y}(\theta, \tau)$ then the option should be exercised.

2.5 Performance of the approximation

In this section we perform several numerical experiments to study the accuracy of the approximation of the American put option introduced in the previous section. The approximation error has two possible sources: the asymptotic expansion and the suboptimal exercise rule. Hereafter we find that the convergence of the asymptotic expansion is extremely fast, meaning that the major source of the approximation error is the suboptimal exercise rule. This error also appears to be small.

2.5.1 Convergence of the asymptotic expansion

To illustrate the speed of convergence of the asymptotic expansion we find an approximation of the American put (15) using a 2000-step binomial tree to value the barrier options $P(\theta, \tau; y)$. Then we compute the same approximation using expansion (14) truncated at different orders ($N = 2, 3, 4, 5$). Assuming that 2000 steps are sufficient to compute option prices with high precision, the difference between the two approximations is only due to series truncation. For the numerical experiment we assume $r = 0.05$, $\delta = 0$, $\sigma = 0.2$. The optimal \tilde{y} is found using a simple search algorithm. We start with $y = \theta$ and then move in the direction of increasing y with a step size of 0.1. When this preliminary search is terminated, we refine the search with a smaller step size of 0.01. This procedure allows us to find \tilde{y} with a precision of 0.01.

Figure 1 compares errors of the two approximations with the true American put price being computed on a 2000-step binomial tree. Observe that the convergence of the short-maturity asymptotic expansion is very fast at all maturities. The 4th order expansion, given explicitly in Appendix B as an example, already appears to be sufficiently close to the tree-based approximation. The higher order expansion terms appear to be negligibly small relative to the error stemming from the suboptimal exercise rule.

2.5.2 Comparison with existing methods

In this section we compare our approach with other analytical approximations developed for the Black-Scholes model. We perform the analysis using model parameters chosen in the corresponding paper.

Broadie and Detemple (1996) suggest simple lower and upper bounds on the American call price. The lower bound is computed as the maximum over prices of call options that are exercised as the price level reaches some critical value (capped call options). The upper bound is derived from the lower bound using a formula for the early exercise premium. The same procedure can be applied using our approximation, which also provides a lower

bound.

The difference between our approach and that of Broadie and Detemple (1996) is that we use this rule for the normalized moneyness rather than the price of the underlying asset. We believe that expressing the suboptimal exercise rule in terms of normalized moneyness is more appealing (at least in the domain where time-to-maturity is not large), and we expect our approach to be more accurate. Although the price of the capped option in Broadie and Detemple (1996) admits an exact analytical expression, our setup is nevertheless given by an asymptotic expansion with a fast convergence rate, as shown in the previous section.

Table 1 reports lower bounds on American call prices from Broadie and Detemple (1996) (Tables 1 and 2), along with our results. To gauge the early exercise premium we also give European option prices. We compute American call prices using the put-call symmetry (see e.g. Broadie and Detemple (2004)). The call option price is equal to the put option price with S replaced by K and, vice-versa, and r replaced by δ , and vice-versa.

The first part of Table 1 reports option values corresponding to time-to-maturity equal to 6 months. Here the convergence of the asymptotic expansion is sufficiently fast and the 4th order expansion is largely satisfactory. The accuracy of our approximation is clearly superior to the lower bound of Broadie and Detemple (1996). The relative error does not exceed 0.2%, which is more than sufficient for applications. In the second part of Table 1 we compare different approximations of option prices with long time-to-maturity (3 years). The convergence of the series here is much slower. This is to be expected since we rely on a short-maturity expansion. Accuracy is still reasonably good even if we limit ourselves, for example, to a 5th order expansion with a relative error not exceeding half a percent. Our approximation based on the 5th order expansion is again more accurate than the lower bound of Broadie and Detemple (1996).

With respect to computational efficiency, our method is equivalent to the lower bound approximation of Broadie and Detemple (1996). Both methods involve similar maximization procedure, and formulas have comparable complexities. To give an idea of the computational speed, a Matlab code requires only 0.002 seconds to compute an approximation with a 4th order expansion. This is comparable to a 35-step binomial tree on the same computer. Note that Broadie and Detemple (1996) approach may still be preferable for pricing long maturity options. In this case, our approximation requires a higher order expansion, which increases the computational time.

Bunch and Johnson (2000) propose an alternative fast method for American option pricing based on an analytical approximation of the early exercise boundary (see Zhu and He (2007) for a modification of this approach better suited for approximating long term options). Table 2 reports option values from Bunch and Johnson (2000) (Table II) and our results based on a 4th order asymptotic expansion (see Appendix B). The accuracy of our approximation is comparable with that of Bunch and Johnson (2000), and is roughly equivalent to a 300-step

tree.

3 Three-factor model

The true power of the method introduced in the previous sections lies in the possibility of its extension to a more general model with stochastic volatility and stochastic interest rates. Let us consider the following risk-neutral dynamics:

$$\begin{aligned} dS_t &= (r_t - \delta)S_t dt + \sigma_t S_t dW_t^{(1)}, \\ d\sigma_t &= a(\sigma_t)dt + b(\sigma_t)dW_t^{(2)}, \\ dr_t &= \alpha(r_t, t)dt + \beta(r_t)dW_t^{(3)}, \end{aligned} \tag{20}$$

with $dW_t^{(i)} dW_t^{(j)} = \rho_{ij} dt$, $i, j = 1, 2, 3$. Model (20) nests most models used in applications. To allow for negative ρ_{12} and ρ_{23} covers potential leverage and flight-to-quality effects. The empirical literature on European option pricing tends to assume no correlation between stock prices and interest rates. Bakshi, Cao and Chen (1997) admit that this is a potentially limiting assumption (see footnote on page 2009), since economic theory suggests a negative correlation. As we will see in the numerical analysis below correlations ρ_{12} and ρ_{13} may have a sizable effect, while ρ_{23} does not.

3.1 Modified problem and its solution

The PDE for the put option price $\mathbf{P}(S, \sigma, r, t)$ is:

$$\begin{aligned} 0 &= \mathbf{P}_t + \mathbf{P}_S S(r - \delta) + \mathbf{P}_\sigma a(\sigma) + \mathbf{P}_r \alpha(r, t) + \frac{1}{2} \mathbf{P}_{SS} S^2 \sigma^2 + \frac{1}{2} \mathbf{P}_{\sigma\sigma} b^2(\sigma) \\ &\quad + \frac{1}{2} \mathbf{P}_{rr} \beta^2(r) + \mathbf{P}_{S\sigma} \sigma S b(\sigma) \rho_{12} + \mathbf{P}_{Sr} \sigma S \beta(\sigma) \rho_{13} + \mathbf{P}_{\sigma r} b(\sigma) \beta(\sigma) \rho_{23} - r \mathbf{P}, \end{aligned} \tag{21}$$

with the boundary conditions given in (2).

As in the Black-Scholes case, we skip the "smooth-pasting" condition, go from $\mathbf{P}(S, \sigma, r, t)$ to $P(\theta, \sigma, r, \tau)$, and derive a recursive system of PDEs to characterize the coefficients P_n , $n = 1, 2, \dots$, in the short-maturity asymptotic expansion of P .

Let us make the change of variables from (S, t) to $\theta = \frac{\ln(K/S)}{\sigma\sqrt{T-t}}$ and $\tau = T - t$, and make use of the following relationships:

$$\mathbf{P}_S = -\frac{1}{\sigma S \sqrt{\tau}} P_\theta, \quad \mathbf{P}_{SS} = \frac{1}{\sigma^2 S^2 \tau} P_{\theta\theta} + \frac{1}{\sigma S^2 \sqrt{\tau}} P_\theta, \quad \mathbf{P}_t = \frac{\theta}{2\tau} P_\theta - P_\tau, \quad \mathbf{P}_\sigma = P_\sigma - \frac{\theta}{\sigma} P_\theta,$$

$$\mathbf{P}_{\sigma S} = -\frac{1}{\sigma S\sqrt{\tau}}P_{\sigma\theta} + \frac{1}{\sigma^2 S\sqrt{\tau}}P_{\theta} + \frac{\theta}{\sigma^2 S\sqrt{\tau}}P_{\theta\theta}, \quad \mathbf{P}_{\sigma\sigma} = P_{\sigma\sigma} - \frac{2\theta}{\sigma}P_{\sigma\theta} + \frac{2\theta}{\sigma^2}P_{\theta} + \frac{\theta^2}{\sigma^2}P_{\theta\theta},$$

$$\alpha(r, t) = \alpha(r, T - \tau) = \alpha_0(r) + \tau\alpha_1(r) + \tau^2\alpha_2(r) \dots$$

