Optimal asset management for pension funds

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Research Paper

Abstract

Purpose of this paper: we study the asset allocation problem for a pension fund which maximizes the expected present value of its wealth augmented by the prospective mathematical reserve at the death time of a representative member.

Design/methodology/approach: we apply the stochastic optimization technique in continuous time. In order to present an explicit solution we consider the case of both deterministic interest rate and market price of risk.

Findings: we demonstrate that the optimal portfolio is always less risky than the Merton's (1969-1971) one. In particular, the asset allocation is less and less risky until the pension date while, after retirement of the fund's representative member, it becomes riskier and riskier.

Practical implications: the paper shows the best way for managing a pension fund portfolio during both the accumulation and the decumulation phases.

Originality/value: the paper fills a gap in the optimal portfolio literature about the joint analysis of both the actuarial and the financial framework. In particular, we show that the actuarial part strongly affects the behaviour of the optimal asset allocation.

JEL classification: G23, G11.

Key words: pension fund, asset allocation, mortality risk, inflation risk.

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1 Introduction

Up to now, most countries that have undertaken a reform of their pension system have primarily focused on the accumulation phase and paid less attention to the decumulation phase. This is also true in the academic literature (see for instance, Deelstra *et al.*, 2000, Boulier *et al.*, 2001, and Battocchio and Menoncin, 2004). There are only few works dealing with the decumulation phase (see for instance, Charupat and Milevsky, 2002, and Battocchio *et al.*, 2003).

Even if old age pensions will not arise for many years, both accumulation and decumulation phases need to be well organized and efficient to guarantee full success of a new pension system. Indeed, the failure or not of a new pension system depends on its ability to use whatever capital has been amassed at the end of the active life of covered workers to supply them with a reasonably sufficient regular income.

In this work, we present a closed-form solution for optimal asset allocation during the accumulation and decumulation phases under mortality risk. The main difficulty in designing a dedicated framework for pension funds is the presence of non-tradeable endowment processes. Some closed-form solutions without any non-tradeable income sources have already been derived in the literature. After the seminal papers of Merton (1969, 1971), we mainly refer to the works of Kim and Omberg (1996), Chacko and Viceira (1999), Deelstra *et al.* (2000), Boulier *et al.* (2001), and Wachter (2002). In these papers the market structure is as follows: (i) there exists only one stochastic state variable (the riskless interest rate or the risk premium) following the Vasiček (1977) model or the Cox *et al.* (1985) model, (ii) there exists only one risky asset, (iii) a discount bond may exist. Some works consider a complete financial market (Deelstra *et al.*, 2000, Boulier *et al.*, 2001, and Wachter, 2002) while others deal with an incomplete market (Kim and Omberg, 1996, and Chacko and Viceira, 1999). Our setting, instead, is very general and it does not take into account any particular functional form for both the drift and diffusion terms of the stochastic processes involved in the framework (we mainly refer to Menoncin, 2002).

In order to be able to present a closed form solution for the optimal asset allocation, we assume that the risk sources of both contributions and pensions can be spanned in the financial market (as in Bodie *et al.,* 1992). Cuoco (1997) and El Karoui and Jeanblanc-Picqué (1998) offer an existence result for the optimal portfolio for a constrained investor who is endowed with a stochastic labor income flow.

We further assume that the pension fund is able to borrow against its prospective mathematical reserve. Thus, its objective is to maximize the intertemporal utility of its real wealth, augmented by the expected value of all the future contributions and diminished by the expected value of all the future pensions (let us call this modified wealth as "disposable wealth"). Furthermore, the objective function takes the form of a HARA (Hyperbolic Absolute Risk Aversion index) utility function which coincides with the power of fund disposable wealth. This assumption allows us to reach a closed-form solution that will be used in numerical simulations to provide useful guidelines for optimal asset allocation in a pension fund context.

Note that the case of a pension fund is different from the standard case of an investor having labor income. Indeed, the revenues (contributions) and expenses (pensions) of the fund must be linked by a condition ("feasibility condition") guaranteeing that it is profitable to contract for both the subscriber and the pension fund. To further enrich our framework, we introduce a deterministic profit sharing rule. This means that a proportion of the fund nominal surplus (i.e. the difference between the managed wealth and the contributions) is redistributed to the members, who thus share profits induced by the exposure to the risky assets.

Moreover, the link between contributions and pensions can be established inside one of the two following frameworks: the so-called defined-benefit pension plan (hereafter DB) or the so-called defined-contribution pension plan (hereafter DC). In a DB plan benefits are fixed in advance by the sponsor and contributions are initially set and subsequently adjusted in order to maintain the fund in balance. In a DC plan contributions are fixed and benefits depend on the returns on the fund portfolio. In particular, DC plans allow contributors to know, at each time, the value of their retirement accounts. Historically, fund managers have mainly proposed DB plans, which are definitely preferred by workers. Indeed, the financial risks associated with DB plans are supported by the plan sponsor rather than by individual members of the plan. Nowadays, most of the proposed pension plans are based on DC schemes involving a considerable transfer of risks to workers. Accordingly, DC pension funds provide contributors with a service of savings management, even if they do not guarantee any minimum performance. As we have already highlighted, only contributions are fixed in advance, while the final state of the retirement account depends fundamentally on the administrative and financial skills of the fund manager. Therefore, an efficient financial management is essential to gain contributor trust.

