TESTING FOR STOCHASTIC DOMINANCE EFFICIENCY

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Abstract

We consider consistent tests for stochastic dominance efficiency at any order of a given portfolio with respect to all possible portfolios constructed from a set of assets. We justify block bootstrap approaches to achieve valid inference in a time series setting. The test statistics are computed using linear and mixed integer programming formulations. Monte Carlo results show that the bootstrap procedure performs well in finite samples. The empirical application reveals that the Fama and French market portfolio is first and second order stochastic dominance efficient, although it is mean-variance inefficient.

Key words and phrases: Nonparametric, Stochastic Ordering, Dominance Efficiency, Linear Programming, Mixed Integer Programming, Block Bootstrap.

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1 Introduction

Stochastic dominance is a central theme in a wide variety of applications in economics, finance and statistics, see e.g. the review papers by Kroll and Levy (1980) and Levy (1992), the classified bibliography by Mosler and Scarsini (1993), and the books by Shaked and Shanthikumar (1994) and Levy (1998). It aims at comparing random variables in the sense of stochastic orderings expressing the common preferences of rational decision-makers. Stochastic orderings are binary relations defined on classes of probability distributions. They translate mathematically intuitive ideas like "being larger" or "being more variable" for random quantities. They extend the classical mean-variance approach to compare riskiness.

The main attractiveness of the stochastic dominance approach is that it is non-parametric, in the sense that its criteria do not impose explicit specifications of an investor preferences or restrictions on the functional forms of probability distributions. Rather, they rely only on general preference and belief assumptions. Thus, no specification is made about the return distribution, and the empirical distribution estimates the underlying unknown distribution.

Traditionally, stochastic dominance is tested pairwise. Only recently Kuosmanen (2004) and Post (2003) have introduced the notion of stochastic dominance efficiency. This notion is a direct extension of stochastic dominance to the case where full diversification is allowed. In that setting both authors derive statistics to test for stochastic dominance efficiency of a given portfolio with respect to all possible portfolios constructed from a set of financial assets. Such a derivation relies intrinsically on using ranked observations under an i.i.d. assumption on the asset returns. Contrary to the initial observations, ranked observations, i.e., order statistics, are no more i.i.d.. Besides, each order statistic has a different mean since expectations of order statistics correspond to quantiles. However the approach suggested by Post (2003) based on the bootstrap is valid (see Nelson and Pope (1992) for an early use of bootstrap in stochastic dominance tests). Indeed bootstrapping the ranked observations or the initial observations does not affect bootstrap distributions of test statistics, at least in an i.i.d. framework.

The goal of this paper is to develop consistent tests for stochastic dominance efficiency at *any* order for *time-dependent* data. Serial correlation is known to pollute financial data (see the empirical section), and to alter, often severely, the size and power of testing procedures when neglected. We rely on weighted Kolmogorov-Smirnov type statistics in testing for stochastic dominance. They are inspired by the consistent procedures developped by Barrett and Donald (2003) and extended by Horvath, Kokoszka, and Zitikis (2006) to accommodate noncompact support. Other stochastic dominance tests are suggested in the literature; see e.g. Anderson (1996), Beach and Davidson (1983), Davidson and Duclos (2000). However these tests rely on pairwise comparisons made at a fixed number of arbitrary chosen points. This is not a desirable feature since it introduces the possibility of test inconsistency. We develop general stochastic dominance efficiency tests that compare a given portfolio with an optimal diversified portfolio formed from a given finite set of assets. We build on the general distribution definition of stochastic dominance in contrast to the traditional expected utility framework.

Note that De Giorgi (2005) solves a portfolio selection problem based on reward-risk measures consistent with second order stochastic dominance. If investors have homogeneous expectations and optimally hold reward-risk efficient portfolios, then in the absence of market frictions, the portfolio of all invested wealth, or the market portfolio, will itself be a reward-risk efficient portfolio. The market portfolio should therefore be itself efficient in the sense of second order stochastic dominance according to that theory (see De Giorgi and Post (2005) for a rigorous derivation of this result). This reasoning is similar to the one underlying the derivation of the CAPM (Sharpe (1964), Lintner (1965)), where investors optimally hold mean-variance efficient portfolios. A direct test for second order stochastic dominance efficiency of the market portfolio can be viewed as a nonparametric way to test empirically for such a theory.

The paper is organized as follows. In Section 2, we recall the notion of stochastic dominance efficiency introduced by Kuosmanen (2004) and Post (2003), and discuss the general hypotheses for testing stochastic dominance efficiency at any order. We describe the test statistics, and analyse the asymptotic properties of the testing procedures. We follow Barrett and Donald (2003) and Horvath, Kokoszka, and Zitikis (2006), who extend and justify the procedure of McFadden (1989) (see also Klecan, McFadden and McFadden (1991), Abadie (2002)) leading to consistent tests of stochastic dominance. We also use simulation based procedures to compute p-values. From a technical point of view, we modify their work to accommodate the presence of full diversification and time-dependent data. We rely on a block bootstrap method, and explain this in Section 3.

Note that other resampling methods such as subsampling are also available (see Linton, Maasoumi and Whang (2005) for the standard stochastic dominance tests). Linton, Post and Whang (2005) follow this route in the context of testing procedures for stochastic dominance efficiency. They use subsampling to estimate the p-values, and discuss power issues of the testing procedures. We prefer block bootstrap to subsampling since the former uses the full sample information. The block bootstrap is better suited to samples with a limited number of time-dependent data: we have 460 monthly observations in our empirical application.

Linton, Post and Whang (2005) focus on the dominance criteria of order two and three. In our paper, we also test for first order stochastic dominance efficiency as formulated in Kuosmanen (2004), although it gives necessary and not sufficient conditions for optimality (Post (2005)). The first order stochastic dominance criterion places on the form of the utility function no restriction beyond the usual requirement that it is nondecreasing, i.e., investors prefer more to less. Thus, this criterion is appropriate for both risk averters and risk lovers since the utility function may contain concave as well as convex segments. Owing to its generality, the first order stochastic dominance permits a preliminary screening of investment alternatives eliminating those which no rational investor will ever choose. The second order stochastic dominance criterion adds the assumption of global risk aversion. This criterion is based on a stronger assumption and therefore, it permits a more sensible selection of investments. The test statistic for second order stochastic dominance efficiency is formulated in terms of standard linear programming. Numerical implementation of first order stochastic dominance efficiency tests is much more difficult since we need to develop mixed integer programming formulations. Nevertheless, widely available algorithms can be used to compute both test statistics. We discuss in detail the computational aspects of mathematical programming formulations corresponding to the test statistics in Section 4.

In Section 5 we design a Monte Carlo study to evaluate actual size and power of the proposed tests in finite samples. In Section 6 we provide an empirical illustration. We analyze whether the Fama and French market portfolio can be considered as efficient according to first and second order stochastic dominance criteria when confronted to diversification principles made of six Fama and French benchmark portfolios formed on size and book-to-market equity ratio (Fama and French (1993)). The motivation to test for the efficiency of the market portfolio is that many institutional investors invest in mutual funds. These funds track value-weighted equity indices which strongly resemble the market portfolio. We find that the market portfolio is first and second order stochastic dominance efficient. We give some concluding remarks in Section 7. Proofs and detailed mathematical programming formulations are gathered in an appendix.

2 Tests of stochastic dominance efficiency

We consider a strictly stationary process $\{\boldsymbol{Y}_t; t \in \mathbb{Z}\}$ taking values in $\mathbb{R}^n$. The observations consist in a realization of $\{\boldsymbol{Y}_t;\ t = 1, ..., T\}$. These data correspond to observed returns of n financial assets. For inference we also need the process being strong mixing (α-mixing) with mixing coefficients α_t such that $\alpha_T = O(T^{-a})$ for some $a > 1$ as $T \to \infty$ (see Doukhan (1994) for relevant definition and examples). In particular returns generated by various stationary ARMA, GARCH and stochastic volatility models meet this requirement (Carrasco and Chen (1998)). We denote by $F(\boldsymbol{y})$, the continuous cdf of $\boldsymbol{Y} = (Y_1, ..., Y_n)'$ at point $\boldsymbol{y} = (y_1, ..., y_n)'$.

Let us consider a portfolio $\boldsymbol{\lambda} \in \mathbb{L}$ where $\mathbb{L} := \{\boldsymbol{\lambda} \in \mathbb{R}^n_+ : \boldsymbol{e}'\boldsymbol{\lambda} = 1\}$ with $\boldsymbol{e}$ for a vector made of ones. This means that short sales are not allowed and that the portfolio weights sum to one. Let us denote by $G(z, \boldsymbol{\lambda}; F)$ the cdf of the portfolio return $\boldsymbol{\lambda}'\boldsymbol{Y}$ at point z given by $G(z, \boldsymbol{\lambda}; F) := \int_{\mathbb{R}^n} \mathbb{I}\{\boldsymbol{\lambda}'\boldsymbol{u} \leq z\} dF(\boldsymbol{u})$.

Further define for $z \in \mathbb{R}$:

$$
\begin{aligned}
\mathcal{J}_1(z, \boldsymbol{\lambda}; F) &:= G(z, \boldsymbol{\lambda}; F), \\
\mathcal{J}_2(z, \boldsymbol{\lambda}; F) &:= \int_{-\infty}^{z} G(u, \boldsymbol{\lambda}; F) du = \int_{-\infty}^{z} \mathcal{J}_1(u, \boldsymbol{\lambda}; F) du, \\
\mathcal{J}_3(z, \boldsymbol{\lambda}; F) &:= \int_{-\infty}^{z} \int_{a}^{u} G(v, \boldsymbol{\lambda}; F) dv du = \int_{-\infty}^{z} \mathcal{J}_2(u, \boldsymbol{\lambda}; F) du,
\end{aligned}
$$

and so on. The integral $\mathcal{J}_j(z, \boldsymbol{\lambda}; F)$ is finite if $E[(-\boldsymbol{\lambda}'\boldsymbol{Y})_+^{j-1}]$ is finite for $j \geq 2$, where $(x)_+ = \max(x, 0)$ (Horvath, Kokoszka, and Zitikis (2006)).