This allows us to transform (21) into:

$$\begin{aligned} 0 &= \frac{\theta}{2}P_{\theta} + \frac{1}{2}P_{\theta\theta} - \tau P_{\tau} + \sqrt{\tau} \left[\frac{1}{2\sigma} (\sigma^2 + 2(\delta - r)) P_{\theta} + b\rho_{12} \left(-P_{\sigma\theta} + \frac{1}{\sigma}P_{\theta} + \frac{\theta}{\sigma}P_{\theta\theta} \right) \right. \\ &\quad \left. - \beta\rho_{13}P_{\theta r} \right] + \tau \left[a \left(P_{\sigma} - \frac{\theta}{\sigma}P_{\theta} \right) + b\beta\rho_{23} \left(P_{\sigma r} - \frac{\theta}{\sigma}P_{\theta r} \right) \right. \\ &\quad \left. + \frac{b^2}{2} \left(P_{\sigma\sigma} - \frac{2\theta}{\sigma}P_{\sigma\theta} + \frac{2\theta}{\sigma^2}P_{\theta} + \frac{\theta^2}{\sigma^2}P_{\theta\theta} \right) + \frac{\beta^2}{2}P_{rr} - rP + \alpha_0(r)P_r \right] \\ &\quad + \tau^2\alpha_1(r)P_r + \tau^3\alpha_2(r)P_r \dots \end{aligned} \quad (22)$$

Further, as in (14), let us consider an asymptotic expansion of the option price near maturity of the form:

$$P(\theta, \sigma, r, \tau) = P_1(\theta, \sigma, r, \tau)\sqrt{\tau} + P_2(\theta, \sigma, r, \tau)\tau + P_3(\theta, \sigma, r, \tau)\tau\sqrt{\tau} + \dots$$

Substituting this into (22), we obtain the following PDEs for P_n , $n = 1, 2, \dots$:

$$\begin{aligned} 0 &= P_{n\theta\theta} + \theta P_{n\theta} - nP_n + \frac{1}{\sigma} (\sigma^2 + 2(\delta - r)) P_{n-1\theta} + 2b\rho_{12} \left(-P_{n-1\sigma\theta} + \frac{1}{\sigma}P_{n-1\theta} + \frac{\theta}{\sigma}P_{n-1\theta\theta} \right) \\ &\quad - 2\beta\rho_{13}P_{n-1\theta r} + 2b\beta\rho_{23} \left(P_{n-2\sigma r} - \frac{\theta}{\sigma}P_{n-2\theta r} \right) + 2a \left(P_{n-2\sigma} - \frac{\theta}{\sigma}P_{n-2\theta} \right) + b^2 \left(P_{n-2\sigma\sigma} - \frac{2\theta}{\sigma}P_{n-2\sigma\theta} \right. \\ &\quad \left. + \frac{2\theta}{\sigma^2}P_{n-2\theta} + \frac{\theta^2}{\sigma^2}P_{n-2\theta\theta} \right) + P_{n-2rr}\beta^2 - 2rP_{n-2} + 2(\alpha_0(r)P_{n-2r} + \alpha_1(r)P_{n-4r} + \alpha_2(r)P_{n-6r} + \dots), \end{aligned} \quad (23)$$

with $P_m = 0$ for $m \leq 0$.

Proposition 3 describes the solution to this system of PDEs. We do not provide the proof of the proposition, which is lengthy but straightforward. The proof parallels the proof of Proposition 1. The results may be verified by a direct substitution of the solution with unknown coefficients.

Proposition 3 Consider partial differential equation (22), the boundary condition: $P(-\infty, \sigma, r, t) = 0$, and regular asymptotic expansion:

$$P(\theta, \sigma, r, \tau) = \sum_{n=1}^{\infty} P_n(\theta, \sigma, r)\tau^{\frac{n}{2}}, \quad (24)$$

with (θ, τ) in the vicinity of $(0, 0)$. For any solution to this problem there exist functions $C_1(\sigma, r)$, $C_2(\sigma, r)$, ... such that for each n : $P_n(\theta, \sigma, r) = C_n(\sigma, r) [p_n^0(\theta, \sigma, r)\Phi(\theta) + q_n^0(\theta, \sigma, r)\phi(\theta)] + p_n^1(\theta, \sigma, r)\Phi(\theta) + q_n^1(\theta, \sigma, r)\phi(\theta)$, where $p_n^0 \in \Pi^0(n, \theta)$, $p_n^1 \in \Pi^1(n - 2, \theta)$, $q_n^0 \in \Pi^0(n - 1, \theta)$, and $q_n^1 \in \Pi^1(3n - 5, \theta)$ with coefficients depending on

model parameters and $C_1(\sigma, r)$, $C_2(\sigma, r)$, ..., $C_{n-1}(\sigma, r)$.

To get the coefficients C_n we proceed in the same manner as in the Black-Scholes case. We impose an explicit early exercise rule by requiring that the put option is exercised as soon as it hits the barrier level $\theta = y$. This condition allows us to uniquely identify the coefficients $C_n(\sigma, r)$, $n = 1, 2, \dots$. Indeed, we equate expansion (24) evaluated at $\theta = y$ to the expansion of the put option payoff: $K [1 - \exp(-\sigma y \sqrt{\tau})] = \sigma y K \sqrt{\tau} - \frac{\sigma^2 y^2 K}{2} \tau + \dots$. After equating P_1 to $\sigma y K$ we determine $C_1(\sigma, r)$. Then we find p_2^0 , p_2^1 , q_2^0 and q_2^1 , and equating P_2 to $\frac{\sigma^2 y^2 K}{2}$, we obtain $C_2(\sigma, r)$. The recursive formulas for C_n can be obtained easily from a symbolic calculus software package and copy-pasted in a code for option pricing. As an illustrative example we give the 3rd order expansion of the solution to the modified problem under an affine three-factor model (see (28) below) in Appendix D. For the 5th order expansion the formula is much lengthier but is still straightforward to derive and implement. This is the formula we use in the numerical experiments of Section 3.3. We do not reproduce it here to save space.

Let us denote the put price with barrier level y by $P(\theta, \tau; y)$, and the American put price by $P(\theta, \tau)$, where we skip the dependencies on σ and r to ease notation. Then we have

$$P(\theta, \tau) \simeq P_1(\theta, \tau; \tilde{y}(\theta, \tau))\sqrt{\tau} + P_2(\theta, \tau; \tilde{y}(\theta, \tau))\tau + \dots, \quad (25)$$

where

$$\tilde{y}(\theta, \tau) = \arg \max_{y \geq \theta} P(\theta, \tau; y). \quad (26)$$

3.2 Approximation of the early exercise premium

Our numerical experiments suggest that the convergence of asymptotic expansion (24) under stochastic volatility is slower than in the Black-Scholes model. To improve our approximation we suggest using closed-form solutions for European option prices whenever they are available. Well-known examples are found in the class of affine and quadratic multifactor models, where European options can be valued quickly and accurately via the inverse Fourier transform (see e.g. Duffie, Pan and Singleton (2000), Leippold and Wu (2002), Cheng and Scaillet (2007)). Indeed, the American put option price is the sum of the European put price and the early exercise premium, and the approximation error can be decomposed in a similar way. If the two errors in such a decomposition are relatively independent then the accuracy of our approximation can be improved if European put option prices are available in closed-form expressions.

Recall that the European put price is the solution to the modified problem with $y = \infty$. Consequently, the early exercise premium is approximately given by $P(\theta, \tau; \tilde{y}(\theta, \tau)) - P(\theta, \tau; \infty)$, where $\tilde{y}(\theta, \tau)$ is defined in (26). Now suppose that the price of the European put is available. Then the American put price may be approximated

by

$$P(\theta, \tau) \simeq P(\theta, \tau; \infty) + [P_1(\theta; \tilde{y}(\theta, \tau)) - P_1(\theta; \infty)] \sqrt{\tau} + [P_2(\theta; \tilde{y}(\theta, \tau)) - P_2(\theta; \infty)] \tau + \dots \quad (27)$$

To distinguish between the two approximations presented so far, we refer to (25) as approximation 1 and to (27) as approximation 2.