The continuous time model studied in this paper is able to describe both DB and DC pension plans since we take into account two different stochastic variables for contributions and pensions. Note that we do not require one of them to be necessarily deterministic. In order to reduce the model to a pure DB plan it is sufficient to equate the diffusion term of pensions to zero, while in a pure DC plan it is the diffusion term of contributions which must be equated to zero. We demonstrate, in a simplified framework, that the optimal portfolio for a pension providing a DC plan is almost always less risky than those of a pension providing its members with a DB plan.

Rudolf and Ziemba (2003) study a framework which is very similar to ours. Nevertheless, the points that distinguish our work are the following ones: (i) we study the effect of the mortality risk by explicitly taking into account a mortality law, (ii) we disentangle the impact of the accumulation and decumulation phases on the asset allocation.

Along the paper we consider agents trading continuously in a frictionless, arbitrage-free and complete market.

The paper is structured as follows. In Section 2 we first outline the general economic background and give the stochastic differential equations describing the dynamics of asset prices, state variables, contributions, and pensions. Then, we determine the evolution of the fund real wealth, and present the objective function to be maximized. In Section 3 the optimal portfolio allocation is computed. There we also present our main result: a closed-form solution of the problem if the financial market is complete and the market price of risk and the riskless interest rate are both independent of the state variables. Section 4 provides a numerical illustration based on a simple market structure. Section 5 concludes.

2 The model

In this paper we study how the manager of a pension fund can invest the fund wealth in order to optimize a given objective function. In order to identify the model, we show how we represent both the financial market and the objective function of the pension fund.

As for all the other investment funds, a pension fund must cope with a set of non-financial risks given by the contributions and withdrawals from the fund wealth. This risk is typically identified with the so-called background risk, which cannot be spanned (replicated) on the financial market. In a dynamic optimization framework, both contributions and withdrawals are state variables. In fact, the pension fund manager cannot control them.

A major difference between investment funds and pension funds is that these last ones must explicitly cope with the mortality risk. The introduction of such a risk in the analysis complicates a lot the computations.

2.1 The financial market

On the financial market there are m assets whose values (S) follow the stochastic differential equation

$$\underset{m\times 1}{dS} = \underset{m\times 1}{\mu (S,X,t)}dt + \underset{m\times d}{\Sigma (S,X,t)}'\underset{d\times 1}{dW}, \quad S(t_0) = S_0, \tag{1}$$

where W is a $d-$dimensional Wiener process, and the prime denotes transposition. The drift and diffusion terms μ and Σ are supposed to satisfy the usual Lipschitz conditions guaranteeing that Equation (1) has a unique strong solution (see Karatzas and Shreve, 1991). Furthermore, μ and Σ are $\mathcal{F}_t$-measurable, where $\mathcal{F}_t$ is the $\sigma-$algebra through which the Wiener processes are measured on the complete probability space $(\Theta, \mathcal{F}, \mathbb{P})$. All processes below are supposed to satisfy the same properties as those stated for Equation (1). Values of all variables are known at the initial date t_0 and are equal to the non-stochastic variable S_0.

The variable vector X contains all the state variables affecting the asset prices. It is assumed to satisfy the stochastic differential equation

$$\underset{s\times 1}{dX} = \underset{s\times 1}{\mu_X (X,t)}dt + \underset{s\times d}{\Sigma_X (X,t)'}\underset{d\times 1}{dW}, \quad X(t_0) = X_0. \tag{2}$$

Finally, there exists a riskless asset whose value G follows

$$dG = Gr(X,t)\,dt, \quad G(t_0) = 1.$$

The financial market is assumed to be complete ($\exists!\Sigma^{-1} \Rightarrow m = d$). Thus, there exits only one market price of risk given by

$$\xi = \Sigma'^{-1}(\mu - rS), \tag{3}$$

through which we can define the martingale equivalent measure

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_{t_0}^{H} \xi' dW_t - \frac{1}{2}\int_{t_0}^{H} \|\xi\|^2\, dt\right). \tag{4}$$

Furthermore, according to Girsanov Theorem, the stochastic process

$$dW^{\mathbb{Q}} = \xi dt + dW, \tag{5}$$

is a Wiener process with respect to $\mathbb{Q}$.

2.2 The contributions and pensions

During the life of a pension fund we can easily distinguish two phases:

1. the so-called "accumulation" phase (hereafter APh) during which the contributions are paid by the members of the fund during their work life.
2. the so-called "decumulation" phase during (hereafter DPh) which the fund pays the pensions to its members who retire.

Let T be the retirement date of a representative fund member, then the contributions (L_c) and pensions (L_p) can be represented through the same process as follows:

$$dL = \mathbb{I}_{t<T}dL_c - (1 - \mathbb{I}_{t<T})\,dL_p,$$

where $\mathbb{I}_{\mathcal{E}}$ is the indicator function whose value is 1 if the event $\mathcal{E}$ occurs and 0 otherwise.