From Davidson and Duclos (2000) Equation (2), we know that

$$
\mathcal{J}_j(z, \boldsymbol{\lambda}; F) = \int_{-\infty}^{z} \frac{1}{(j-1)!} (z-u)^{j-1} dG(u, \boldsymbol{\lambda}, F),
$$

which can be rewritten as

$$
\mathcal{J}_j(z, \boldsymbol{\lambda}; F) = \int_{\mathbb{R}^n} \frac{1}{(j-1)!} (z - \boldsymbol{\lambda}'\boldsymbol{u})^{j-1} \mathbb{I}\{\boldsymbol{\lambda}'\boldsymbol{u} \leq z\} dF(\boldsymbol{u}). \tag{2.1}
$$

The general hypotheses for testing the stochastic dominance efficiency of order j of $\boldsymbol{\tau}$, hereafter SDE_j, can be written compactly as:

$$
\begin{aligned}
H_0^j: \quad & \mathcal{J}_j(z, \boldsymbol{\tau}; F) \leq \mathcal{J}_j(z, \boldsymbol{\lambda}; F) \quad \text{for all}\, z \in \mathbb{R} \,\text{and for all}\, \boldsymbol{\lambda} \in \mathbb{L}, \\
H_1^j: \quad & \mathcal{J}_j(z, \boldsymbol{\tau}; F) > \mathcal{J}_j(z, \boldsymbol{\lambda}; F) \quad \text{for some}\, z \in \mathbb{R} \,\text{or for some}\, \boldsymbol{\lambda} \in \mathbb{L}.
\end{aligned}
$$

In particular we get first and second order stochastic dominance efficiency when $j = 1$ and $j = 2$, respectively. The hypothesis for testing the stochastic dominance of order j of the distribution of portfolio $\boldsymbol{\tau}$ over the distribution of portfolio $\boldsymbol{\lambda}$ take analoguous forms but for a given $\boldsymbol{\lambda}$ instead of several of them. The notion of stochastic dominance efficiency is a straightforward extension where full diversification is allowed (Kuosmanen (2004), Post (2003)).

The empirical counterpart to (2.1) is simply obtained by integrating with respect to the empirical distribution $\hat{F}$ of F, which yields:

$$\mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F}) \quad = \quad \frac{1}{T} \sum_{t=1}^{T} \frac{1}{(j-1)!} (z - \boldsymbol{\lambda}' \boldsymbol{Y}_t)^{j-1} \mathbb{I}\{\boldsymbol{\lambda}' \boldsymbol{Y}_t \leq z\}, \tag{2.2}$$

and can be rewritten more compactly for $j \geq 2$ as:

$$\mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F}) \quad = \quad \frac{1}{T} \sum_{t=1}^{T} \frac{1}{(j-1)!} (z - \boldsymbol{\lambda}' \boldsymbol{Y}_t)_+^{j-1}.$$

Since $\sqrt{T}(\hat{F} - F)$ tends weakly to a mean zero Gaussian process $\mathcal{B} \circ F$ in the space of continuous functions on $\mathbb{R}^n$ (see e.g. the multivariate functional central limit theorem for stationary strongly mixing sequences stated in Rio (2000)), we may derive the limiting behaviour of (2.2) using the Continuous Mapping Theorem (as in Lemma 1 of Barrett and Donald (2003)).

Lemma 2.1. $\sqrt{T}[\mathcal{J}_j(\cdot; \hat{F}) - \mathcal{J}_j(\cdot; F)]$ *tends weakly to a Gaussian process* $\mathcal{J}_j(\cdot; \mathcal{B} \circ F)$ *with mean zero and covariance function given by:*

- for $j = 1$*:*

$$\Omega_1(z_1, z_2, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) := E[G(z_1, \boldsymbol{\lambda}_1; \mathcal{B} \circ F) G(z_2, \boldsymbol{\lambda}_2; \mathcal{B} \circ F)]$$

$$= \sum_{t \in \mathbb{Z}} E\ [\mathbb{I}\{\boldsymbol{\lambda}_1' \boldsymbol{Y}_0 \leq z_1\} \mathbb{I}\{\boldsymbol{\lambda}_2' \boldsymbol{Y}_t \leq z_2\}] - G(z_1, \boldsymbol{\lambda}_1; F) G(z_2, \boldsymbol{\lambda}_2; F),$$

- for $j \geq 2$*:*

$$\Omega_j(z_1, z_2, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) := E\left[\mathcal{J}_j(z_1, \boldsymbol{\lambda}_1; \mathcal{B} \circ F)\mathcal{J}_j(z_2, \boldsymbol{\lambda}_2; \mathcal{B} \circ F)\right]$$

$$= \sum_{t \in \mathbb{Z}} \frac{1}{((j-1)!)^2} E\left[(z_1 - \boldsymbol{\lambda}_1' \boldsymbol{Y}_0)_+^{j-1}(z_2 - \boldsymbol{\lambda}_2' \boldsymbol{Y}_t)_+^{j-1}\right] - \mathcal{J}_j(z_1, \boldsymbol{\lambda}_1; F)\mathcal{J}_j(z_2, \boldsymbol{\lambda}_2; F),$$

with $(z_1, z_2)' \in \mathbb{R}^2$ *and* $(\boldsymbol{\lambda}_1', \boldsymbol{\lambda}_2')' \in \mathbb{L}^2$.

For i.i.d. data the covariance kernel reduces to

$$\Omega_1(z_1, z_2, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = E\left[\mathbb{I}\{\boldsymbol{\lambda}_1' \boldsymbol{Y} \leq z_1\}\mathbb{I}\{\boldsymbol{\lambda}_2' \boldsymbol{Y} \leq z_2\}\right] - G(z_1, \boldsymbol{\lambda}_1; F)G(z_2, \boldsymbol{\lambda}_2; F),$$

and for $j \geq 2$:

$$\Omega_j(z_1, z_2, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$$

$$= \frac{1}{((j-1)!)^2} E\left[(z_1 - \boldsymbol{\lambda}_1' \boldsymbol{Y})_+^{j-1}(z_2 - \boldsymbol{\lambda}_2' \boldsymbol{Y})_+^{j-1}\right] - \mathcal{J}_j(z_1, \boldsymbol{\lambda}_1; F)\mathcal{J}_j(z_2, \boldsymbol{\lambda}_2; F).$$

Let us consider the weighted Kolmogorov-Smirnov type test statistic

$$\hat{S}_j := \sqrt{T} \sup_{z, \boldsymbol{\lambda}} q(z) \left[\mathcal{J}_j(z, \boldsymbol{\tau}; \hat{F}) - \mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F})\right],$$

and a test based on the decision rule:

$$\text{“ reject } \quad H_0^j \quad \text{if} \quad \hat{S}_j > c_j \text{”},$$

where c_j is some critical value that will be discussed in a moment. We introduce the same positive weighting function $q(z)$ as in Horvath, Kokoszka, and Zitikis (2006), because we have no compactness assumption on the support of $\boldsymbol{Y}$. However Horvath, Kokoszka, and Zitikis (2006) show that there is no need to introduce it for $j = 1, 2$. Then we can take $q(z) = \mathbb{I}\{z < +\infty\}$. Under that choice of q the statement of Proposition 2.2 below remains true for $j = 1, 2$, without the technical assumption on G needed for the general result with a higher j.

The following result characterizes the properties of the test, where

$$\bar{S}_j \quad := \quad \sup_{z,\boldsymbol{\lambda}} q(z)\left[\mathcal{J}_j(z,\boldsymbol{\tau};\mathcal{B}\circ F)-\mathcal{J}_j(z,\boldsymbol{\lambda};\mathcal{B}\circ F)\right].$$

Proposition 2.2. *Let c_j be a positive finite constant. Let the weighting function $q:\mathbb{R}\to[0,+\infty)$ satisfy $\sup_z q(z)(1+(z)_+)^{j-2}<+\infty$, $j\geq 2$, and assume that the distribution function G satifies $\int_{\mathbb{R}}(1+(-z)_+)^{j-2}\sqrt{G(z,\boldsymbol{\lambda};F)(1-G(z,\boldsymbol{\lambda};F))}dz<+\infty$ for all $\boldsymbol{\lambda}\in\mathbb{L}$, $j\geq 2$, then:*

(i) if H_0^j is true,

$$\lim_{T\to\infty} P[\textit{reject}\, H_0^j]\leq P[\bar{S}_j>c_j]:=\alpha(c_j),$$

with equality when $G(z,\boldsymbol{\lambda};F)=G(z,\boldsymbol{\tau};F)$ for all $z\in\mathbb{R}$ and some $\boldsymbol{\lambda}\in\mathbb{L}$;

(ii) if H_0^j is false,

$$\lim_{T\to\infty} P[\textit{reject}\, H_0^j]=1.$$

The result provides a random variable that dominates the limiting random variable corresponding to the test statistic under the null hypothesis. The inequality yields a test that never rejects more often than $\alpha(c_j)$ for any portfolio τ satisfying the null hypothesis. As noted in the result the probability of rejection is asymptotically exactly $\alpha(c_j)$ when $G(z,\boldsymbol{\lambda};F)=G(z,\boldsymbol{\tau};F)$ for all $z\in\mathbb{R}$ and some $\boldsymbol{\lambda}\in\mathbb{L}$. The first part implies that if we can find a c_j to set the $\alpha(c_j)$ to some desired probability level (say the conventional 0.05 or 0.01) then this will be the significance level for composite null hypotheses in the sense described by Lehmann (1986). The second part of the result indicates that the test is capable of detecting any violation of the full set of restrictions of the null hypothesis.

We conjecture that a similar result holds with $\sup_{z,\boldsymbol{\lambda}} q(z)\mathcal{J}_j(z,\boldsymbol{\lambda};\mathcal{B}\circ F)$ substituted for $\bar{S}_j$. We have not been able to show this because of the complexity of the covariance of the empirical process which impedes us to exploit the Slepian-Fernique-Marcus-Shepp inequality (see Proposition A.2.6 of van der Vaart and Wellner (1996)) as in Barrett and Donald (2003). In stochastic dominance efficiency tests this second result would not bring a significant improvement in the numerical tractability as opposed to the context of Barrett and Donald (2003).

In order to make the result operational, we need to find an appropriate critical value c_j. Since the distribution of the test statistic depends on the underlying distribution, this is not an easy task, and we decide hereafter to rely on a block bootstrap method to simulate p-values. The other method suggested by Barrett and Donald (2003), namely a simulation based multiplier method, would only provide an approximation in our case since it does not work for dependent data.

3 Simulations of p-values with block bootstrap

Block bootstrap methods extend the nonparametric i.i.d. bootstrap to a time series context (see Barrett and Donald (2003) and Abadie (2002) for use of the nonparametric i.i.d. bootstrap in stochastic dominance tests). They are based on "blocking" arguments, in which data are divided into blocks and those, rather than individual data, are resampled in order to mimick the time dependent structure of the original data. An alternative resampling technique could be subsampling, for which similar results can be shown to hold as well (see Linton, Maasoumi, and Whang (2005) for use in stochastic dominance tests and comparison between the two techniques in terms of asymptotic and finite sample properties). We focus on block bootstrap since we face moderate sample sizes in the empirical applications, and wish to exploit the full sample information. Besides a Monte Carlo investigation of the finite sample properties of subsampling based tests is too time consuming in our context (see Section 5 on how we solve that problem in a bootstrap setting based on a suggestion of Davidson and MacKinnon (2006a,b)).