3.3 Numerical analysis

3.3.1 Accuracy of the approximation

In this section we illustrate the accuracy of our method when both volatility and interest rates are stochastic. We assume the price S of the underlying asset to have the following risk-neutral dynamics:

$$\begin{aligned} dS_t &= (r_t - \delta)S_t dt + \sqrt{v_t}S_t dW_t^{(1)}, \\ dv_t &= \kappa_v(\bar{v} - v_t)dt + \sigma_v\sqrt{v_t}dW_t^{(2)}, \\ dr_t &= \kappa_r(\bar{r} - r_t)dt + \sigma_r\sqrt{r_t}dW_t^{(3)}, \end{aligned} \quad (28)$$

with $r_0 = \bar{r}$, $v_0 = \bar{v}$,⁴ and where interest rate r is uncorrelated with both the volatility and the price of the underlying asset, namely $\rho_{13} = \rho_{23} = 0$. The advantage of the affine three-factor model (28) is that it admits a closed-form solution for the European option price. The closed-form allows us to gauge the accuracy of our approach, to compare performance of approximation 1 versus approximation 2, and to assess potential bias in simulation-based approaches with discretized paths.

Table 3 reports the European and American put values for different combination of model parameters. The choice of parameter values is influenced by possible applications of our methodology, which include stock index and currency options. Empirical estimates of Bates (1996) and Carr and Wu (2007) for currency options suggest that the volatility mean-reversion κ_v does not exceed 1.5 and the volatility of volatility σ_v is below 0.3. This is also in line with the findings of Bakshi, Cao and Chen (2000) for stock index options. Currency markets are characterized by very low correlation between exchange rates and volatility, which is typically between -0.1 and 0.1 . On the contrary, stock indices are highly negatively correlated with volatility. We take this into account by introducing a set of parameters with a high negative correlation $\rho_{12} = -0.5$. The specification of the interest rate process is consistent with findings of Bakshi, Cao and Chen (2000).

American put prices are computed using the simulation-based approach of Longstaff and Schwartz (2001) with 1,000,000 sample paths⁵, 500 time steps, and 50 exercise dates. We use the Euler discretization method with

⁴This assumption is needed to make sure that r_t and σ_t are constant when $\sigma_r = \sigma_v = 0$, and that the Black-Scholes model is a natural benchmark for comparison purposes (see the next section for numerical examples).

⁵To reduce the variance, we generate 500'000 random paths of the price of the underlying asset, while the other 500'000 paths are

the full truncation method suggested by Lord, Koekoek and van Dijk (2009). To check for possible bias in the simulation-based results due to the truncation, we compare European put prices given by a closed-form solution (see Bakshi, Cao and Chen (2000)) and by the Monte-Carlo approach. Approximations 1 and 2 of American put prices are computed using a 5th order expansion with the search algorithm described in Section 2.5.1. To give an idea of the computational advantage of our method, a Matlab code implementing the algorithm of Longstaff and Schwartz (2001) takes dozens of minutes to compute a single option price while our approximation takes roughly a tenth of a second.

The comparison of European put prices based on Monte-Carlo simulations with those based on the closed-form solution suggests that, if any, the truncation bias is almost negligible. Indeed, in 33 out of 36 cases, the absolute difference between the European option prices does not exceed the standard error of the Monte-Carlo simulation.

Overall approximation 2 appears to be more accurate than approximation 1 in the valuation of American options. Both approximations provide accurate and similar option values for in-the-money ($K = 110$) and at-the-money put options ($K = 100$). The reported errors for approximation 2 relative to the simulated prices do not exceed half a percent, and can be as small as one basis point. The corresponding absolute errors are of a magnitude of 10^{-3} . For some out-of-the-money put options ($K = 90$) the two approximations lead to mild distortions in option valuation with approximation 2 being closer to the Monte-Carlo pricing. The reported relative errors for approximation 2 can be larger than two percent for a limited number of cases. For those cases absolute errors are, however, small in the range of $10^{-4} - 10^{-5}$.

To explain the observed differences yielded by the two approximations in Table 3, note that an approximation based on short-maturity asymptotics is essentially an expansion of the option price around the initial values of the stochastic factors. In the Black-Scholes model, where only the price of the underlying asset is stochastic, the short-maturity asymptotics expansion converges very quickly (see the analysis of the previous section). Such an approximation should be less accurate in models with multiple stochastic factors that are likely to move away from their initial values during the lifetime of the option. Consequently, we expect a short-maturity asymptotic approximation to be less accurate for out-of-the-money options, which have longer expected time-to-maturity. Here approximation 2 performs better as it relies on the exact European put price given by a closed-form formula.

The same reasoning also suggests that approximation errors of in-the-money put options should be lower than those of at-the-money put options. An observation of the relative errors in Table 3 confirms this conjecture. Based on this last evidence, we further guess that our methodology may prove to be very accurate in approximating the early exercise boundary. A combination of an approximation of the early exercise boundary and a Monte-Carlo simulation can result in a more accurate pricing scheme at the expense of a slight increase in computational time. We leave the development of such an approach for future research.

obtained by reflection.

3.3.2 Effect of stochastic volatility and stochastic interest rates

In this section we take advantage of our fast pricing algorithm to study the effects of stochastic volatility and stochastic interest rates on the American put price. In particular, we are interested in their effect on the early exercise premium. A simple alternative approximation approach to pricing American options in a multifactor setting is to compute the European option price using known closed-form solution while adding the early exercise premium evaluated under the Black-Scholes model.⁶ In this section we explain why such an approach may result in an economically significant mispricing. Apart from illustrating the advantage of our approach for option pricing, this study provides new insights on the key determinants of the American put price.

The effect of a change in model parameters on the American put price ($\Delta\mathbf{P}$) can be decomposed into the effect on the early exercise premium ($\Delta\mathbf{EEP}$) and the European put price ($\Delta\mathbf{P}^E$): $\Delta\mathbf{P} = \Delta\mathbf{EEP} + \Delta\mathbf{P}^E$. For a price S below the early exercise price, the American put price is equal to its payoff. Therefore, for deep-in-the-money options, $\Delta\mathbf{P} \simeq 0$ and $\Delta\mathbf{EEP} \simeq -\Delta\mathbf{P}^E$. This means that deep-in-the-money American put options are not affected by a model specification, while the effect on the early exercise premium is comparable with the impact on the European put price. For deep-out-of-money options, the early exercise premium is negligible and $\Delta\mathbf{P} \simeq \Delta\mathbf{P}^E$. This means that the model specification has an identical effect on both American and European put price. Taking into account these two extremes, we can guess that the effect of model specification on near-at-the-money American put options will be equal in sign but smaller in magnitude than the impact on European put options. Another justification for this guess is that the possibility of an early exercise reduces the expected lifetime of the put option, thus diminishing the impact of stochastic factors. The subsequent numerical analysis confirms our conjecture.

Consider a generalization of the affine specification of the previous section by allowing for an additional correlation ρ_{13} between the price of the underlying asset and the interest rate. The effect of introducing stochastic volatility and stochastic interest rates will be measured relative to the benchmark case of the Black-Scholes model, which corresponds to $\sigma_v = \sigma_r = 0$ in (28). Hereafter we assume zero correlation between volatility and interest rates ($\rho_{23} = 0$) since unreported results show that it appears to have a negligible impact on put prices.

Figure 2 illustrates the effect of generalizations of the Black-Scholes model on the put prices and the early exercise premium. Figure 2a shows the effect of the introduction of a stochastic volatility uncorrelated with the price of the underlying asset. The European put under uncorrelated volatility is equal to the expected Black-Scholes price with volatility being the variable to be integrated (Hull and White (1987)), and the European put price decreases due to the convexity of the Black-Scholes option price with respect to volatility. While American put prices also decrease, the magnitude of the impact is smaller as expected. The discrepancy between American

⁶We thank Liuren Wu for pointing out this approach sometimes used in practice.

and European prices increases as options go deeper in-the-money (the price of the underlying asset goes down), which is reflected by a larger impact on the early exercise premium. Here at-the-money $\Delta\mathbf{EEP}$ is 1.1% of the Black-Scholes price.