Remark 1 *Observe that we indicate with L the cumulated contribution and pension processes. Thus, the contribution instantaneously paid (or the pension instantaneously received) between time t and time $t + dt$ is given by $dL(t)$.*

The management of a pension fund can be set in one of the following cases.

1. Defined contributions (hereafter DC): the contributions are set in advance while the pensions that will be paid after retirement will be set in order to keep the fund in balance. In this framework, the variable L_c is deterministic while the variable L_p is stochastic and depends on the performances of the pension fund (there could be a minimum guaranteed pension).

2. Defined benefits (hereafter DB): the pensions are set in advance while the contributions are adjusted during the accumulation phase. In this case the variable L_p is deterministic while L_c is stochastic and can depend on the performances of assets on the financial market.

In order to take into account both cases of DC and DB, we model both dL_c and dL_p as stochastic process

$$\begin{aligned} dL_c &= \mu_c (L_c, X, t)\, dt + \underset{1\times d}{\Sigma_c (L_c, X, t)'} \underset{d\times 1}{dW}, \qquad (6) \\ dL_p &= \mu_p (L_p, X, t)\, dt + \underset{1\times d}{\Sigma_p (L_p, X, t)'} \underset{d\times 1}{dW}, \qquad (7) \end{aligned}$$

where μ_c is the (positive) contribution rate while μ_p is the (positive) pension rate. In a DC scheme, we must put μ_c independent of X and $\Sigma_c = \mathbf{0}$, while in a DB scheme we must put μ_p independent of X and $\Sigma_p = \mathbf{0}$.

Here, dW is the same process as those driving the asset price risks. This hypothesis, together with those of a complete market, implies that the contributions and pensions can be spanned in the financial market. Accordingly, the contributions and pensions can be evaluated as any other asset. This means that, at each time t, the balance between contributions and pensions (let us call it Δ) can be written as

$$\Delta (t) = \mathbb{E}_t^{\mathbb{Q},\tau} \left[\int_t^{\tau} \frac{G(t)}{G(s)} dL(s) \right], \qquad (8)$$

where τ is the stochastic death time of a representative subscriber and the expectation is taken with respect to the martingale equivalent measure $\mathbb{Q}$ and to the death time τ. We underline that τ is assumed to be independent of all the other stochastic variables. Thus, by using the indicator function, the value of Δ can be written as

$$\begin{aligned} \Delta (t) &= \mathbb{E}_t^{\mathbb{Q},\tau} \left[\int_t^{\infty} \mathbb{I}_{s<\tau} \frac{G(t)}{G(s)} dL(s) \right] \\ &= \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{\infty} \mathbb{E}_t^{\tau} \left[\mathbb{I}_{s<\tau} \right] \frac{G(t)}{G(s)} dL(s) \right], \end{aligned}$$

and, since the expected value of an indicator function coincides with a probability, we have

$$\Delta (t) = \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{\infty} \mathbb{P}(\tau > s |\, x) \frac{G(t)}{G(s)} dL(s) \right],$$

where we have indicated with x the current age of the fund member. Let us call ${}_sp_x$ the conditional probability that an individual aged of x survives for s period.[1] Then, the function Δ becomes

$$\Delta(t) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\infty}({}_sp_x)\frac{G(t)}{G(s)}dL(s)\right]. \tag{9}$$

We underline that the variable Δ has an important role in the actuarial economics. In fact, it coincides with the prospective mathematical reserve. In the case of insurance companies this reserve is given by the present value of all the premia that will be received, diminished by the present value of all the indemnities.

Now, after some algebraic manipulation and substituting in (6) and (7) the stochastic process $dW^{\mathbb{Q}}$ defined in (5), we have

$$\begin{aligned}
\Delta(t) &= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{T}({}_sp_x)\frac{G(t)}{G(s)}dL_c + \int_T^{\infty}({}_sp_x)\frac{G(t)}{G(s)}dL_p\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{T}({}_sp_x)\frac{G(t)}{G(s)}(\mu_c - \Sigma_c'\xi)\,ds\right] \\
&\quad -\mathbb{E}_t^{\mathbb{Q}}\left[\int_T^{\infty}({}_sp_x)\frac{G(t)}{G(s)}\left(\mu_p - \Sigma_p'\xi\right)ds\right]
\end{aligned}$$

The present value of all pensions must equate the present value of all contributions if we want the pension scheme to be suitable for both the pension fund and the member. Thus, in order to guarantee a balance in t_0 when the pension fund is subscribed, the value of $\Delta(t_0)$ must be zero.