Let b, l denote integers such that $T = bl$. We distinguish hereafter two different ways of proceeding, depending on whether the blocks are overlapping or nonoverlapping. The overlapping rule (Kunsch (1989)) produces $T - l + 1$ overlapping blocks, the kth being $\boldsymbol{B}_k = (\boldsymbol{Y}'_k, ..., \boldsymbol{Y}'_{k+l-1})'$ with $k \in \{1, ..., T-l+1\}$. The nonoverlapping rule (Carlstein (1986)) just asks the data to be divided into b disjoint blocks, the kth being $\boldsymbol{B}_k = (\boldsymbol{Y}'_{(k-1)l+1}, ..., \boldsymbol{Y}'_{kl})'$ with $k \in \{1, ..., b\}$. In either case the block bootstrap method requires that we choose blocks $\boldsymbol{B}^*_1, ..., \boldsymbol{B}^*_b$ by resampling randomly, with replacement, from the set of overlapping or nonoverlapping blocks. If $\boldsymbol{B}^*_i = (\boldsymbol{Y}^{*\prime}_{i1}, ..., \boldsymbol{Y}^{*\prime}_{il})'$, a block bootstrap sample $\{\boldsymbol{Y}^*_t;\ t = 1, ..., T\}$ is made of $\{\boldsymbol{Y}^*_{11}, ..., \boldsymbol{Y}^*_{1l}, \boldsymbol{Y}^*_{21}, ..., \boldsymbol{Y}^*_{2l}, ..., \boldsymbol{Y}^*_{b1}, ..., \boldsymbol{Y}^*_{bl}\}$, and we let $\hat{F}^*$ denote its empirical distribution.

Let $\hat{E}^*$ denote the expectation operator with respect to the probability measure induced by block bootstrap sampling. If the blocks are nonoverlapping, then $\hat{E}^* \mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F}^*) = \mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F})$. In contrast $\hat{E}^* \mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F}^*) \neq \mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F})$ under an overlapping scheme (Hall, Horowitz, and Jing (1995)). The resulting bias decreases the rate of convergence of the estimation errors of the block bootstrap with overlapping blocks. Fortunately this problem can be solved easily by recentering the test statistic as discussed in Linton, Maasoumi and Whang (2005) and the review paper of Haerdle, Horowitz and Kreiss (2003) (see also Hall and Horowitz (1996), Andrews (2002) for a discussion of the need for recentering to avoid excessive bias in tests based on extremum estimators). Let us consider

$$S^*_j := \sqrt{T} \sup_{z, \boldsymbol{\lambda}} q(z) \left\{ \left[\mathcal{J}_j(z, \boldsymbol{\tau}; \hat{F}^*) - \hat{E}^* \mathcal{J}_j(z, \boldsymbol{\tau}; \hat{F}^*) \right] - \left[\mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F}^*) - \hat{E}^* \mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F}^*) \right] \right\},$$

with, for the overlapping rule,

$$\hat{E}^* \mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F}^*) = \frac{1}{T-l+1} \sum_{t=1}^{T} w(t, l, T) \frac{1}{(j-1)!} (z - \boldsymbol{\lambda}' \boldsymbol{Y}_t)^{j-1} \mathbb{I}\{\boldsymbol{\lambda}' \boldsymbol{Y}_t \leq z\},$$

where

$$w(t, l, T) = \begin{cases} t/l & \text{if} \quad t \in \{1, ..., l-1\}, \\ 1 & \text{if} \quad t \in \{l, ..., T-l+1\}, \\ (T-t+1)/l & \text{if} \quad t \in \{T-l+2, ..., T\}, \end{cases}$$

and with, for the nonoverlapping rule, $\hat{E}^* \mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F}^*) = \mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F})$.

From Hall, Horowitz, and Jing (1995) we have in the overlapping case:

$$\hat{E}^* \mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F}^*) - \mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F}) = [l(T-l-1)]^{-1}[l(l-1)\mathcal{J}_j(z, \boldsymbol{\lambda}; \hat{F}) - V_1 - V_2],$$

where $V_1 = \sum_{t=1}^{l-1}(l-t)\frac{1}{(j-1)!}(z-\boldsymbol{\lambda}'\boldsymbol{Y}_t)^{j-1}\mathbb{I}\{\boldsymbol{\lambda}'\boldsymbol{Y}_t \le z\}$, and $V_2 = \sum_{t=T-l-2}^{T}[t-(T-l+1)]\frac{1}{(j-1)!}(z-\boldsymbol{\lambda}'\boldsymbol{Y}_t)^{j-1}\mathbb{I}\{\boldsymbol{\lambda}'\boldsymbol{Y}_t \le z\}$, and thus the difference between the two rules vanishes asymptotically.

Let us define $p_j^* := P[S_j^* > \hat{S}_j]$. Then the block bootstrap method is justified by the next statement.

Proposition 3.1. *Assuming that $\alpha < 1/2$, a test for SDE_j based on the rule:*

$$\text{“ } \textit{reject}\, H_0^j \quad \textit{if} \quad p_j^* < \alpha \text{ ”},$$

satisfies the following

$$\lim P[\textit{reject}\, H_0^j] \le \alpha \quad \textit{if} \quad H_0^j \textit{ is true},$$

$$\lim P[\textit{reject}\, H_0^j] = 1 \quad \textit{if} \quad H_0^j \textit{ is false}.$$

In practice we need to use Monte-Carlo methods to approximate the probability. The p-value is simply approximated by $\tilde{p}_j = \frac{1}{R}\sum_{r=1}^{R}\mathbb{I}\{\tilde{S}_{j,r} > \hat{S}_j\}$, where the averaging is made on R replications. The replication number can be chosen to make the approximations as accurate as we desire given time and computer constraints.

4 Implementation with mathematical programming

In this section we present the final mathematical programming formulations corresponding to the test statistics for first and second order stochastic dominance efficiency. In the appendix we provide the detailed derivation of the formulations.

4.1 Formulation for first order stochastic dominance

The test statistic $\hat{S}_1$ for first order stochastic dominance efficiency is derived using mixed integer programming formulations. The following is the full formulation of the model:

$$
\begin{aligned}
\max_{\mathbf{z},\boldsymbol{\lambda}} \quad & \hat{S}_1 = \sqrt{T}\frac{1}{T}\sum_{t=1}^{T}(L_t - W_t) && && (4.1a)\\
\text{s.t.} \quad & M(L_t - 1) \leq z - \boldsymbol{\tau}'\boldsymbol{Y}_t \leq ML_t, && \forall\, t && (4.1b)\\
& M(W_t - 1) \leq z - \boldsymbol{\lambda}'\boldsymbol{Y}_t \leq MW_t, && \forall\, t && (4.1c)\\
& \boldsymbol{e}'\boldsymbol{\lambda} = 1, && && (4.1d)\\
& \boldsymbol{\lambda} \geq 0, && && (4.1e)\\
& W_t \in \{0,1\},\ L_t \in \{0,1\}, && \forall\, t && (4.1f)
\end{aligned}
$$

with M being a large constant.

The model is a mixed integer program maximizing the distance between the sum over all scenarios of two binary variables, $\frac{1}{T}\sum_{t=1}^{T} L_t$ and $\frac{1}{T}\sum_{t=1}^{T} W_t$ which represent $\mathcal{J}_1(z, \boldsymbol{\tau}; \hat{F})$ and $\mathcal{J}_1(z, \boldsymbol{\lambda}; \hat{F})$, respectively (the empirical cdf of portfolios $\boldsymbol{\tau}$ and $\boldsymbol{\lambda}$ at point z). According to Inequalities (4.1b), L_t equals 1 for each scenario $t \in T$ for which $z \geq \boldsymbol{\tau}'\boldsymbol{Y}_t$, and 0 otherwise. Analogously, Inequalities (4.1c) ensure that W_t equals 1 for each scenario for which $z \geq \boldsymbol{\lambda}'\boldsymbol{Y}_t$. Equation (4.1d) defines the sum of all portfolio weights to be unity, while Inequality (4.1e) disallows for short positions in the available assets.

This formulation permits to test the dominance of a given portfolio $\boldsymbol{\tau}$ over any potential linear combination $\boldsymbol{\lambda}$ of the set of other portfolios. So we implement a test of efficiency and not simple stochastic dominance.

When some of the variables are binary, corresponding to mixed integer programming, the problem becomes NP-complete (non-polynomial, i.e., formally intractible).

The best and most widely used method for solving mixed integer programs is branch and bound (an excellent introduction to mixed integer programming is given by Nemhauser and Wolsey (1999)). Subproblems are created by restricting the range of the integer variables. For binary variables, there are only two possible restrictions: setting the variable to 0, or setting the variable to 1. To apply branch and bound, we must have a means of computing a lower bound on an instance of the optimization problem and a means of dividing the feasible region of a problem to create smaller subproblems. There must also be a way to compute an upper bound (feasible solution) for at least some instances; for practical purposes, it should be possible to compute upper bounds for some set of nontrivial feasible regions.

The method starts by considering the original problem with the complete feasible region, which is called the root problem. The lower-bounding and upper-bounding procedures are applied to the root problem. If the bounds match, then an optimal solution has been found, and the procedure terminates. Otherwise, the feasible region is divided into two or more regions, each strict subregions of the original, which together cover the whole feasible region. Ideally, these subproblems partition the feasible region. They become children of the root search node. The algorithm is applied recursively to the subproblems, generating a tree of subproblems. If an optimal solution is found to a subproblem, it is a feasible solution to the full problem, but not necessarily globally optimal. Since it is feasible, it can be used to prune the rest of the tree. If the lower bound for a node exceeds the best known feasible solution, no globally optimal solution can exist in the subspace of the feasible region represented by the node. Therefore, the node can be removed from consideration. The search proceeds until all nodes have been solved or pruned, or until some specified threshold is met between the best solution found and the lower bounds on all unsolved subproblems.

In our case, the number of nodes is several hundreds of millions. Under this form, this is a very difficult problem to solve. It takes more than two days to find the optimal solution for relatively small time series. We reformulate the problem in order to reduce the solving time and to obtain a tractable formulation.

We can see that there is a set of at most T values, say $\mathcal{R} = \{r_1, r_2, ..., r_T\}$, containing the optimal value of the variable z. A direct consequence is that we can solve first order stochastic dominance efficiency by solving the smaller problems $P(r)$, $r \in \mathcal{R}$, in which z is fixed to r. Then we can take the value for z that yields the best total result. The advantage is that the optimal values of the L_t variables are known in $P(r)$.