The comparison between Figures 2a and 2b shows the effect of having a negative correlation between volatility and the price of the underlying asset. Here we observe a well-known effect of the implied volatility skew: out-of-the-money put options become relatively more expensive. The difference between Figures 2b and 2c illustrates the impact of volatility mean-reversion. In our case with the spot variance v_0 equal to its long-term average \bar{v} , the volatility mean-reversion diminishes the impact of stochastic volatility on option prices. In practice, the long-term average is likely to be higher than the spot level due to the volatility risk premium, meaning that the volatility mean-reversion will tend to increase option prices. The comparison between Figures 2c and 2d reveals the additional effect of stochastic interest rates. Often an increase in domestic interest rates results in a decrease of the stock price or the price of the foreign currency. To emphasize this effect we assume a high negative correlation $\rho_{13} = -0.5$. The negative correlation between the price of the underlying asset and the spot interest rate implies that the states where the European put yields larger payoffs (low price) are discounted more heavily due to higher interest rates. As a result, put option prices decrease after the introduction of stochastic interest rates.

4 Concluding remarks

In this paper we have described a new approach to pricing American options in a general setting with stochastic volatility and stochastic interest rates. Although the analytical approximation is based on a short-maturity asymptotic expansion, it performs extremely well in the Black-Scholes context with time-to-maturity up to several years. Under stochastic volatility, the convergence of the asymptotic expansion is slower. This problem is dealt with by considering the approximation of the early exercise premium instead of the American put. Then the convergence is achieved much faster: across all moneyness degrees, the approximation remains accurate for options with time-to-maturity up to half a year. Using our method, we have run several numerical experiments to study the effect of model specification on the American put. We have found that effects stemming from stochastic volatility and stochastic interest rates can be substantial.

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APPENDIX A. Proof of Proposition 1.

Substituting (14) into (11) we arrive at:

$$-nP_n + \theta P_{n\theta} + P_{n\theta\theta} + \frac{1}{\sigma} [\sigma^2 + 2(\delta - r)] P_{n-1\theta} - 2rP_{n-2} = 0, \quad n = 1, 2, \dots \quad (29)$$

with $P_0 = P_{-1} = 0$. The homogeneous solutions of equation (29) form a two dimensional space. One dimension is spanned by a polynomial solution which does not satisfy the boundary condition (8) at infinity. The other independent solution has the form:

$$P_n^0(\theta) = p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta). \quad (30)$$

Let us substitute (30) in the homogeneous part of (29). After some rearrangements we find:

$$\left(\frac{d^2 P_n^0}{d\theta^2} + \theta \frac{d p_n^0}{d\theta} - n p_n^0 \right) \Phi(\theta) + \left(-(n+1)q_n^0 - \theta \frac{d q_n^0}{d\theta} + \frac{d^2 q_n^0}{d\theta^2} + 2 \frac{d p_n^0}{d\theta} \right) \phi(\theta) = 0.$$

It is easy to verify that PDE $\frac{d^2 P_n^0}{d\theta^2} + \theta \frac{d p_n^0}{d\theta} - n p_n^0 = 0$, has polynomial solution $p_n^0(\theta) = \pi_{n0}^0 \theta^n + \pi_{n1}^0 \theta^{n-2} + \pi_{n2}^0 \theta^{n-4} + \dots$, with $\pi_{n0}^0 = 1, \pi_{ni+1}^0 = \frac{(n-2i)(n-2i-1)}{2i+2} \pi_i^0$. The polynomial solution to $-(n+1)q_n^0 - \theta \frac{d q_n^0}{d\theta} + \frac{d^2 q_n^0}{d\theta^2} + 2 \frac{d p_n^0}{d\theta} = 0$, has the form $q_n^0(\theta) = \varkappa_{n0}^0 \theta^{n-1} + \varkappa_{n1}^0 \theta^{n-3} + \varkappa_{n2}^0 \theta^{n-5} + \dots$ with $\varkappa_{ni+1}^0 = \frac{\varkappa_{ni}^0 (n-1-2i)(n-2-2i) + 2\pi_{ni+1}^0 (n-2i-2)}{2n-2i-2}$.

Let us now find a particular solution P_n^1 of (29), which satisfies the boundary condition at infinity. Any solution of (29) with appropriate behavior at the boundary is given by $P_n(\theta) = C_n P_n^0(\theta) + P_n^1(\theta)$, where C_n is some constant. Let us look for a particular solution P_n^1 in the form $P_n^1(\theta) = p_n^1(\theta)\Phi(\theta) + q_n^1(\theta)\phi(\theta)$. This implies that the general solution is:

$$P_n(\theta) = C_n [p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta)] + p_n^1(\theta)\Phi(\theta) + q_n^1(\theta)\phi(\theta). \quad (31)$$

Let us guess that polynomials p_n^1 and q_n^1 are as follows:

$$p_n^1(\theta) = \pi_{n0}^1 \theta^n + \pi_{n1}^1 \theta^{n-2} + \pi_{n2}^1 \theta^{n-4} + \dots, \quad q_n^1(\theta) = \varkappa_{n0}^1 \theta^{n-1} + \varkappa_{n1}^1 \theta^{n-3} + \varkappa_{n2}^1 \theta^{n-5} + \dots$$

After substituting P_{n-1} and P_{n-2} of the form (31) into equation (29) for P_n^1 we obtain a system of two equations:

$$\frac{d^2 p_n^1}{d\theta^2} + \theta \frac{d p_n^1}{d\theta} - n p_n^1 + \tilde{\sigma} C_{n-1} \frac{d p_{n-1}^0}{d\theta} + \tilde{\sigma} \frac{d p_{n-1}^1}{d\theta} - 2r C_{n-2} p_{n-2}^0 - 2r p_{n-2}^1 = 0,$$

$$\begin{aligned}
& -(n+1)q_n^1 - \theta \frac{dq_n^1}{d\theta} + \frac{d^2 q_n^1}{d\theta^2} + 2 \frac{dp_n^1}{d\theta} + \tilde{\sigma} C_{n-1} p_{n-1}^0 + \tilde{\sigma} C_{n-1} \frac{dq_{n-1}^0}{d\theta} - \tilde{\sigma} C_{n-1} \theta q_{n-1}^0 \\
& + \tilde{\sigma} p_{n-1}^1 + \tilde{\sigma} \frac{dq_{n-1}^1}{d\theta} - \tilde{\sigma} \theta q_{n-1}^1 - 2r C_{n-2} q_{n-2}^0 - 2r C_{n-2} q_{n-2}^1 = 0,
\end{aligned}$$

where $\tilde{\sigma} = \frac{1}{\sigma} [\sigma^2 + 2(\delta - r)]$. These equations can be solved as before. In particular we may assume $\pi_{n0}^1 = \varkappa_{n0}^1 = 0$ since we can safely subtract a homogeneous solution. Here we do not write down the lengthy recursive relationship. In practice the PDE for P_n^1 can be solved directly by the substitution of its guessed form.

APPENDIX B. 4th order expansion of the solution to the modified problem under the Black-Scholes model.

The solution to the modified problem has the 4th order short-maturity expansion:

$$P(\theta, \tau) = \sum_{n=1}^4 \tau^{\frac{n}{2}} \{ C_n [p_n^0(\theta) \Phi(\theta) + q_n^0(\theta) \phi(\theta)] + p_n^1(\theta) \Phi(\theta) + q_n^1(\theta) \phi(\theta) \},$$

where

$$p_1^0(\theta) = \theta, \quad p_1^1(\theta) = 0, \quad q_1^0(\theta) = 1, \quad q_1^1(\theta) = 0,$$

$$p_2^0(\theta) = \theta^2 + 1, \quad p_2^1(\theta) = \frac{1}{2\sigma} C_1 (\sigma^2 - 2\mu), \quad q_2^0(\theta) = \theta, \quad q_2^1(\theta) = 0,$$

$$p_3^0(\theta) = \theta^3 + 3\theta, \quad p_3^1(\theta) = \frac{1}{\sigma} [C_2 \sigma^2 - 2C_2 \mu - r C_1 \sigma] \theta, \quad q_3^0 = \theta^2 + 2,$$

$$q_3^1(\theta) = \frac{1}{8\sigma^2} ([8C_2 \sigma^3 - 16C_2 \sigma \mu - 8r C_1 \sigma^2 - 4C_1 \sigma^2 \mu + C_1 \sigma^4 + 4C_1 \mu^2]),$$