Definition 2 *A pair of positive contribution and pension rates (μ_c^*, μ_p^*) is said to be feasible if it satisfies the condition*

$$\int_{t_0}^{T}({}_sp_x)\,\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\frac{G(t_0)}{G(s)}(\mu_c^* - \Sigma_c'\xi)\right]ds = \int_T^{\infty}({}_sp_x)\,\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\frac{G(t_0)}{G(s)}\left(\mu_p^* - \Sigma_p'\xi\right)\right]ds. \tag{10}$$

A similar feasibility condition is imposed in Josa-Fombellida and Rincón-Zapatero (2001) where the authors examine the problem of a firm which must pay both wages (before its workers retire) and pensions (after they retire). Accordingly, the feasibility condition implies the equality between the total expected value of wages and pensions paid with the total expected value of worker productivity (according to the usual economic rule equating the optimal wage with the marginal product of labour).

[1] Formally ${}_sp_x = \exp\left(-\int_x^{x+s}\lambda(t)\,dt\right)$, where $\lambda(t)$ is the *instantaneous hazard rate*. As Merton (1990, Section 18.2) underlines, $\lambda(t)$ takes the usual interpretation of the force measuring the probability that the person will die between t and $t+dt$.

Of course, there are an infinite number of combinations of positive μ_c and μ_p which satisfy (10). This means that a pension fund can supply its members with a lot of possible pension schemes. Let us show two particular cases:

1. in a DC framework ($\Sigma_c = \mathbf{0}$) when the contribution rate is kept constant, the pension fund can pay a pension rate such that

$$\mu_c^* = \frac{\int_T^\infty ({}_s p_x)\, \mathbb{E}_{t_0}^{\mathbb{Q}} \left[G(s)^{-1} \left(\mu_p^* - \Sigma_p' \xi \right) \right] ds}{\int_{t_0}^T ({}_s p_x)\, \mathbb{E}_t^{\mathbb{Q}} \left[G(s)^{-1} \right] ds},$$

   and $\mu_c^* > 0$;

2. in a DB framework ($\Sigma_p = \mathbf{0}$) when the pension rate is kept constant, the pension fund can ask for a contribution rate such that

$$\mu_p^* = \frac{\int_{t_0}^T ({}_s p_x)\, \mathbb{E}_{t_0}^{\mathbb{Q}} \left[G(s)^{-1} \left(\mu_c^* - \Sigma_c' \xi \right) \right] ds}{\int_T^\infty ({}_s p_x)\, \mathbb{E}_{t_0}^{\mathbb{Q}} \left[G(s)^{-1} \right] ds},$$

   and $\mu_p^* > 0$.

It is important to stress that Equation (10), as already confirmed by the previous two cases, states that higher contribution rates (must) correspond to higher pension rates. In fact, since we are in a fully funded pension system, the pensioners just take the cumulated amount of what they have put in the pension fund during their work-life. Accordingly, it is obvious that there must be a positive relationship between contributions and pensions.

In particular, when contributions and pensions are both constant, the relationship (10) implies that contributions must be an affine transformation of pensions. Furthermore, when we are in a DB-DC scheme (i.e. $\Sigma_c = \Sigma_p = \mathbf{0}$), the feasible pension rate is just proportional to the feasible contribution rate. As we will show in the next subsections, in this last case any positive contribution is available to the member since the feasible ratio is always positive.

Since it will be very useful in what follows, we want now to determine the sign of the prospective mathematical reserve. The feasible contributions and pensions are set in such a way that the prospective reserve is zero when the member enters the fund (i.e. $\Delta(t_0) = 0$). Just after receiving the first contribution, the prospective reserve becomes negative since it still contains the present value of all the future pensions that will have to be paid, but it contains one contribution less. So, until the retirement date (T) while time goes on and the contributions start being received by the fund, the prospective reserve contains less and less positive values (contributions) and it decreases accordingly. After the retirement date, the prospective reserve just contains negative values. In fact, there are no more contributions and it coincides with the present value of all the future pensions that the fund will pay to the members. Nevertheless, while these pensions are being paid, the present value of all the other (future) pensions becomes lower and lower. In fact, the death probability becomes higher and higher. This means that, after T, the prospective reserve starts increasing and it tends toward zero. In the next subsection, after presenting a particular form for the conditional survival probability ${}_sp_x$, we will show a graph representing this behaviour of the prospective reserve.

2.3 The mortality law

One of the most used distribution function for the survival event is the so-called Gompertz-Makeham distribution. The probability to be alive after t periods for an individual aged of x is given by[2]

$$
{}_tp_x = \exp\left(-\phi t + e^{\frac{x-m}{b}}\left(1 - e^{\frac{t}{b}}\right)\right), \tag{11}
$$

where ϕ is a positive constant measuring accidental deaths linked to non-age factors, while m and b are modal and scaling parameters of the distribution, respectively. When either b or m tend to infinity we have the exponential distribution of the form

$$
{}_tp_x = e^{-\phi t},
$$

whose force of mortality (ϕ) is constant and does not depend on the agent's age (in fact ϕ measures the non-age factors).

As it will be useful in what follows, we now define the "mortality force" (λ) as the opposite of the elasticity of ${}_tp_x$ with respect to time. In differential form, the mortality force must solve the following differential equation

$$
d\left({}_tp_x\right) = -\lambda\left(t\right)\left({}_tp_x\right)dt, \quad \left({}_0p_x\right) = 1. \tag{12}
$$

In order to understand better the role of the two parameters m and b we have plotted in Figures (1) and (2) the values of Function (11) for different values of m and b, respectively.