The reduced form of the problem is as follows (see the appendix for the derivation of this formulation and details on practical implementation):

$$\begin{aligned} \min \quad & \sum_{t=1}^{T} W_t \\ \text{s.t.} \quad & \boldsymbol{\lambda}'\boldsymbol{Y}_t \geq r - (r - M_t)W_t, \quad \forall t, \\ & \boldsymbol{e}'\boldsymbol{\lambda} = 1, \\ & \boldsymbol{\lambda} \geq 0, \\ & W_t \in \{0, 1\}, \quad \forall t. \end{aligned} \tag{4.2a}$$

4.2 Formulation for second order stochastic dominance

The model to derive the test statistic $\hat{S}_2$ for second order stochastic dominance efficiency is the following:

$$\max_{\mathbf{z},\boldsymbol{\lambda}} \quad \hat{S}_2 = \sqrt{T}\frac{1}{T}\sum_{t=1}^{T}(L_t - W_t) \tag{4.3a}$$

$$\text{s.t.} \quad M(F_t - 1) \leq z - \boldsymbol{\tau}'\boldsymbol{Y}_t \leq MF_t, \qquad \forall t, \tag{4.3b}$$

$$-M(1 - F_t) \leq L_t - (z - \boldsymbol{\tau}'\boldsymbol{Y}_t) \leq M(1 - F_t), \qquad \forall t, \tag{4.3c}$$

$$-MF_t \leq L_t \leq MF_t, \qquad \forall t \tag{4.3d}$$

$$W_t \geq z - \boldsymbol{\lambda}'\boldsymbol{Y}_t, \qquad \forall t, \tag{4.3e}$$

$$\boldsymbol{e}'\boldsymbol{\lambda} = 1, \tag{4.3f}$$

$$\boldsymbol{\lambda} \geq 0, \tag{4.3g}$$

$$W_t \geq 0,\ F_t \in \{0, 1\}, \qquad \forall t \tag{4.3h}$$

with M being a large constant.

The model is a mixed integer program maximizing the distance between the sum over all scenarios of two variables, $\frac{1}{T}\sum_{t=1}^{T} L_t$ and $\frac{1}{T}\sum_{t=1}^{T} W_t$ which represent the $\mathcal{J}_2(z, \boldsymbol{\tau}; \hat{F})$ and $\mathcal{J}_2(z, \boldsymbol{\lambda}; \hat{F})$, respectively. This is difficult to solve since it is the maximization of the difference of two convex functions. We use a binary variable F_t, which, according to Inequalities (4.3b), equals 1 for each scenario $t \in T$ for which $z \geq \boldsymbol{\tau}'\boldsymbol{Y}_t$, and 0 otherwise. Then, Inequalities (4.3c) and (4.3d) ensure that the variable L_t equals $z - \boldsymbol{\tau}'\boldsymbol{Y}_t$ for the scenarios for which this difference is positive, and 0 for all the other scenarios. Inequalities (4.3e) and (4.3h) ensure that W_t equals exactly the difference $z - \boldsymbol{\lambda}'\boldsymbol{Y}_t$ for the scenarios for which this difference is positive, and 0 otherwise. Equation (4.3f) defines the sum of all portfolio weights to be unity, while Inequality (4.3g) disallows for short positions in the available assets.

Again, this is a very difficult problem to solve. It takes more than a day for the optimal solution. The model is easily transformed to a linear one, which is very easy to solve.

We solve second order stochastic dominance efficiency by solving again smaller problems $P(r)$, $r \in \mathcal{R}$, in which z is fixed to r, before taking the value for z that yields the best total result.

The new model is the following:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{T}\sum_{t=1}^{T} W_{i,t} \\
\text{s.t.} \quad & W_{i,t} \geq r_i - \boldsymbol{\lambda}_i' \boldsymbol{Y}_t, \quad \forall i, \quad \forall t \in T \\
& \boldsymbol{e}'\boldsymbol{\lambda}_i = 1, \quad \forall i, \\
& \boldsymbol{\lambda}_i \geq 0, \quad \forall i, \\
& W_{i,t} \geq 0, \quad \forall i, \quad \forall t. \qquad (4.4a)
\end{aligned}
$$

The optimal portfolio $\boldsymbol{\lambda}_i$ and the optimal value r_i of variable z are for that i, that gives $\min \sum_{t=1}^{T} W_{i,t}$. Now, the computational time for this formulation of the problem is less than a minute.

5 Monte Carlo study

In this section we design Monte Carlo experiments to evaluate actual size and power of the proposed tests in finite samples. Because of the computational burden of evaluating bootstrap procedures in a highly complex optimization environment, we implement the suggestion of Davidson and McKinnon (2006a,b) to get approximate rejection probabilities. We consider two financial assets in a time series context. We assume that their returns behave like a stationary vector autoregressive process of order one $\boldsymbol{Y}_t = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{Y}_{t-1} + \boldsymbol{\nu}_t$, where $\boldsymbol{\nu}_t \sim N(0, \boldsymbol{\Sigma})$ and all the eigenvalues of $\boldsymbol{B}$ are less than one in absolute value. The marginal distribution of $\boldsymbol{Y}$ is Gaussian with $\boldsymbol{\mu} = (\boldsymbol{Id} - \boldsymbol{B})^{-1}\boldsymbol{A}$ and covariance matrix $\boldsymbol{\Omega}$ satisfying vec $\boldsymbol{\Omega} = (\boldsymbol{Id} - \boldsymbol{B} \otimes \boldsymbol{B})^{-1}$ vec $\boldsymbol{\Sigma}$ (vec $\boldsymbol{D}$ is the stack of the columns of matrix $\boldsymbol{D}$).

5.1 Approximate rejection probabilities

According to Davidson and MacKinnon (2006a,b), a simulation estimate of the rejection probability of the bootstrap test at order j and for significance level α is

$\hat{RP}_j(\alpha) = \frac{1}{R}\sum_{r=1}^{R} \mathbb{I}\{\hat{S}_{j,r} < \hat{Q}_j^*(\alpha)\}$ where the test statistics $\hat{S}_{j,r}$ are obtained under the true data generating process on R subsamples, and $\hat{Q}_j^*(\alpha)$ is the α-quantile of the bootstrap statistics $\hat{S}_{j,r}^*$. So, for each one of the two cases (first and second order), the data generating process DGP_0 is used to draw realizations of the two asset returns, using the vector autoregressive process described above (with different parameters for each case to evaluate size and power). We generate $R = 300$ original samples with size $T = 460$. For each one of these original samples we generate a block bootstrap (nonoverlapping case) data generating process $\widehat{DGP}$. Once $\widehat{DGP}$ is obtained for each replication r, a new set of random numbers, independent of those used to obtain $\widehat{DGP}$, is drawn. Then, using these numbers we draw R original samples and R block bootstrap samples to compute $\hat{S}_{j,r}$ and $\hat{S}_{j,r}^*$ to get the estimate $\hat{RP}_j(\alpha)$.

5.2 Data generating process and results

To evaluate the actual size, we test for first and second order stochastic dominance efficiency of one portfolio $\boldsymbol{\tau}$ containing the first asset only, i.e., $\boldsymbol{\tau} = (1, 0)'$, and compare it to all other possible portfolios $\boldsymbol{\lambda}$ containing both assets with positive weights summing to one.

According to Levy (1973,1982), portfolio $\boldsymbol{\tau}$ dominates portfolio $\boldsymbol{\lambda}$ at the first order if $\mu_\tau > \mu_\lambda$ and $\sigma_\tau = \sigma_\lambda$ (with obvious notations). Otherwise we get a crossing. To compute the crossing we solve $(x - \mu_\tau)/\sigma_\tau = (x - \mu_\lambda)/\sigma_\lambda$. Then we test with a truncation weighting before the crossing, i.e., the lowest point across all combinations. By doing so we test on the part of the support where we have strict dominance (see Levy (1982) for the definition of the truncated distributions). We take $\boldsymbol{A} = (0.05, 0.05)'$, $\text{vec}\,\boldsymbol{B} = (0.9, 0.1, -0.1, -0.9)'$, and $\text{vec}\,\boldsymbol{\Sigma} = (0.1, 0, 0, 0.1)'$ for the parameters of the vector autoregressive process. The parameters of the Gaussian stationary distribution are then $\boldsymbol{\mu} = (0.5, 0)'$, and $\text{vec}\,\boldsymbol{\Omega} = (0.5, 0, 0, 0.5)'$.

On the other hand, portfolio $\boldsymbol{\tau}$ dominates portfolio $\boldsymbol{\lambda}$ at second order if $(\mu_\tau -$

$\mu_\lambda)/(\sigma_\lambda - \sigma_\tau) > 0$ when $\mu_\tau > \mu_\lambda$ and $\sigma_\tau < \sigma_\lambda$. We choose here $\boldsymbol{A} = (0.05, 0.05)'$, $\text{vec}\,\boldsymbol{B} = (0.9, 0.1, -0.1, -0.9)'$, and $\text{vec}\,\boldsymbol{\Sigma} = (0.1, 0, 0, 0.2)'$. The parameters of the Gaussian stationary distribution are then $\boldsymbol{\mu} = (0.5, 0)'$, and $\text{vec}\,\boldsymbol{\Omega} = (0.5, 0, 0, 1)'$.

We set the significance level α equal to 5%, and the block size to $l = 10$. We get $\hat{R}P_1(5\%) = 4.33\%$ for the first order stochastic dominance efficiency test, while we get $\hat{R}P_2(5\%) = 4.00\%$ for the second order stochastic dominance efficiency test. Hence we may conclude that both bootstrap tests perform well in terms of size properties.

To evaluate the actual power, we take an inefficient portfolio as the benchmark portfolio $\boldsymbol{\tau}$, and we compare it to all other possible portfolios $\boldsymbol{\lambda}$ containing both assets with positive weights summing to one. Since portfolio $\boldsymbol{\tau}$ is not the efficient one the algorithm should find the first asset of the size design as the efficient one. We use two different inefficient portfolios: the equally weighted portfolio, i.e., $\boldsymbol{\tau} = (0.5, 0.5)'$, and the portfolio containing the second asset only, i.e., $\boldsymbol{\tau} = (0, 1)'$.

We find that the power of both tests is large. Indeed, we find $\hat{R}P_1(5\%) = 96.66\%$ for the first order stochastic dominance efficiency test when we take wrongly as efficient the equally weighted portfolio. Similarly we find $\hat{R}P_1(5\%) = 97.33\%$ when we take wrongly the second asset as efficient. In the case of the second order stochastic dominance efficiency, we find $\hat{R}P_2(5\%) = 98.33\%$ and $\hat{R}P_2(5\%) = 98.66\%$ under the two different null hypotheses, respectively.