$$p_4^0(\theta) = \theta^4 + 6\theta^2 + 3,$$

$$\begin{aligned}
p_4^1(\theta) &= \frac{1}{2\sigma} [3C_3 \sigma^2 - 6C_3 \mu - 2r\sigma C_2] \theta^2 \\
&+ \frac{1}{4\sigma^2} [3C_3 \sigma^3 + \sigma^4 C_2 - 6C_3 \sigma \mu - \sigma (-3C_3 \sigma^2 + 6C_3 \mu + 2r\sigma C_2) \\
&\quad + 4C_2 \mu^2 - 2\sigma^3 r C_1 - 4\sigma^2 C_2 \mu - 2r\sigma^2 C_2 + 4\sigma r C_1 \mu],
\end{aligned}$$

$$q_4^0(\theta) = \theta^3 + 5\theta,$$

$$q_4^1(\theta) = \frac{1}{48\sigma^3} [8C_1 \mu^3 + 72C_3 \sigma^4 - C_1 \sigma^6 - 48r\sigma^3 C_2 + 6C_1 \sigma^4 \mu - 12C_1 \sigma^2 \mu^2 - 144C_3 \sigma^2 \mu] \theta,$$

and $C_1 = (Ky\sigma)(\Phi_0y + \phi_0)^{-1}$, $C_2 = -(\Phi_0C_1\sigma^2 - 2\Phi_0C_1\mu + Ky^2\sigma^3) [2\sigma(\Phi_0y^2 + \Phi_0 + \phi_0y)]^{-1}$,

$$\begin{aligned} C_3 &= [24\sigma^2(\Phi_0y^3 + 3\Phi_0y + \phi_0y^2 + 2\phi_0)]^{-1} \\ &\times (-24\Phi_0y\sigma^3C_2 + 48\Phi_0y\sigma C_2\mu + 24\Phi_0y\sigma^2rC_1 \\ &- 24\phi_0C_2\sigma^3 + 48\phi_0C_2\sigma\mu + 24\phi_0rC_1\sigma^2 \\ &+ 12\phi_0C_1\sigma^2\mu - 3\phi_0C_1\sigma^4 - 12\phi_0C_1\mu^2 + 4Ky^3\sigma^5), \end{aligned}$$

$$\begin{aligned} C_4 &= -[48\sigma^3(\Phi_0y^4 + 6\Phi_0y^2 + 3\Phi_0 + \phi_0y^3 + 5\phi_0y)]^{-1} \\ &\times (72\Phi_0\sigma^4y^2C_3 - 144\Phi_0\sigma^2y^2C_3\mu - 48\Phi_0\sigma^3y^2rC_2 + 48\Phi_0\sigma C_2\mu^2 \\ &+ 12\Phi_0\sigma^5C_2 + 72\Phi_0\sigma^4C_3 - 144\Phi_0\sigma^2C_3\mu - 48\Phi_0\sigma^3rC_2 - 24\Phi_0\sigma^4rC_1 \\ &- 48\Phi_0\sigma^3C_2\mu + 48\Phi_0\sigma^2rC_1\mu + 8\phi_0yC_1\mu^3 + 72\phi_0yC_3\sigma^4 - \phi_0yC_1\sigma^6 \\ &- 48\phi_0yr\sigma^3C_2 - 12\phi_0yC_1\mu^2\sigma^2 - 144\phi_0yC_3\sigma^2\mu + 6\phi_0yC_1\sigma^4\mu + 2Ky^4\sigma^7), \end{aligned}$$

with $\mu = r - \delta$, $\Phi_0 = \Phi(y)$, $\phi_0 = \phi(y)$.

APPENDIX C. Proof of Proposition 2.

By definition of the barrier put option, its value at the early exercise boundary is equal to the payoff g . Hence, $P(\theta, \tau; \theta) = K(1 - e^{-\sigma\theta\sqrt{\tau}}) = g(\theta, \tau, K)$, which implies that $P_\theta(\tilde{\theta}, \tau; \tilde{\theta}) + P_y(\tilde{\theta}, \tau; \tilde{\theta}) = g_\theta(\tilde{\theta}, \tau, K)$. Here and further in the proof the subscripted θ refers to the left derivative with respect to θ . Recall that $\tilde{\theta}$ is an argument of the maximum of $P(\tilde{\theta}, \tau; y)$ as a function of y . From (19) by continuity we have:

$$P_y(\tilde{\theta}, \tau; \tilde{\theta}) = 0, \tag{32}$$

which yields the first result of the proposition: $P_\theta(\tilde{\theta}, \tau; \tilde{\theta}) = g_\theta(\tilde{\theta}, \tau, K)$.

Using the notations of Appendix A, we can write the barrier option price as: $P(\theta, \tau; y) = \frac{\sigma y K}{y\Phi(y) + \phi(y)} (\theta\Phi(\theta) + \phi(\theta)) \sqrt{\tau} + R(\theta, \tau; y)\tau$, where we have used the notation $R(\theta, \tau; y) = \sum_{n=2}^{\infty} \tau^{\frac{n-2}{2}} \{C_n(y) [p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta)] + p_n^1(\theta)\Phi(\theta) + q_n^1(\theta)\phi(\theta)\}$. The first order condition (32) implies:

$$\frac{\sigma K \phi(\tilde{\theta})}{\tilde{\theta}\Phi(\tilde{\theta}) + \phi(\tilde{\theta})} = -R_y(\tilde{\theta}, \tau; \tilde{\theta})\sqrt{\tau}, \tag{33}$$

with

$$R_y(\tilde{\theta}, \tau; \tilde{\theta}) = \sum_{n=2}^{\infty} \tau^{\frac{n-2}{2}} \frac{\partial}{\partial y} \left\{ C_n(y) \left[p_n^0(\tilde{\theta}) \Phi(\tilde{\theta}) + q_n^0(\tilde{\theta}) \phi(\tilde{\theta}) \right] + p_n^1(\tilde{\theta}) \Phi(\tilde{\theta}) + q_n^1(\tilde{\theta}) \phi(\tilde{\theta}) \right\}. \quad (34)$$

Using the expressions given in Appendix A we observe that the leading term in (34) has asymptotics of order $\tilde{\theta}^{-1}$. The other terms in expansion (34) converge to zero faster than $\tilde{\theta}^{-1}$ due to the multiplication by a positive power of τ and the slow growth of $\tilde{\theta}(\tau) \leq \bar{\theta}(\tau) \sim \sqrt{\ln(1/\tau)}$. Substituting $R_y(\tilde{\theta}, \tau; \tilde{\theta}) \sim \tilde{\theta}^{-1}$ in (33) yields $\frac{\sigma K \tilde{\theta} \phi(\tilde{\theta})}{\tilde{\theta} \Phi(\tilde{\theta}) + \phi(\tilde{\theta})} \sim \sqrt{\tau}$. Using the approximation $\Phi(\tilde{\theta}) = 1 - \phi(\tilde{\theta}) \tilde{\theta}^{-1} + O(\tilde{\theta}^{-2})$ we have $\sigma K \phi(\tilde{\theta}) \sim \sqrt{\tau}$, which gives the second result $\tilde{\theta} \sim \sqrt{\ln(1/\tau)}$ of the proposition.

APPENDIX D. 3rd order expansion of the solution to the modified problem under an affine three-factor model.

Let us consider the affine three-factor model (28) with $\rho_{23} = 0$, i.e., an Heston model with stochastic interest rates. With condensed notations the solution to the modified problem has the 3rd order short-maturity expansion:

$$\begin{aligned} P = & [\theta C_1 \Phi + C_1 \phi] \sqrt{\tau} + \frac{1}{2\sigma} \left[(2C_2 \sigma \theta^2 + 2C_2 \sigma + C_1 \sigma^2 - 2C_1 q) \Phi + \theta (2C_2 \sigma + \frac{1}{2} \sigma_v \rho_{12} C_1) \phi \right] \tau \\ & + \left[-\frac{\theta}{\sigma} (-C_3 \sigma \theta^2 - 3C_3 \sigma + 2\sigma_r \sqrt{r} \rho_{13} \sigma \frac{\partial C_2}{\partial r} + 2C_2 q - 2\sigma_v \rho_{12} C_2 + \sigma_v \rho_{12} \frac{\partial C_2}{\partial \sigma} \sigma + r C_1 \sigma - C_2 \sigma^2) \Phi \right. \\ & + \frac{1}{24\sigma^2} \left(24C_3 \sigma^2 \theta^2 + 48C_3 \sigma^2 + \frac{3}{4} \sigma_v^2 \rho_{12}^2 C_1 \theta^4 - 3\sigma_v C_1 \theta^2 \rho_{12} \sigma^2 - \frac{5}{2} \sigma_v^2 C_1 \theta^2 \rho_{12}^2 + 6\sigma_v C_1 \theta^2 \rho_{12} q \right. \\ & + \sigma_v^2 C_1 \theta^2 + 12C_1 q^2 - 12C_1 \sigma^2 q + 12a\sigma C_1 - 24r C_1 \sigma^2 + \frac{1}{2} \sigma_v^2 C_1 - 6\sigma_v \rho_{12} C_1 q + 12\sigma_r \sqrt{r} \rho_{13} \sigma C_1 \\ & \left. - 24\sigma_v \rho_{12} \frac{\partial C_2}{\partial \sigma} \sigma^2 + 36\sigma_v \rho_{12} C_2 \sigma - 48\sigma_r \sqrt{r} \rho_{13} \sigma^2 \frac{\partial C_2}{\partial r} - 48C_2 \sigma q + 24C_2 \sigma^3 + 3C_1 \sigma^4 + \frac{1}{4} \sigma_v^2 \rho_{12}^2 C_1 \right. \\ & \left. - 3\sigma_v \rho_{12} C_1 \sigma^2 \right) \phi \left. \right] \tau \sqrt{\tau} + O(\tau^2), \end{aligned}$$