For the sake of simplicity, we have plotted the so-called "pure Gompertz" case by taking $\phi = 0$. The central values of parameters m (88.18 for male and 92.63 for female) and b (10.5 for male and 8.78 for females) have been estimated in Milevsky (2001) where the author prices all annuities using the Individual Annuity Mortality (IAM) 2000 table, dynamically adjusted using scale G, published by the Society of Actuaries. The different behaviour of Function (11) for males (solid line) and females (dashed line) are represented in Figure 3.

We can observe that the survival probability till 50 years for males and 60 years for females is very high and close to 1. Then there is a sudden decrease and we reach a probability of surviving till 100 years that is almost zero for males but still positive for females.

Since it will be very useful in what follows, we now present in this framework how to compute the present value of a life annuity.

[2] It can be immediately checked that ${}_0p_x = 1$, ${}_\infty p_x = 0$ for any age x.

Figure 1: Behaviour of the Gompertz function with respect to different values of m [80 the solid line, 88 the dashed line, and 95 the pointed line]

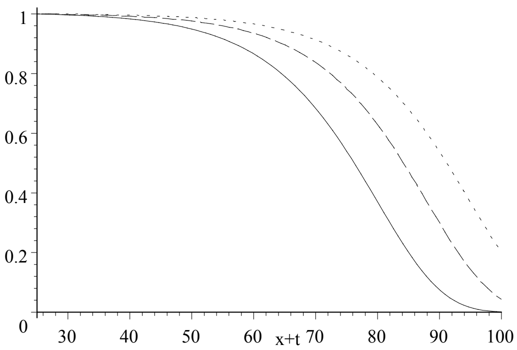


Figure 2: Behaviour of the Gompertz function with respect to different values of b [8 the solid line, 10 the dashed line, and 15 the pointed line]

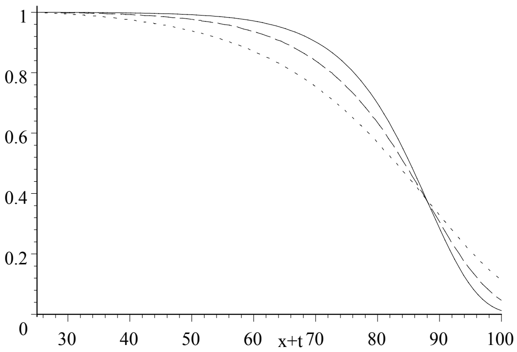
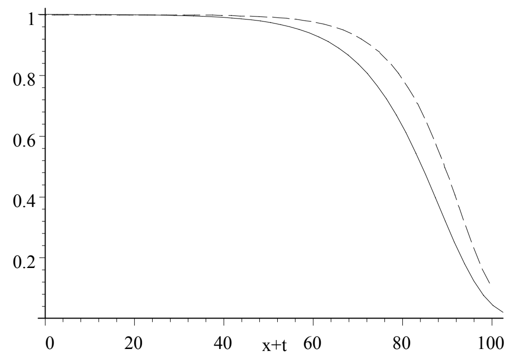


Figure 3: Survival probability for an individual aged of 20 (the pure Gompertz case)



Proposition 3 *Given the survival probability for t periods in Equation (11), when the riskless interest rate is constant, the value of a life annuity can be written as*

$$\int_{t_0}^{\infty} ({}_tp_x)\, e^{-r(t-t_0)} dt = b e^{(\phi+r)(x-m)+rt_0+e^{\frac{x-m}{b}}} \Gamma\left(-(\phi+r)\, b, e^{\frac{t_0+x-m}{b}}\right),$$

where

$$\Gamma(y_1, y_2) = \int_{y_2}^{\infty} e^{-t} t^{y_1-1} dt,$$

is the so-called incomplete gamma function.

Proof. Let us make the following variable substitution:

$$\begin{aligned} z &= e^{\frac{t+x-m}{b}} \Leftrightarrow t = -x + m + b \ln z, \\ dz &= \frac{1}{b} z dt \Leftrightarrow dt = b \frac{1}{z} dz, \end{aligned}$$

and so the integral can be written as

$$\int_{t_0}^{\infty} ({}_tp_x)\, e^{-r(t-t_0)} dt = b e^{(r+\phi)(x-m)+rt_0+e^{\frac{x-m}{b}}} \int_{e^{\frac{t_0+x-m}{b}}}^{\infty} z^{-(r+\phi)b-1} e^{-z} dz.$$

■

In the case mentioned by Proposition 3 the feasible ratio for constant contribution and pension rates can be simply represented as a ratio between two Gamma functions.