Finally we present Monte Carlo results in Table 5.1 on the sensitivity to the choice of block length. We investigate block sizes ranging from $l = 4$ to $l = 16$ by step of 4. This covers the suggestions of Hall, Horowitz, and Jing (1995), who show that optimal block sizes are multiple of $T^{1/3}$, $T^{1/4}$, $T^{1/5}$, depending on the context. According to our experiments the choice of the block size does not seem to dramatically alter the performance of our methodology.

Block size l	4	8	10	12	16
Size: $\tau = (1, 0)'$					
$\hat{R}P_1(5\%)$	4.00%	4.00%	4.33%	4.00%	5.33%
$\hat{R}P_2(5\%)$	3.66%	4.00%	4.00%	4.33%	4.66%
Power: $\tau = (.5, .5)'$					
$\hat{R}P_1(5\%)$	97.66%	97.00%	96.66%	96.66%	95.33%
$\hat{R}P_2(5\%)$	98.66%	98.33%	98.33%	98.00%	96.33%
Power: $\tau = (0, 1)'$					
$\hat{R}P_1(5\%)$	97.66%	97.00%	97.33%	98.00%	96.66%
$\hat{R}P_2(5\%)$	98.33%	98.33%	98.66%	98.66%	97.66%

Table 5.1: Sensitivity analysis of size and power to the choice of block length. We compute the actual size and power of the first and second order stochastic dominance efficiency tests for block sizes ranging from $l = 4$ to $l = 16$. The efficient portfolio is $\tau = (1, 0)'$ for the size analysis, and either $\tau = (.5, .5)'$ or $\tau = (0, 1)'$ for the power analysis.

6 Empirical application

In this section we present the results of an empirical application. To illustrate the potential of the proposed test statistics, we test whether different stochastic dominance efficiency criteria (first and second order) rationalize the market portfolio. Thus, we test for the stochastic dominance efficiency of the market portfolio with respect to all possible portfolios constructed from a set of assets, namely six risky assets. Although we focus the analysis on testing second order stochastic dominance efficiency of the market portfolio, we additionally test for first order stochastic dominance efficiency to examine the degree of the subject rationality (in the sense that they prefer more to less).

6.1 Description of the data

We use six Fama and French benchmark portfolios as our set of risky assets. They are constructed at the end of each June, and correspond to the intersections of two portfolios formed on size (market equity, ME) and three portfolios formed on the ratio of book equity to market equity (BE/ME). The size breakpoint for year t is the median NYSE market equity at the end of June of year t. BE/ME for June of year t is the book equity for the last fiscal year end in $t-1$ divided by ME for December of $t-1$. Firms with negative BE are not included in any portfolio. The annual returns are from January to December. We use data on monthly excess returns (month-end to month-end) from July 1963 to October 2001 (460 monthly observations) obtained from the data library on the homepage of Kenneth French (http://mba.turc.dartmouth.edu/pages/faculty/ken.french). The test portfolio is the Fama and French market portfolio, which is the value-weighted average of all non-financial common stocks listed on NYSE, AMEX, and Nasdaq, and covered by CRSP and COMPUSTAT.

First we analyze the statistical characteristics of the data covering the period from July 1963 to October 2001 (460 monthly observations) that are used in the test statistics. As we can see from Table 6.1, portfolio returns exhibit considerable variance in comparison to their mean. Moreover, the skewness and kurtosis indicate that normality cannot be accepted for the majority of them. These observations suggest adopting the first and second order stochastic dominance efficiency tests which account for the full return distribution and not only the mean and the variance.

One interesting feature is the comparison of the behavior of the market portfolio with that of the individual portfolios. Figure 6.1 shows the mean-standard deviation efficient frontier of the six Fama and French benchmark portfolios. The plot also shows the mean and standard deviation of the individual benchmark portfolio returns and of the Fama and French market (M) portfolio return. We observe that the test portfolio (M) is mean-standard deviation inefficient. It is clear that we can construct portfolios

Descriptive Statistics						
No.	Mean	Std. Dev.	Skewness	Kurtosis	Minimum	Maximum
Market Portfolio	0.462	4.461	-0.498	2.176	-23.09	16.05
1	0.316	7.07	-0.337	-1.033	-32.63	28.01
2	0.726	5.378	-0.512	0.570	-28.13	26.26
3	0.885	5.385	-0.298	1.628	-28.25	29.56
4	0.323	4.812	-0.291	-1.135	-23.67	20.48
5	0.399	4.269	-0.247	-0.706	-21.00	16.53
6	0.581	4.382	-0.069	-0.929	-19.46	20.46

Table 6.1: Descriptive statistics of monthly returns in % from July 1963 to October 2001 (460 monthly observations) for the Fama and French market portfolio and the six Fama and French benchmark portfolios formed on size and book-to-market equity ratio. Portfolio 1 has low BE/ME and small size, portfolio 2 has medium BE/ME and small Size, portfolio 3 has high BE/ME and small size, ..., portfolio 6 has high BE/ME and large size.

that achieve a higher expected return for the same level of standard deviation, and a lower standard deviation for the same level of expected return. If the investor utility function is not quadratic, then the risk profile of the benchmark portfolios cannot be totally captured by the variance of these portfolios. Generally, the variance is not a satisfactory measure. It is a symmetric measure that penalizes gains and losses in the same way. Moreover, the variance is inappropriate to describe the risk of low probability events. Figure 6.1 is silent on return moments other than mean-variance (such as higher-order central moments and lower partial moments). Finally, the mean-variance approach is not consistent with second order stochastic dominance. This is

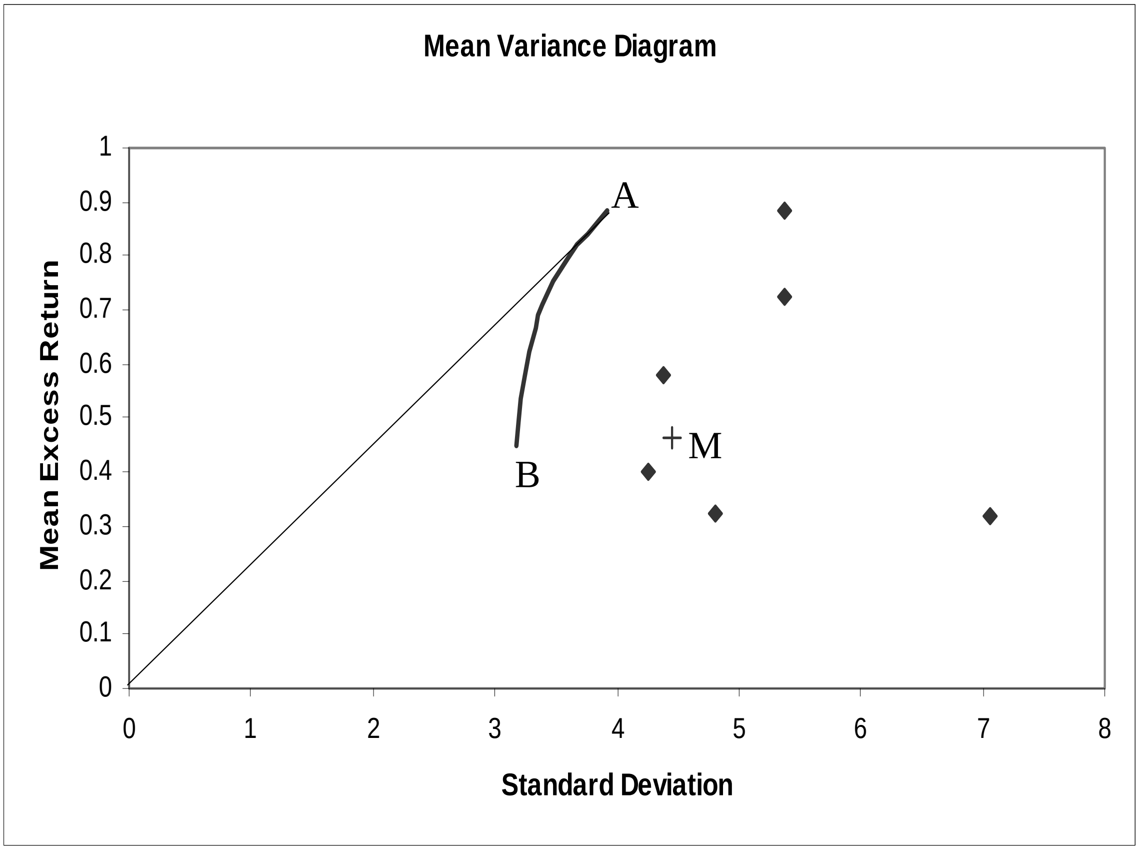

Figure 6.1: Mean-standard deviation efficient frontier of six Fama and French benchmark portfolios. The plot also shows the mean and standard deviation of the individual benchmark portfolio returns and of the Fama and French market (M) portfolio return, which is the test portfolio.

well illustrated by the mean-variance paradox, and motivates us to test whether the market portfolio is first and second order stochastic dominance efficient. These criteria avoid parameterization of investor preferences and the return distribution, and at the same time ensures that the regularity conditions of nonsatiation (first order stochastic dominance efficiency) and risk aversion (second order stochastic dominance efficiency) are satisfied. In brief, the market portfolio must be first order stochastic dominance efficient for all asset pricing models that use a nonsatiable representative investors. It must be second order stochastic dominance efficient for all asset pricing models that use nonsatiable and additionally risk-averse representative investor. This must hold regardless of the specific functional form of the utility function and the return distribution.

6.2 Results of the stochastic dominance efficiency tests

We find a significant autocorrelation of order one at a 5% significance level in benchmark portfolios 1 to 3, while ARCH effects are present in benchmark portfolio 4 at a 5% significance level. This indicates that a block bootstrap approach should be favoured over a standard i.i.d. bootstrap approach. Since the autocorrelations die out quickly, we may take a block of small size to compute the p-values of the test statistics. We choose a size of 10 observations. We use the nonoverlapping rule because we need to recenter the test statistics in the overlapping rule. The recentering makes the test statistics very difficult to compute, since the optimization involves a large number of binary variables. The p-values are approximated with an averaging made on $R = 300$ replications. This number guarantees that the approximations are accurate enough, given time and computer constraints.

For the first order stochastic dominance efficiency, we cannot reject that the market portfolio is efficient. The p-value $\tilde{p}_1 = 0.55$ is way above the significance level of 5%. We also find that the market portfolio is highly and significantly second order stochastic dominance efficient since $\tilde{p}_2 = 0.59$. Although Figure 6.1 shows that the market portfolio is inefficient compared to the benchmark portfolios in the mean-variance scheme, the first and second stochastic dominance efficiency of the market portfolio prove the opposite under more general schemes. These results indicate that the whole distribution rather than the mean and the variance plays an important role in comparing portfolios. This efficiency of the market portfolio is interesting for investors. If the market portfolio is not efficient, individual investors could diversify across diverse asset portfolios and outperform the market.