$$\text{where } C_1 = Ky\sigma / (\Phi_0 y + \phi_0), \quad C_2 = \frac{1}{2\sigma} (-\Phi_0 C_1 \sigma^2 + 2\Phi_0 C_1 q - \frac{1}{2} \phi_0 y \sigma_v \rho_{12} C_1 - Ky^2 \sigma^3) / (\Phi_0 y^2 + \Phi_0 + \phi_0 y),$$

$$\begin{aligned} C_3 = & \frac{1}{24\sigma^2} (-3\Phi_0 C_1 \sigma^4 + 24\Phi_0 y \sigma_v \rho_{12} \frac{\partial C_2}{\partial \sigma} \sigma^2 + 24\Phi_0 \sigma_v \rho_{12} \frac{\partial C_2}{\partial \sigma} \sigma^2 + 24\Phi_0 r C_1 \sigma^2 - \Phi_0 \sigma_v^2 y^2 C_1 \\ & - 24\Phi_0 y C_2 \sigma^3 + 24\Phi_0 y r C_1 \sigma^2 - \frac{3}{4} \Phi_0 \sigma_v^2 \rho_{12}^2 C_1 y^4 + 3\Phi_0 \sigma_v y^2 \rho_{12} C_1 \sigma^2 - 24\Phi_0 C_2 \sigma^3 \\ & - 48\Phi_0 y \sigma_v \rho_{12} C_2 \sigma - 36\Phi_0 \sigma_v \rho_{12} C_2 \sigma - 12\Phi_0 a \sigma C_1 - 6\Phi_0 \sigma_v y^2 \rho_{12} C_1 q + 48\Phi_0 y C_2 \sigma q \\ & + 6\Phi_0 \sigma_v \rho_{12} C_1 q + 48\Phi_0 y \sigma_r \sqrt{r} \rho_{13} \sigma^2 \frac{\partial C_2}{\partial r} + 48\Phi_0 \sigma_r \sqrt{r} \rho_{13} \sigma^2 \frac{\partial C_2}{\partial r} - 12\Phi_0 C_1 q^2 \\ & - 12\Phi_0 \sigma_r \sqrt{r} \rho_{13} \sigma C_1 + 12\Phi_0 C_1 \sigma^2 q + 48\Phi_0 C_2 \sigma q - \frac{1}{2} \Phi_0 \sigma_v^2 C_1 + 3\Phi_0 \sigma_v \rho_{12} C_1 \sigma^2 \\ & - \frac{1}{4} \Phi_0 \sigma_v^2 \rho_{12}^2 C_1 + \frac{5}{2} \Phi_0 \sigma_v^2 y^2 \rho_{12}^2 C_1 + 4Ky^3 \sigma^5) / (\Phi_0 y^3 + 3\Phi_0 y + \Phi_0 y^2 + 2\Phi_0), \end{aligned}$$

$$\text{and } q = r - \delta, \quad a = \frac{\kappa_v (\bar{v} - v) - \sigma_v^2 / 4}{2\sigma}, \quad \phi_0 = \phi(y), \quad \Phi_0 = \Phi(y).$$

Table 1. American call option prices and their approximations under the Black-Scholes model.

The table compares option price bounds of Broadie and Detemple (1996) with our approximation based on asymptotic expansions of different orders. Broadie&Detemple refers to option price lower bounds reported in Tables 1 and 2 of Broadie and Detemple (1996). “True value” is a 15’000-step binomial tree approximation computed in Broadie and Detemple (1996). Here all options have strike price $K = 100$. Time-to-maturity τ , asset price S , interest rate r , volatility σ , and dividend rate δ are indicated in the table.

Option parameters S	$r = 0.03, \sigma = 0.2, \delta = 0.07$					$r = 0.03, \sigma = 0.4, \delta = 0.07$					$r = 0, \sigma = 0.3, \delta = 0.07$					$r = 0.07, \sigma = 0.3, \delta = 0.03$				
	80	90	100	110	120	80	90	100	110	120	80	90	100	110	120	80	90	100	110	120
$\tau = 0.5$ years																				
European	0.215	1.345	4.578	10.421	18.302	2.651	5.622	10.021	15.768	22.650	1.006	3.004	6.694	12.166	19.155	1.664	4.495	9.251	15.798	23.706
4th order	0.219	1.384	4.774	11.083	20.000	2.686	5.718	10.228	16.166	23.341	1.036	3.118	7.022	12.934	20.696	1.665	4.495	9.248	15.792	23.698
Broadie&Detemple	0.218	1.376	4.750	11.049	20.000	2.676	5.694	10.190	16.110	23.271	1.029	3.098	6.985	12.882	20.650	1.664	4.495	9.251	15.798	23.706
True value	0.219	1.386	4.783	11.098	20.000	2.689	5.722	10.239	16.181	23.360	1.037	3.123	7.035	12.955	20.717	1.664	4.495	9.251	15.798	23.706
$\tau = 3$ years																				
European	2.241	4.355	7.386	11.331	16.117	10.309	14.162	18.532	23.363	28.598	4.644	7.269	10.542	14.430	18.882	12.133	17.343	23.301	29.882	36.973
4th order	2.552	5.108	8.974	14.333	21.320	11.157	15.518	20.563	26.249	32.532	5.460	8.752	13.023	18.313	24.644	12.082	17.200	23.052	29.536	36.560
5th order	2.574	5.148	9.028	14.388	21.358	11.291	15.676	20.735	26.424	32.698	5.496	8.802	13.082	18.375	24.702	12.089	17.288	23.249	29.853	36.991
Broadie&Detemple	2.553	5.121	9.002	14.371	21.354	11.238	15.609	20.656	26.337	32.607	5.463	8.766	13.048	18.347	24.685	12.145	17.367	23.347	29.961	37.099
True value	2.580	5.167	9.066	14.443	21.414	11.326	15.722	20.793	26.495	32.781	5.518	8.842	13.142	18.453	24.791	12.145	17.369	23.348	29.964	37.104

Table 2. Put option prices and their approximations under the Black-Scholes model.

The table compares the approach of Bunch and Johnson (2000) with our approximation based on a 4th order asymptotic expansion. Bunch&Johnson refers to results reported in Table II of Bunch and Johnson (2000). "True value" is a 10,000-step binomial tree approximation computed in Bunch and Johnson (2000). Here asset price $S = 40$, interest rate $r = 0.0488$ and dividend yield is zero. Time-to-maturity τ and asset price volatility σ are indicated in the table.