Corollary 4 *Given the survival probability for t periods as in Equation (11), when the riskless interest rate is constant, the constant contribution and pension rates satisfying Equation (10) for a DC-DB scheme must satisfy*

$$\frac{\mu_p^*}{\mu_c^*} = \frac{\Gamma\left(-(\phi+r)\,b, e^{\frac{t_0+x-m}{b}}\right)}{\Gamma\left(-(\phi+r)\,b, e^{\frac{T+x-m}{b}}\right)} - 1. \tag{13}$$

Proof. Since we have put μ_p^*, μ_c^*, and r constant, then Equation (10) simplifies to

$$\mu_c^* \int_{t_0}^{T} ({}_s p_x)\, e^{-r(s-t_0)} ds = \mu_p^* \int_{T}^{\infty} ({}_s p_x)\, e^{-r(s-t_0)} ds.$$

Furthermore, it is true that

$$\int_{t_0}^{T} ({}_s p_x)\, e^{-r(s-t_0)} ds = \int_{t_0}^{\infty} ({}_s p_x)\, e^{-r(s-t_0)} ds - \int_{T}^{\infty} ({}_s p_x)\, e^{-r(s-t_0)} ds,$$

and so

$$\frac{\mu_p^*}{\mu_c^*} = \frac{\int_{t_0}^{\infty} ({}_s p_x)\, e^{-r(s-t_0)} ds}{\int_{T}^{\infty} ({}_s p_x)\, e^{-r(s-t_0)} ds} - 1.$$

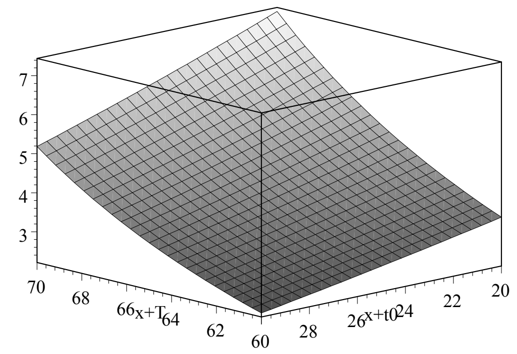
Now, all the integrals are in the form presented in Proposition 3 and, after substituting the value of $({}_s p_x)$ given by Equation (11) we obtain what presented in the corollary. ■

In Figure 4 we show the behaviour of the constant ratio (13) with respect to both t_0 (the time when the worker subscribes the fund) and T (the time when the worker retires). The interest rate is set to 0.02 while the parameters of Equation (11) are set to $(m, b, \phi) = (88.18, 10.5, 0)$.

We immediately see that (for a given value of the agent's age x) the value of μ_p^*/μ_c^* is increasing in T and decreasing in t_0. When the retirement age increases, the fund can afford to pay higher pensions since the contributions are paid during a longer period of time. On the contrary, when t_0 increases and the worker enters the fund at a higher age, the fund requires higher contributions in order to have the same level of pensions.

Thanks to the specification of the mortality law we can also represent the behaviour of the prospective reserve (8). In order to show the effect of a change in the mortality law on the prospective mathematical reserve let us suppose here that we are in a DC-DB scheme with constant contributions and pensions (i.e. $\Sigma_c = \Sigma_p = \mathbf{0}$). In this case the expected value operator in (8) must be computed only with respect to the stochastic death time and the prospective reserve has the closed form representation shown in the following corollary.

Figure 4: Feasible (constant) ratio μ_p^*/μ_c^* for constant interest rate



Corollary 5 *Given the survival probability for t periods as in Equation (11), when the riskless interest rate, the contribution and the pension rates are all constant, then the prospective mathematical reserve for a DC-DB scheme can be written as*

$$\begin{aligned}\Delta(t) &= \mu_c^* b e^{(\phi+r)(x-m)+rt+e^{\frac{x-m}{b}}}\Gamma\left(-(\phi+r)b, e^{\frac{t+x-m}{b}}\right) \\ &\times\left(1-\frac{\Gamma\left(-(\phi+r)b, e^{\frac{t_0+x-m}{b}}\right)}{\Gamma\left(-(\phi+r)b, e^{\frac{\min(t,T)+x-m}{b}}\right)}\right).\end{aligned} \tag{14}$$

Proof. We take into account two different cases:

1. $t < T$ and so some contributions till T must be paid and still all the pensions must be paid; the value of $\Delta(t)$ is, in this case:

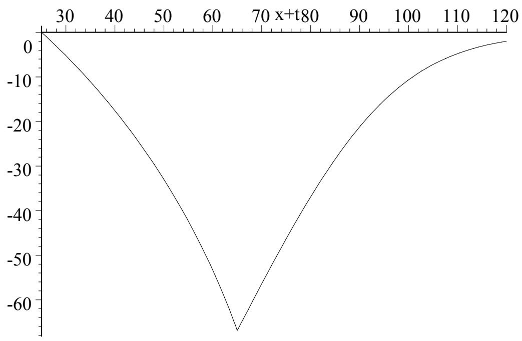
$$\Delta(t) = \mu_c^* \int_t^T ({}_s p_x)\, e^{-r(s-t)} ds - \mu_p^* e^{-r(T-t)} \int_T^\infty ({}_s p_x)\, e^{-r(s-T)} ds,$$

   and, by using the results stated in Proposition 3, we have

$$\begin{aligned}\Delta(t) &= \mu_c^* b e^{(\phi+r)(x-m)+rt+e^{\frac{x-m}{b}}}\Gamma\left(-(\phi+r)b, e^{\frac{t+x-m}{b}}\right) \\ &\times\left(1-\frac{\Gamma\left(-(\phi+r)b, e^{\frac{t_0+x-m}{b}}\right)}{\Gamma\left(-(\phi+r)b, e^{\frac{t+x-m}{b}}\right)}\right),\end{aligned}$$