Our efficiency finding cannot be attributed to a potential lack of power of the testing procedures. Indeed, we use a long enough time series of 460 return observations, and a relatively narrow cross-section of six benchmark portfolios. Further even if our test concerns a necessary and not a sufficient condition for optimality of the market portfolio (Post (2005)), this does not influence the output of our results. Indeed, the conclusion of the test if that the market portfolio dominates all possible combinations of the other portfolios, and this for all nonsatiable decision-makers; thus, it is also true for some of them.

6.3 Rolling window analysis

We carry out an additional test to validate the second order stochastic dominance efficiency of the market portfolio and the stability of the model results. It is possible that the efficiency of the market portfolio changes over time, as the risk and preferences of investors change. Therefore, the market portfolio may be efficient in the total sample, but inefficient in some subsamples. Moreover, the degree of efficiency may change over time, as pointed by Post (2003). To control for that, we perform a rolling window analysis, using a window width of 10 years. The test statistic is calculated separately for 340 overlapping 10-year periods, (July 1963-June 1973), (August 1963-July 1973),...,(November 1991-October 2001).

Figure 6.2 shows the corresponding p-values. Interestingly, we observe that the market portfolio is second order stochastic dominance efficient in the total sample period. The second order stochastic dominance efficiency is not rejected on any subsamples. The p-values are always greater than 15%, and in some cases they reach the $80\% - 90\%$. This result confirms the second order stochastic dominance efficiency that was found in the previous subsection, for the whole period. This means that we cannot form an optimal portfolio from the set of the six benchmark portfolios that dominates the market portfolio by second order stochastic dominance. The line exhibits large fluctuations; thus the degree of efficiency is changing over time, but remains always above the critical level of 5%.

Note that the computational complexity and the associated large solution time of the first order stochastic dominance efficiency test are prohibitive for a rolling window analysis. It involves a large number of optimization models: 340 rolling windows times 300 bootstraps for each one times 460 programs, where 460 is the number of discrete values of z, a discretization that reduces the solution time, as explained in the appendix.

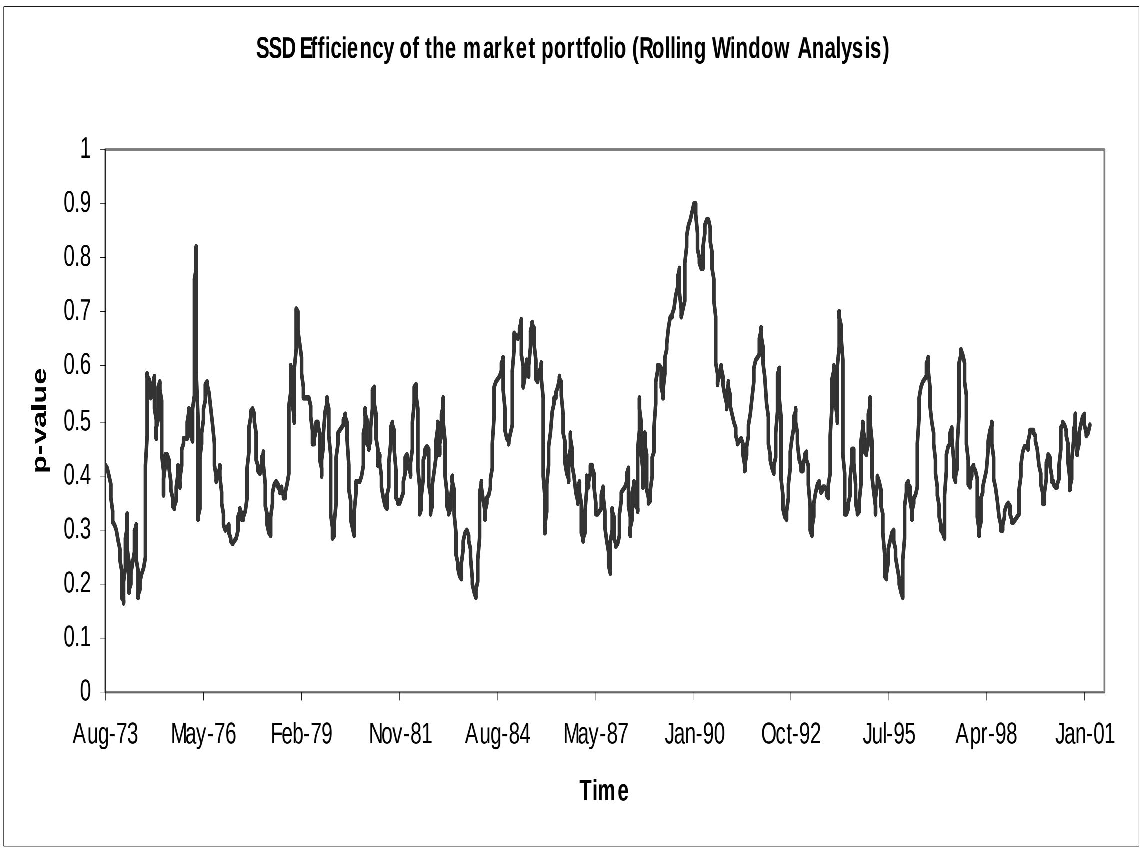

Figure 6.2: p-values for the second order stochastic dominance efficiency test using a rolling window of 120 months. The test statistic is calculated separately for 340 overlapping 10-year periods, (July 1963-June 1973), (August 1963-July 1973),...,(November 1991-October 2001). The second order stochastic dominance efficiency is not rejected once.

7 Concluding remarks

In this paper we develop *consistent* tests for stochastic dominance efficiency at *any* order for *time-dependent* data. We study tests for stochastic dominance efficiency of a given portfolio with respect to all possible portfolios constructed from a set of risky assets. We justify block bootstrap approaches to achieve valid inference in a time series setting. Linear as well as mixed integer programs are formulated to compute the test statistics.

To illustrate the potential of the proposed test statistics, we test whether different stochastic dominance efficiency criteria (first and second order) rationalize the Fama and French market portfolio over six Fama and French benchmark portfolios constructed as the intersections of two ME portfolios and three BE/ME portfolios. Empirical results indicate that we cannot reject that the market portfolio is first and second order stochastic dominance efficient. The result for the second order is also confirmed in a rolling window analysis. In contrast, the market portfolio is mean-variance inefficient, indicating the weakness of the variance to capture the risk.

The next step in this line of work is to develop estimators of efficiency lines as suggested by Davidson and Duclos (2000) for poverty lines in stochastic dominance. For the first order we should estimate the smallest return at which the distribution associated with the portfolio under test and the smallest distribution generated by any portfolios built from the same set of assets intersect. Similarly we could rely on an intersection between integrals of these distributions to determine efficiency line at higher orders.

Another future step in this direction is to extend inference procedures to prospect stochastic dominance efficiency and Markowitz stochastic dominance efficiency. These concepts allows to take into account that investors react differently to gains and losses and have S-shaped or reverse S-shaped utility functions. The development of conditions to be tested for other stochastic dominance efficiency concepts is of great interest.

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APPENDIX

In the proofs we use the shorter notation: $D_j(z, \boldsymbol{\tau}, \boldsymbol{\lambda}; F) := \mathcal{J}_j(z, \boldsymbol{\tau}; F) - \mathcal{J}_j(z, \boldsymbol{\lambda}; F)$. We often remove arguments, and use ˆ to indicate dependence on the empirical distribution. All limits are taken as T goes to infinity.

A Proof of Proposition 2.2

1. Proof of Part (i):

By the definition of $\hat{S}_j$ and $D_j \leq 0$ for all z and for all $\boldsymbol{\lambda}$ under H_0^j, we get: $\hat{S}_j \leq \sqrt{T} \sup q[\hat{D}_j - D_j] + \sqrt{T} \sup q D_j \leq \sqrt{T} \sup q[\hat{D}_j - D_j]$. Writing the latter expression with the sum of the following six quantities:

$$
\begin{aligned}
&\hat{Q}_j^1(L, z, \boldsymbol{\tau}, \boldsymbol{\lambda}) := \\
&\mathbb{I}\{-\infty < z \leq -L\} q(z) \frac{\sqrt{T}}{(j-1)!} \int_{\mathbb{R}^n} d\left[\hat{F}(\boldsymbol{u}) - F(\boldsymbol{u})\right] \\
&\qquad \left[(z - \boldsymbol{\tau}'\boldsymbol{u})^{j-1} \mathbb{I}\{\boldsymbol{\tau}'\boldsymbol{u} \leq z\} - (z - \boldsymbol{\lambda}'\boldsymbol{u})^{j-1} \mathbb{I}\{\boldsymbol{\lambda}'\boldsymbol{u} \leq z\}\right], \\
&\hat{Q}_j^2(L, z, \boldsymbol{\tau}, \boldsymbol{\lambda}) := \\
&\mathbb{I}\{-L < z \leq L\} q(z) \frac{\sqrt{T}}{(j-1)!} \int_{\mathbb{R}^n} d\left[\hat{F}(\boldsymbol{u}) - F(\boldsymbol{u})\right] \\
&\qquad \left[(z - \boldsymbol{\tau}'\boldsymbol{u})^{j-1} \mathbb{I}\{\boldsymbol{\tau}'\boldsymbol{u} \leq -L\} - (z - \boldsymbol{\lambda}'\boldsymbol{u})^{j-1} \mathbb{I}\{\boldsymbol{\lambda}'\boldsymbol{u} \leq -L\}\right], \\
&\hat{Q}_j^3(L, z, \boldsymbol{\tau}, \boldsymbol{\lambda}) := \\
&\mathbb{I}\{-L < z \leq L\} q(z) \frac{\sqrt{T}}{(j-1)!} \int_{\mathbb{R}^n} d\left[\hat{F}(\boldsymbol{u}) - F(\boldsymbol{u})\right] \\
&\qquad \left[(z - \boldsymbol{\tau}'\boldsymbol{u})^{j-1} \mathbb{I}\{\boldsymbol{\tau}'\boldsymbol{u} \leq z\} - (z - \boldsymbol{\lambda}'\boldsymbol{u})^{j-1} \mathbb{I}\{\boldsymbol{\lambda}'\boldsymbol{u} \leq z\}\right], \\
&\hat{Q}_j^4(L, z, \boldsymbol{\tau}, \boldsymbol{\lambda}) := \\
&\mathbb{I}\{L < z \leq +\infty\} q(z) \frac{\sqrt{T}}{(j-1)!} \int_{\mathbb{R}^n} d\left[\hat{F}(\boldsymbol{u}) - F(\boldsymbol{u})\right] \\
&\qquad \left[(z - \boldsymbol{\tau}'\boldsymbol{u})^{j-1} \mathbb{I}\{\boldsymbol{\tau}'\boldsymbol{u} \leq -L\} - (z - \boldsymbol{\lambda}'\boldsymbol{u})^{j-1} \mathbb{I}\{\boldsymbol{\lambda}'\boldsymbol{u} \leq -L\}\right], \\
&\hat{Q}_j^5(L, z, \boldsymbol{\tau}, \boldsymbol{\lambda}) := \\
&\mathbb{I}\{L < z \leq +\infty\} q(z) \frac{\sqrt{T}}{(j-1)!} \int_{\mathbb{R}^n} d\left[\hat{F}(\boldsymbol{u}) - F(\boldsymbol{u})\right]
\end{aligned}
$$