Method	$\sigma = 0.2$			$\sigma = 0.3$			$\sigma = 0.4$		
	$\tau = 1/12$	$\tau = 1/3$	$\tau = 7/12$	$\tau = 1/12$	$\tau = 1/3$	$\tau = 7/12$	$\tau = 1/12$	$\tau = 1/3$	$\tau = 7/12$
$K = 35$									
European put	0.006	0.196	0.417	0.077	0.687	1.189	0.246	1.330	2.113
4th order	0.006	0.200	0.432	0.077	0.697	1.218	0.247	1.345	2.152
Bunch & Johnson	0.006	0.200	0.433	0.077	0.698	1.229	0.247	1.347	2.153
300-step tree	0.006	0.200	0.434	0.078	0.698	1.220	0.247	1.348	2.157
True value	0.006	0.200	0.433	0.077	0.698	1.220	0.247	1.346	2.155
$K = 40$									
European put	0.840	1.522	1.881	1.299	2.428	3.064	1.758	3.334	4.247
4th order	0.852	1.578	1.986	1.310	2.481	3.165	1.768	3.386	4.349
Bunch & Johnson	0.853	1.581	1.992	1.310	2.484	3.171	1.769	3.389	4.354
300-step tree	0.853	1.581	1.990	1.310	2.482	3.169	1.769	3.391	4.357
True value	0.852	1.580	1.990	1.310	2.483	3.170	1.768	3.387	4.353
$K = 45$									
European put	4.840	4.780	4.840	4.980	5.529	5.972	5.236	6.377	7.166
4th order	5.021	5.085	5.261	5.059	5.702	6.237	5.286	6.506	7.376
Bunch & Johnson	5.002	5.091	5.265	5.062	5.708	6.244	5.289	6.512	7.385
300-step tree	5.000	5.088	5.267	5.060	5.706	6.246	5.286	6.511	7.383
True value	5.000	5.088	5.267	5.060	5.706	6.244	5.287	6.510	7.383

Table 3. Put option prices under stochastic volatility and stochastic interest rates

Option prices are computed for an affine model of the asset price with stochastic volatility and stochastic interest rates:

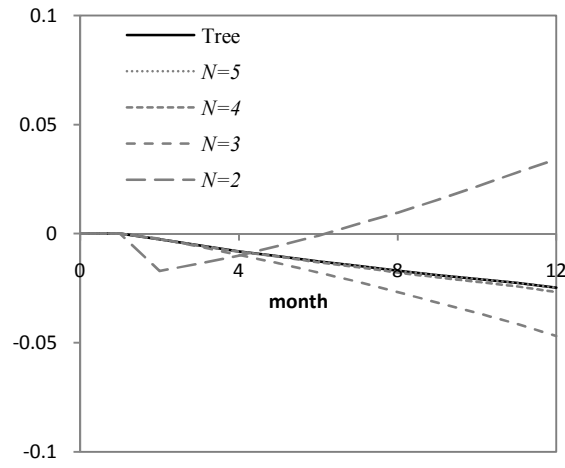
$$dS_t = r_t S_t dt + \sqrt{v_t} S_t dW_t^{(1)},$$

$$dv_t = \kappa_v (0.02 - v_t) dt + \sigma_v \sqrt{v_t} dW_t^{(2)},$$

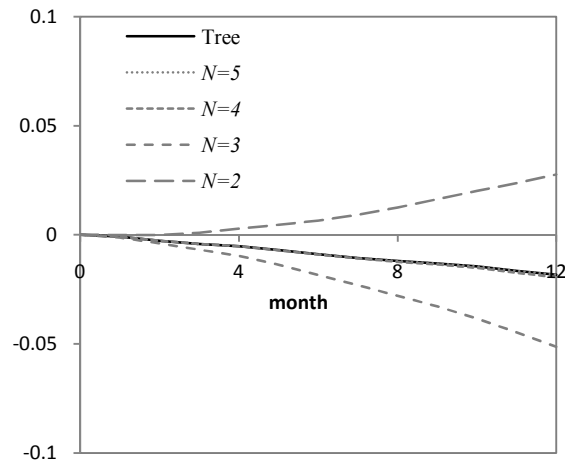
$$dr_t = 0.3(0.04 - r_t) + 0.1 \sqrt{r_t} dW_t^{(3)},$$

with $\rho_{13} = \rho_{23} = 0$. Asset price $S = 100$ and spot interest rate $r = 0.04$. Strike price K , time-to-maturity τ , volatility mean-reversion parameter κ_v , volatility of volatility σ_v , and correlation ρ_{12} take different values. Monte-Carlo refers to the Longstaff and Schwartz (2000) algorithm with 1,000,000 sample paths, 500 time steps, and 50 exercise dates (see the text for more details). Standard errors are shown in parenthesis. European put prices are also computed using a closed-form solution to assess the extent of the bias of Monte-Carlo results due to truncation of the volatility process at zero. Approximations 1 and 2 are computed with a 5th order asymptotic expansion.

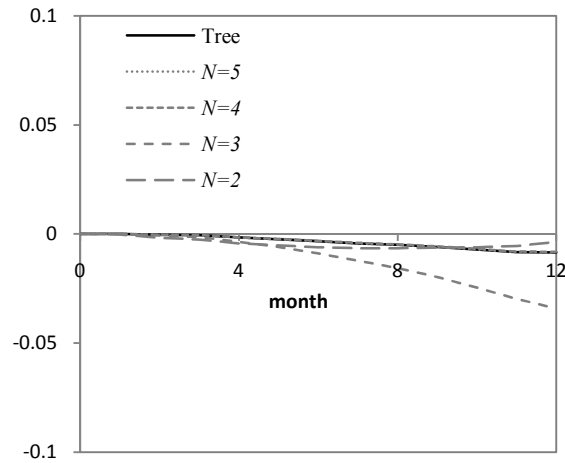
Method	K = 90			K = 100			K = 110		
	$\tau = 1/12$	$\tau = 1/4$	$\tau = 1/2$	$\tau = 1/12$	$\tau = 1/4$	$\tau = 1/2$	$\tau = 1/12$	$\tau = 1/4$	$\tau = 1/2$
$v = 0.01, \kappa_v = 1.5, \sigma_v = 0.15, \rho_{12} = 0.1.$									
European put (Monte-Carlo)	0.0001 (0.0000)	0.0330 (0.0003)	0.1959 (0.0011)	1.0162 (0.0016)	1.6504 (0.0027)	2.2282 (0.0084)	9.6366 (0.0030)	9.0709 (0.0050)	8.6375 (0.0066)
European put (Closed form)	0.0001	0.0335	0.1965	1.0160	1.6492	2.2254	9.6358	9.0701	8.6410
American put (Monte-Carlo)	0.0001 (0.0000)	0.0346 (0.0003)	0.2040 (0.0010)	1.0438 (0.0013)	1.7379 (0.0023)	2.3951 (0.0031)	9.9950 (0.0005)	9.9823 (0.0009)	9.9796 (0.0021)
American put (Approx. 1)	0.0002	0.0303	0.1694	1.0416	1.7332	2.3989	10.000	10.000	10.000
American put (Approx. 2)	0.0001	0.0342	0.2054	1.0415	1.7319	2.3891	10.000	10.000	10.000
Relative Error (Approx. 2)	21.54%	1.16%	0.69%	0.22%	0.35%	0.25%	0.05%	0.18%	0.20%
$v = 0.04, \kappa_v = 0.75, \sigma_v = 0.3, \rho_{12} = 0.1.$									
European put (Monte-Carlo)	0.0602 (0.0005)	0.5188 (0.0019)	1.1417 (0.0034)	2.1021 (0.0031)	3.3195 (0.0050)	4.2042 (0.0066)	9.7901 (0.0054)	10.0212 (0.0079)	10.3198 (0.0096)
European put (Closed form)	0.0603	0.5205	1.1439	2.1009	3.3156	4.1999	9.7904	10.0156	10.3200
American put (Monte-Carlo)	0.0619 (0.0004)	0.5303 (0.0018)	1.1824 (0.0032)	2.1306 (0.0027)	3.4173 (0.0043)	4.4249 (0.0057)	10.0386 (0.0023)	10.4271 (0.0052)	11.0224 (0.0070)
American put (Approx. 1)	0.0608	0.5321	1.1984	2.1249	3.4054	4.4088	10.0135	10.4249	11.0342
American put (Approx. 2)	0.0606	0.5283	1.1788	2.1249	3.4060	4.4139	10.0137	10.4233	11.0178
Relative Error (Approx. 2)	2.10%	0.38%	0.30%	0.27%	0.33%	0.25%	0.25%	0.04%	0.04%
$v = 0.04, \kappa_v = 1.5, \sigma_v = 0.3, \rho_{12} = 0.1.$									
European put (Monte-Carlo)	0.0576 (0.0004)	0.4837 (0.0018)	1.0359 (0.0032)	2.0851 (0.0031)	3.2480 (0.0049)	4.0509 (0.0063)	9.7846 (0.0054)	9.9647 (0.0078)	10.1669 (0.0094)
European put (Closed form)	0.0577	0.4849	1.0383	2.0844	3.2441	4.0467	9.7850	9.9594	10.1657
American put (Monte-Carlo)	0.0592 (0.0004)	0.4950 (0.0017)	1.0752 (0.0029)	2.1138 (0.0026)	3.3478 (0.0042)	4.2732 (0.0055)	10.0372 (0.0023)	10.3825 (0.0050)	10.8964 (0.0067)
American put (Approx. 1)	0.0582	0.4999	1.1243	2.1088	3.3381	4.2848	10.0116	10.3860	10.9363
American put (Approx. 2)	0.0580	0.4925	1.0717	2.1087	3.3368	4.2659	10.0119	10.3825	10.8950
Relative Error (Approx. 2)	2.03%	0.51%	0.33%	0.24%	0.33%	0.17%	0.25%	0.00%	0.01%
$v = 0.04, \kappa_v = 1.5, \sigma_v = 0.15, \rho_{12} = -0.5.$									
European put (Monte-Carlo)	0.0765 (0.0005)	0.5887 (0.0021)	1.2482 (0.0036)	2.0996 (0.0032)	3.3164 (0.0052)	4.2165 (0.0068)	9.7403 (0.0054)	9.8117 (0.0081)	9.9901 (0.0099)
European put (Closed form)	0.0767	0.5903	1.2490	2.0998	3.3147	4.2085	9.7405	9.8073	9.9877
American put (Monte-Carlo)	0.0787 (0.0005)	0.6012 (0.0019)	1.2896 (0.0032)	2.1277 (0.0027)	3.4089 (0.0044)	4.4103 (0.0056)	10.0198 (0.0019)	10.2512 (0.0047)	10.6988 (0.0064)
American put (Approx. 1)	0.0771	0.6000	1.3039	2.1221	3.3970	4.4139	10.000	10.2442	10.7069
American put (Approx. 2)	0.0771	0.5982	1.2823	2.1221	3.3951	4.3927	10.000	10.2432	10.6935
Relative Error (Approx. 2)	2.03%	0.50%	0.57%	0.26%	0.40%	0.40%	0.20%	0.08%	0.05%