2. $t \geq T$ and so all contributions have already been paid and only some

Figure 5: Behaviour of the prospective mathematical reserve $\Delta(t)$ with respect to time



pensions still remain to be paid; the value of $\Delta(t)$ is, in this case:

$$\begin{aligned}
\Delta(t) &= -\mu_p^* \int_t^\infty ({}_s p_x)\, e^{-r(s-t)} ds \\
&= -\mu_p^* b e^{(\phi+r)(x-m)+rt+e^{\frac{x-m}{b}}} \Gamma\left(-(\phi+r)\,b, e^{\frac{t+x-m}{b}}\right) \\
&= \mu_c^* b e^{(\phi+r)(x-m)+rt+e^{\frac{x-m}{b}}} \Gamma\left(-(\phi+r)\,b, e^{\frac{t+x-m}{b}}\right) \\
&\quad \times \left(1 - \frac{\Gamma\left(-(\phi+r)\,b, e^{\frac{t_0+x-m}{b}}\right)}{\Gamma\left(-(\phi+r)\,b, e^{\frac{T+x-m}{b}}\right)}\right).
\end{aligned}$$

The comparison between these two cases immediately give the result in the corollary.

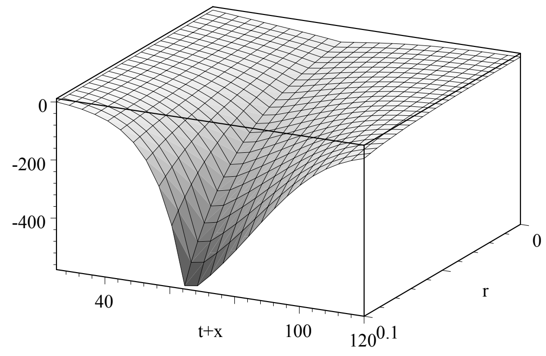
■

With the same parameter values we have already used for the previous numerical simulations, we can represent the value of the prospective reserve (14) as in Figure 5 where $x = 25$ and $T = 40$.

As we already argued, the value of the prospective mathematical reserve always remains negative and, after retirement, while t increases it approaches zero.

The changes in the interest rate affect the prospective reserve as shown in Figure 6. As it is quite intuitive, the behaviour of the reserve value remains the same and the lowest point of the graph increases while the interest rate decreases. In fact, when r increases, the discount factor takes lower values and the present value in T of all the future pensions becomes lower (in absolute value).

Figure 6: Behaviour of the prospective reserve $\Delta(r,t)$ with respect to the interest rate



2.4 The fund wealth

Given the market structure we have already presented in the previous subsections, we can define with $w(t) \in \mathbb{R}^{m\times 1}$ and $w_G(t) \in \mathbb{R}$ the vector containing the amount of risky assets held in the fund portfolio and the scalar representing the amount of riskless asset held, respectively. Thus, the amount of wealth the fund manages at each time is given by

$$R(t) = w(t)' S(t) + w_G(t) G(t), \tag{15}$$

whose differential is

$$dR = \underbrace{w' dS + w_G dG}_{dR_1} + \underbrace{dw'(S + dS) + dw_G \cdot G}_{dR_2},$$

where we suppose to know the initial wealth $R(t_0) = R_0$.

We can see that the change in the fund wealth can be divided into two different sources of change: dR_1 which is the change in R due to the change in the asset prices, and dR_2 which is the change in R due to the change in the portfolio composition. When there are neither consumption nor non-financial revenues, then the self-financing condition on the portfolio asks for dR_2 to equate zero. In our case the change in the portfolio composition must be financed by the contributions during the accumulation phase and must finance the pension payments during the decumulation phase. Furthermore, in our framework, the total amount of contributions and pensions in t are received and paid respectively only if the member is still alive after t periods. This means that the total non-financial flows in (and out) the portfolio must be weighted by the conditional survival probability. Finally, the change in wealth given by dR_2 must equate the change in the non-financial flows (weighted by the conditional survival probability):

$$dw'(S+dS)+dw_G\cdot G=({}_tp_x)\,dL(t).$$

After defining

$$\begin{aligned}\mu_L &\equiv \mathbb{I}_{t<T}\mu_c^*-(1-\mathbb{I}_{t<T})\,\mu_p^*, &(16)\\ \Sigma_L &\equiv \mathbb{I}_{t<T}\Sigma_c-(1-\mathbb{I}_{t<T})\,\Sigma_p, &(17)\end{aligned}$$

the differential of L can be written as

$$dL=\mu_L dt+\Sigma_L' dW,$$

and so the wealth differential, after substituting for the static budget constraint (15) can be written in the following way

$$\begin{aligned}dR &= (Rr+w'(\mu-Sr)+({}_tp_x)\,\mu_L)\,dt \\ &\quad +(w'\Sigma'+({}_tp_x)\,\Sigma_L')\,dW,\end{aligned} \tag{18}$$

where we have eliminated all the trivial functional dependences for the sake of simplicity.