$$\left[(z-\boldsymbol{\tau}'\boldsymbol{u})^{j-1}\mathbb{I}\{-L \leq \boldsymbol{\tau}'\boldsymbol{u} \leq L\} - (z-\boldsymbol{\lambda}'\boldsymbol{u})^{j-1}\mathbb{I}\{-L \leq \boldsymbol{\lambda}'\boldsymbol{u} \leq L\}\right],$$

$$\hat{Q}_j^6(L, z, \boldsymbol{\tau}, \boldsymbol{\lambda}) :=$$

$$\mathbb{I}\{L < z \leq +\infty\}q(z)\frac{\sqrt{T}}{(j-1)!}\int_{\mathbb{R}^n} d\left[\hat{F}(\boldsymbol{u}) - F(\boldsymbol{u})\right]$$
$$\left[(z-\boldsymbol{\tau}'\boldsymbol{u})^{j-1}\mathbb{I}\{L \leq \boldsymbol{\tau}'\boldsymbol{u} \leq z\} - (z-\boldsymbol{\lambda}'\boldsymbol{u})^{j-1}\mathbb{I}\{L \leq \boldsymbol{\lambda}'\boldsymbol{u} \leq z\}\right],$$

we get for $|\theta| \leq 1$:

$$\sqrt{T}\sup q[\hat{D}_j - D_j] = \sup[\hat{Q}_j^3 + \hat{Q}_j^5] + \theta[\sup|\hat{Q}_j^1| + \sup|\hat{Q}_j^2| + \sup|\hat{Q}_j^4| + \sup|\hat{Q}_j^6|]. \quad \text{(A.1)}$$

A similar equality holds true for the limit $\bar{S}_j$ but based on $\mathcal{B}\circ F$ instead of $\sqrt{T}(\hat{F}-F)$, namely for $|\bar{\theta}| \leq 1$

$$\bar{S}_j = \sup[Q_j^3 + Q_j^5] + \bar{\theta}[\sup|Q_j^1| + \sup|Q_j^2| + \sup|Q_j^4| + \sup|Q_j^6|]. \quad \text{(A.2)}$$

Then as in Horvath, Kokoszka, and Zitikis (2006) we deduce the weak convergence of $\hat{S}_j$ to $\bar{S}_j$ by letting L go to infinity since only the first supremum in the right hand side of (A.1) and (A.2) contributes asymptotically under the stated assumptions on q and G.

2. Proof of Part (ii):

If the alternative is true, then there exists some z and some $\boldsymbol{\lambda}$, say $\bar{z} \in \mathbb{R}$ and $\bar{\boldsymbol{\lambda}} \in \mathbb{L}$, for which $D_j(\bar{z}, \boldsymbol{\tau}, \bar{\boldsymbol{\lambda}}; F) =: \delta > 0$. Then the result follows using the inequality $\hat{S}_j \geq q(\bar{z})\sqrt{T}D_j(\bar{z}, \boldsymbol{\tau}, \bar{\boldsymbol{\lambda}}; \hat{F})$, and the weak convergence of $q(\cdot)\sqrt{T}[D_j(\cdot; \hat{F}) - D_j(\cdot; F)]$.

B Proof of Proposition 3.1

Conditionally on the sample, we have that

$$\sqrt{T}(\hat{F}^* - \hat{F}) \overset{p}{\Longrightarrow} \mathcal{B}^* \circ F, \quad \text{(B.1)}$$

where $\mathcal{B}^* \circ F$ is an independent copy of $\mathcal{B} \circ F$ (Bühlmann (1994), Peligrad (1998)).

We can see that the functional $D_j(\cdot;F)$ is Hadamard differentiable at F by induction. Indeed $D_1(\cdot;F)$ is a linear functional, while $D_j(\cdot;F)$ is also a linear functional of a Hadamard differentiable mapping $D_{j-1}(\cdot;F)$. The delta method (van der Vaart and Wellner (1996) Chapter 3.9), the continuous mapping theorem and (B.1) then yields:

$$S_j^* \overset{p}{\Longrightarrow} \sup_{z,\boldsymbol{\lambda}} q(z) D_j(z,\boldsymbol{\tau},\boldsymbol{\lambda};\mathcal{B}^* \circ F), \tag{B.2}$$

where the latter random variable is an independent copy of $\bar{S}_j$.

Note that the median of the distribution $P_j^0(t)$ of $\sup_{z,\boldsymbol{\lambda}} q(z)D_j(z,\boldsymbol{\tau},\boldsymbol{\lambda};\mathcal{B}' \circ F)$ is strictly positive and finite. Since $q(z)D_j(z,\boldsymbol{\tau},\boldsymbol{\lambda};\mathcal{B}' \circ F)$ is a Gaussian process, P_j^0 is absolutely continuous (Tsirel'son (1975)), while $c_j(\alpha)$ defined by $P[\bar{S}_j > c_j(\alpha)] = \alpha$ is finite and positive for any $\alpha < 1/2$ (Proposition A.2.7 of van der Vaart and Wellner (1996)). The event $\{p_j^* < \alpha\}$ is equivalent to the event $\{\hat{S}_j > c_j^*(\alpha)\}$ where

$$\inf\{t : P_j^*(t) > 1-\alpha\} = c_j^*(\alpha) \xrightarrow{p} c_j(\alpha), \tag{B.3}$$

by (B.2) and the aforementioned properties of P_j^0. Then:

$$\begin{aligned}
\lim P[\text{reject } H_0^j | H_0^j] &= \lim P[\hat{S}_j > c_j^*(\alpha)] \\
&= \lim P[\hat{S}_j > c_j(\alpha)] + \lim(P[\hat{S}_j > c_j^*(\alpha)] - P[\hat{S}_j > c_j(\alpha)]) \\
&\leq P[\bar{S}_j > c_j(\alpha)] := \alpha,
\end{aligned}$$

where the last statement comes from (B.3), part i) of Proposition 2.2 and $c_j(\alpha)$ being a continuity point of the distribution of $\bar{S}_j$. On the other hand part ii) of Proposition 2.2 and $c_j(\alpha) < \infty$ ensure that $\lim P[\text{reject } H_0^j | H_1^j] = 1$.

C Mathematical programming formulations

C.1 Formulation for first order stochastic dominance

The initial formulation for the test statistic $\hat{S}_1$ for first order stochastic dominance efficiency is Model (4.1).

We reformulate the problem in order to reduce the solving time and to obtain a tractable formulation. The steps are the following:

1) The factor $\sqrt{T}/T$ can be left out in the objective function, since T is fixed.

2) We can see that there is a set of at most T values, say $\mathcal{R} = \{r_1, r_2, ..., r_T\}$, containing the optimal value of the variable z.

Proof: Vectors $\boldsymbol{\tau}$ and $\boldsymbol{Y}_t$, $t = 1, ..., T$ being given, we can rank the values of $\boldsymbol{\tau}'\boldsymbol{Y}_t, t = 1, ..., T$, by increasing order. Let us call $r_1, ..., r_T$ the possible different values of $\boldsymbol{\tau}'\boldsymbol{Y}_t$, with $r_1 < r_2 < ... < r_T$ (actually there may be less than T different values). Now, for any z such that $r_i \le z \le r_i + 1$, $\sum\limits_{t=1,...,T} L_t$ is constant (it is equal to the number of t such that $\boldsymbol{\tau}'\boldsymbol{Y}_t \le r_i$). Further, when $r_i \le z \le r_i + 1$, the maximum value of $-\sum\limits_{t=1,...,T} W_t$ is reached for $z = r_i$. Hence, we can restrict z to belong to the set $\mathcal{R}$.

3) A direct consequence is that we can solve first order stochastic dominance efficiency by solving the smaller problems $P(r)$, $r \in \mathcal{R}$, in which z is fixed to r. Then we take take the value for z that yields the best total result. The advantage is that the optimal values of the L_t variables are known in $P(r)$. Precisely, $\sum\limits_{t=1,...,T} L_t$ is equal to the number of t such that $\boldsymbol{\tau}'\boldsymbol{Y}_t \le r$. Hence problem $P(r)$ boils down to:

$$\begin{aligned}
\min \quad & \sum_{t=1}^{T} W_t \\
\text{s.t.} \quad & M(W_t - 1) \le r - \boldsymbol{\lambda}'\boldsymbol{Y}_t \le MW_t, \quad \forall t \in T \\
& \boldsymbol{e}'\boldsymbol{\lambda} = 1, \\
& \boldsymbol{\lambda} \ge 0, \\
& W_t \in \{0, 1\}, \quad \forall t \in T.
\end{aligned} \tag{C.1a}$$

Note that this becomes a minimization problem.

Problem $P(r)$ amounts to find the largest set of constraints $\boldsymbol{\lambda}'\boldsymbol{Y}_t \ge r$ consistent with $\boldsymbol{e}'\boldsymbol{\lambda} = 1$ and $\boldsymbol{\lambda} \ge 0$.

Let $M_t = \min \boldsymbol{Y}_{t,i}, i = 1, ..., n$, i.e., the smallest entry of vector $\boldsymbol{Y}_t$.

Clearly, for all $\boldsymbol{\lambda} \geq 0$ such that $\boldsymbol{e}'\boldsymbol{\lambda} = 1$, we have that $\boldsymbol{\lambda}'\boldsymbol{Y}_t \geq M_t$. Hence, Problem $P(r)$ can be rewritten in an even better reduced form:

$$\begin{aligned}
\min \quad & \sum_{t=1}^{T} W_t \\
\text{s.t.} \quad & \boldsymbol{\lambda}'\boldsymbol{Y}_t \geq r - (r - M_t)W_t, \quad \forall t \in T \\
& \boldsymbol{e}'\boldsymbol{\lambda} = 1, \\
& \boldsymbol{\lambda} \geq 0, \\
& W_t \in \{0, 1\}, \quad \forall t \in T.
\end{aligned} \tag{C.2a}$$

We further simplify $P(r)$ by fixing the following variables:

- for all t such that $r \leq M_t$, the optimal value of W_t is equal to 0 since the half space defined by the t-th inequality contains the simplex.