(a) $S = 90$



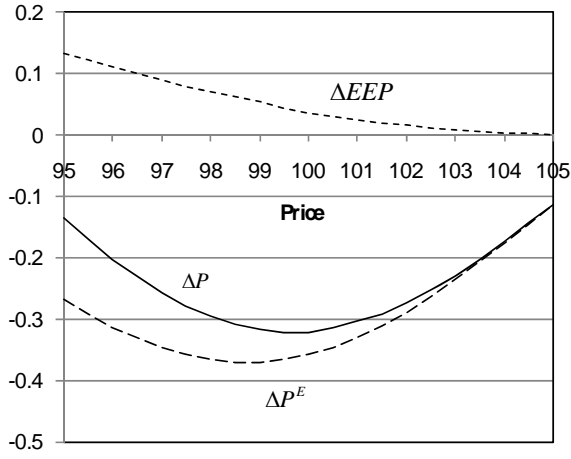
(b) $S = 100$



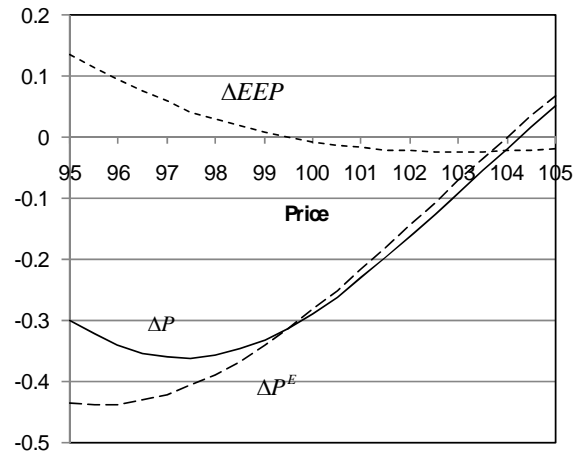
(c) $S = 110$

Figure 1. Convergence of the asymptotic expansion in the Black-Scholes model

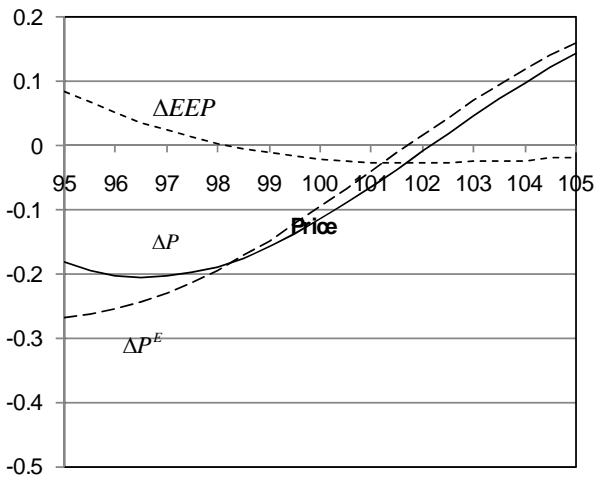
Each graph shows absolute approximation errors of our method based on different orders of asymptotic expansion ($N = 2, 3, 4, 5$). “Tree” refers to the errors of our approximation with barrier option prices being computed on a 2000-step binomial tree instead of asymptotic expansions. Reference American put prices are computed on a 2000-step binomial tree. The Black-Scholes model parameters are $r = 0.05$, $\delta = 0$, $\sigma = 0.2$, $K = 100$. The time unit is one year.



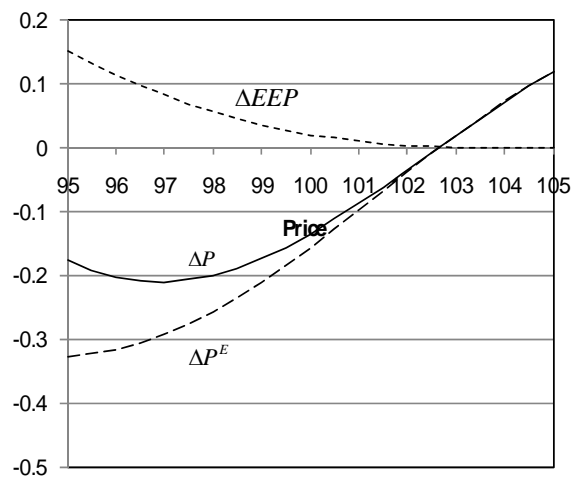
(a) $\kappa_v = 0, \sigma_v = 0.3, \rho_{12} = 0, \kappa_r = 0, \sigma_r = 0, \rho_{13} = 0$



(b) $\kappa_v = 0, \sigma_v = 0.3, \rho_{12} = -0.25, \kappa_r = 0, \sigma_r = 0, \rho_{13} = 0$



(c) $\kappa_v = 1.5, \sigma_v = 0.3, \rho_{12} = -0.25, \kappa_r = 0, \sigma_r = 0, \rho_{13} = 0$



(d) $\kappa_v = 1.5, \sigma_v = 0.3, \rho_{12} = -0.25, \kappa_r = 0.3, \sigma_r = 0.1, \rho_{13} = -0.5$

Figure 2. Impact of stochastic volatility and stochastic interest rates

Graphs show the impact of variation of model parameters on the early exercise premium (ΔEEP), the European put price (ΔP^E) and the American put price (ΔP) relative to the Black-Scholes model. The general model set-up is:

$$\begin{aligned} dS_t &= r_t S_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \\ dv_t &= \kappa_v (0.02 - v_t) dt + \sigma_v \sqrt{v_t} dW_t^{(2)}, \\ dr_t &= \kappa_r (0.04 - r_t) + \sigma_r \sqrt{r_t} dW_t^{(3)}, \\ dW_t^{(1)} dW_t^{(2)} &= \rho_{12} dt, \quad dW_t^{(1)} dW_t^{(3)} = \rho_{13} dt, \quad v_0 = 0.02, \quad r_0 = 0.04, \end{aligned}$$

with uncorrelated $W_t^{(2)}$ and $W_t^{(3)}$. Time-to-maturity is 6 months, strike price $K = 100$. Graph (a) shows the effect of the introduction of stochastic volatility uncorrelated with the price of the underlying asset. The comparison between Graphs (a) and (b) shows the effect of having a negative correlation between volatility and the price of the underlying asset. The difference between Graphs (b) and (c) illustrates the impact of volatility mean-reversion. The comparison between Graphs (c) and (d) reveals the additional effect of stochastic interest rates.