- for all t such that $r \geq M_t$, the optimal value of W_t is equal to 1 since the half space defined by the t-th inequality has an empty intersection with the simplex.

The computational time for this mixed integer programming formulation is significantly reduced. For the optimal solution (which involves 460 mixed integer optimization programs, one for each discrete value of z) it takes less than two hours. The problems are optimized with IBM's CPLEX solver on an Intel Xeon workstation (with a 2*2.4 GHz Power, 6Gb of RAM). We note the almost exponential increase in solution time with the increasing number of observations. We stress here the computational burden that is managed for these tests. The optimization problems are modelled using two different optimization languages: GAMS and AMPL. The General Algebraic Modeling System (GAMS) is a high-level modeling system for mathematical programming and optimization. It consists of a language compiler and a stable of integrated high-performance solvers. GAMS is tailored for complex, large scale modeling applications. A Modeling LAnguage for Mathematical Programming (AMPL) is a comprehensive and powerful algebraic modeling language for linear and nonlinear optimization problems, in discrete or continuous variables

We solve the problem using both GAMS and AMPL. These languages call special solvers (CPLEX in our case) that are specialized in linear and mixed integer programs. CPLEX uses the branch and bound technique to solve the MIP program. The Matlab code (where the simulations run) calls the AMPL or GAMS program, which calls the CPLEX solver to solve the optimization. This procedure is repeated thousand times for the needs of the Monte Carlo experiments and of the empirical application. The procedure codes are available on request from the authors.

The problems could probably be solved more efficiently by developing specialized algorithms that exploit the structure of the mixed integer programming models. However, issues of improving computational efficiency beyond what we manage are not of primary concern in this study.

C.2 Formulation for second order stochastic dominance

The initial formulation for the test statistic $\hat{S}_2$ for second order stochastic dominance efficiency is Model (4.3).

We reformulate the problem, following the same steps as for first order stochastic dominance efficiency. Then the model is transformed to a linear program, which is very easy to solve.

We solve second order stochastic dominance efficiency by solving again smaller problems $P(r)$, $r \in \mathcal{R}$, in which z is fixed to r, before taking the value for z that yields the best total result. The advantage is that the optimal values of the L_t variables are known in $P(r)$. Precisely, $L_t = r - \boldsymbol{\tau}'\boldsymbol{Y}_t$, for the scenarios for which this difference is positive, and zero otherwise. Hence problem $P(r)$ boils down to the linear problem:

$$\begin{aligned}
\min \quad & \sum_{t=1}^{T} W_t \\
\text{s.t.} \quad & W_t \geq r - \boldsymbol{\lambda}'\boldsymbol{Y}_t, \quad \forall t \in T \\
& \boldsymbol{e}'\boldsymbol{\lambda} = 1, \\
& \boldsymbol{\lambda} \geq 0, \\
& W_t \geq 0, \quad \forall t \in T.
\end{aligned} \tag{C.3a}$$

The computational time for this linear programming formulation is very small. To get the optimal solution (which involves 460 linear optimization programs, one for each discrete value of z) using the CPLEX solver, it takes three minutes on average. We can have an even better formulation of this latter model. Instead of solving it for each discrete time of z, we can reformulate the model in order to solve for all discrete values $r_i, i = 1, ..., T$ simultaneously. The new model is the following:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{T}\sum_{t=1}^{T} W_{i,t} \\
\text{s.t.} \quad & W_{i,t} \geq r_i - \boldsymbol{\lambda}_i'\boldsymbol{Y}_t, \quad \forall i, \quad \forall t, \\
& \boldsymbol{e}'\boldsymbol{\lambda}_i = 1, \quad \forall i, \\
& \boldsymbol{\lambda}_i \geq 0, \quad \forall i, \\
& W_{i,t} \geq 0, \quad \forall i, \quad \forall t.
\end{aligned} \tag{C.4a}$$

The optimal portfolio $\boldsymbol{\lambda}_i$ and the optimal value r_i of variable z are for that i, that gives $\min \sum_{t=1}^{T} W_{i,t}$. Now, the computational time for this formulation of the problem is less than a minute.

References

[1] Abadie, A. (2002). Bootstrap Tests for Distributional Treatment Effects in Instrumental Variable Models. Journal of the American Statistical Association 97, 284-292.

[2] Anderson, G. (1996). Nonparametric Tests of Stochastic Dominance in Income Distributions. Econometrica 64, 1183-1193.

[3] Andrews, D. (2002). Higher-Order Improvements of a Computationally Attractive k-Step Bootstrap for Extremum Estimators. Econometrica 64, 891-916.

[4] Barrett, G. and Donald, S. (2003). Consistent Tests for Stochastic Dominance. Econometrica 71, 71-104.

[5] Beach, C. and Davidson, R. (1983). Distribution-Free Statistical Inference with Lorenz Curves and Income Shares. Review of Economic Studies 50, 723-735.

[6] Bühlmann, P. (1994). Blockwise Bootstrapped Empirical Process for Stationary Sequences. Annals of Statistics 22, 995-1012.

[7] Carlstein, E. (1986). The Use of Subseries Methods for Estimating the Variance of a General Statistic from a Stationary Time Series. Annals of Statistics 14, 1171-1179.

[8] Carrasco, M. and Chen, X. (1998). Mixing and Moment Properties of Various GARCH and Stochastic Volatility Models. Econometric Theory 18, 17-39.

[9] Davidson, R. and Duclos, J.-Y. (2000). Statistical Inference for Stochastic Dominance and for the Measurement of Poverty and Inequality. Econometrica 68, 1435-1464.

[10] Davidson, R. and MacKinnon, J. (2006a). The Power of Bootstrap and Asymptotic Tests. Journal of Econometrics 127, 421-441.

[11] Davidson, R. and MacKinnon, J. (2006b). Improving the Reliability of Bootstrap Tests with the Fast Double Bootstrap. Working paper.

[12] De Giorgi, E. (2005). Reward-Risk Portfolio Selection and Stochastic Dominance. Journal of Banking and Finance 29, 895-926.

[13] De Giorgi, E. and Post, T. (2005). Second Order Stochastic Dominance, Reward-Risk Portfolio Selection and the CAPM, forthcoming in Journal of Financial and Quantitative Analysis.

[14] Doukhan, P. (1994). Mixing: Properties and Examples. Lecture Notes in Statistics, 85. Springer-Verlag, Berlin.

[15] Fama, E. and French, K. (1993). Common Risk Factors in the Returns on Stocks and Bonds. Journal of Financial Economics 33, 3-56.

[16] Hall, P. and Horowitz, J. (1996). Bootstrap Critical Values for Tests Based on Generalized-Method-of-Moment Estimators. Econometrica 64, 891-916.

[17] Hall, P., Horowitz, J. and B.-Y. Jing (1995). On Blocking Rules for the Bootstrap with Dependent Data. Biometrika 82, 561-574.

[18] Hansen, B. (1996). Inference when a Nuisance Parameter is not Identified under the Null Hypothesis. Econometrica 64, 413-430.

[19] Horvath, L. and Kokoszka, P. and R. Zitikis (2006). Testing for Stochastic Dominance Using the Weighted McFadden-type Statistic. Journal of Econometrics 133, 191-205.

[20] Klecan, L., McFadden, R. and D. McFadden (1991). A Robust Test for Stochastic Dominance. Working Paper, Department of Economics, MIT.

[21] Kroll, Y. and H. Levy (1980). Stochastic Dominance Criteria : a Review and Some New Evidence. Research in Finance Vol. II, 263-227. JAI Press, Greenwich.

[22] Kunsch, H. (1989). The Jackknife and the Bootstrap for General Stationary Observations. Annals of Statistics 17, 1217-1241.

[23] Kuosmanen, T. (2004). Efficient Diversification According to Stochastic Dominance Criteria. Management Science 50, 1390-1406.

[24] Lehmann, E. (1986). Testing Statistical Hypotheses. 2nd Ed.. John Wiley & Sons, New York.

[25] Levy, H. (1973). Stochastic Dominance among Lognormal Prospects. International Economic Review 14, 601-614.

[26] Levy, H. (1982). Stochastic Dominance Rules for Truncated Normal Distributions: a Note. Journal of Finance 37, 1299-1303.

[27] Levy, H. (1992). Stochastic Dominance and Expected Utility : Survey and Analysis. Management Science 38, 555-593.

[28] Levy, H. (1998). Stochastic Dominance. Kluwer Academic Publisher, Boston.

[29] Lintner, J. (1965). Security Prices, Risk and Maximal Gains from Diversification. Journal of Finance 20, 587-615.

[30] Linton, O., Maasoumi, E. and Whang, Y.-J. (2005). Consistent Testing for Stochastic Dominance under General Sampling Schemes. Review of Economic Studies 72, 735-765.

[31] Linton, O., Post, T. and Y.-J. Whang (2005). Testing for Stochastic Dominance Efficiency. LSE working paper.

[32] McFadden, D. (1989). Testing for Stochastic Dominance. Studies in the Economics of Uncertainty eds Fomby T. and Seo T., 113-134. Springer-Verlag, New York.

[33] Mosler, K. and M. Scarsini (1993). Stochastic Orders and Applications, a Classified Bibliography. Springer-Verlag, Berlin.

[34] Nelson, R. and R. Pope (1992). Bootstrapped Insights into Empirical Applications of Stochastic Dominance. Management Science 37, 1182-1194.

[35] Nemhauser, G. and Wolsey, L. (1999). Integer and Combinatorial Optimization. John Wiley & Sons, New York.

[36] Peligrad, M. (1998). On the Blockwise Bootstrap for Empirical Processes for Stationary Sequences. Annals of Probability 26, 877-901.

[37] Post, T. (2003). Empirical Tests for Stochastic Dominance Efficiency. Journal of Finance 58, 1905-1031.

[38] Post, T. (2005). Wanted: A Test for FSD Optimality of a Given Portfolio. Erasmus University working paper.

[39] Rio, E. (2000). Théorie Asymptotique des Processus Aléatoires Faiblement Dépendants. Mathématiques et Applications, 31. Springer-Verlag, Berlin.

[40] Shaked, M. and Shanthikumar, J. (1994). Stochastic Orders and their Applications. Academic Press, New York.

[41] Sharpe, W. (1964). Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. Journal of Finance 19, 425-442.

[42] Tsirel'son, V. (1975). The Density of the Distribution of the Maximum of a Gaussian Process. Theory of Probability and their Applications 16, 847-856.

[43] van der Vaart, A. and Wellner, J. (1996). Weak Convergence and Empirical Processes. Springer Verlag, New York.