It is evident from Equation (18) that the drift of the fund wealth is increased by the contribution rate (μ_c) and decreased by the pension rate (μ_p). In the same way, the diffusion term of fund wealth is increased by the contribution diffusion (Σ_c) and decreased by the pension diffusion (Σ_p).

2.5 Minimizing the instantaneous variance

The dynamic equation (18) for the fund wealth is computed in a time t given the value of the fund wealth in t_0. This means that the portfolio which minimizes the fund instantaneous variance (w_σ^*) is given by[3]

$$w_\sigma^*=\arg\min_w\left(w(t)'\,\Sigma(t)'+({}_tp_x)\,\Sigma_L(t)'\right)(\Sigma(t)\,w(t)+({}_tp_x)\,\Sigma_L(t)),$$

which gives

$$w_\sigma^*(t)=-({}_tp_x)\left(\Sigma(t)'\,\Sigma(t)\right)^{-1}\Sigma(t)'\,\Sigma_L(t). \tag{19}$$

This allows us to conclude that the asset allocation that minimizes the variance of fund wealth is given by the opposite of the ratio between the exogenous volatility and the asset price volatility. This portfolio must of course be weighted by the probability that an agent is still alive after t periods.

The prescription of such an optimal strategy is very easy to interpret: it is necessary to buy the assets whose volatility is negatively correlated with those of the exogenous risks while it is optimal to sell the assets which are positively correlated with the exogenous risks.

We will show in what follows that the optimal portfolio (19) is a component of the optimal portfolio maximizing the expected utility of fund wealth.

[3] The minimization problem is well defined since the variance-covariance matrix $\Sigma'\Sigma$ is positive definite by construction.

2.6 The objective function

The pension fund is supposed to maximize the expected utility of its wealth at the death time of its subscriber. Thus, the problem can be formulated as

$$\max_{w} \mathbb{E}_{t_0}^{\tau} \left[e^{-\rho(\tau - t_0)} U(R, z, \tau) \right],$$

where ρ is a subjective discount factor, τ is the stochastic death time of the subscriber, $U(R, \tau)$ is the fund utility function, and z is a vector containing all the stochastic variables presented in the previous subsections (i.e. asset values, state variables, contributions, and pensions). The expected value is taken with respect to the joint distribution of τ and all the other risks driving the asset values. Of course the fund wealth R behaves as in (18).

Now, since τ is supposed to be independent of all the other risk sources, that we can decompose the expected value in the following way (as in Richard, 1975):

$$\mathbb{E}_{t_0}^{\tau} \left[e^{-\rho(\tau - t_0)} U(R, z, \tau) \right] = \mathbb{E}_{t_0} \left[\int_{t_0}^{\infty} \pi(t|x)\, e^{-\rho(t - t_0)} U(R, z, t)\, dt \right],$$

where we have indicated with $\pi(t|x)$ the conditional density function of the death time. From the actuarial literature we recall that this density function can be easily derived from the conditional survival probability:

$$\pi(t|x) = -\frac{d\left({}_t p_x\right)}{dt},$$

and, by using the ODE (12) we have

$$\pi(t|x) = \lambda(t)\left({}_t p_x\right),$$

so the objective function can be written as

$$\mathbb{E}_{t_0} \left[\int_{t_0}^{\infty} \lambda(t)\left({}_t p_x\right) e^{-\rho(t - t_0)} U(R, z, t)\, dt \right],$$

which means that the original problem is equivalent to the maximization of the intertemporal fund utility, discounted by a mixed actuarial-financial discount factor.

In this work we allow the utility function to depend on time and stochastic variables different from wealth. This means that our utility function is not time separable. In this case, we can take into account the so-called "habit formation" approach where the time passing can change the form of the utility function.

3 The optimization problem

As shown in the previous subsections, the fund problem can be alternatively formulated as an intertemporal optimization problem as follows:

$$\begin{cases} \max_w \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} \lambda(t)\,({}_tp_x)\,e^{-\rho(t-t_0)}U(R,z,t)\,dt\right] \\ \begin{bmatrix} dz \\ dR \end{bmatrix} = \begin{bmatrix} \mu_z \\ Rr + w'(\mu - Sr) + ({}_tp_x)\,\mu_L \end{bmatrix} dt \\ \qquad + \begin{bmatrix} \Omega' \\ w'\Sigma' + ({}_tp_x)\,\Sigma_L' \end{bmatrix} dW, \\ R(t_0) = R_0, \quad z(t_0) = z_0, \quad \forall t_0 < t < H, \end{cases} \tag{20}$$

where

$$\begin{aligned} \underset{(s+m+1)\times 1}{z} &\equiv \begin{bmatrix} X' & S' & L' \end{bmatrix}', \\ \underset{(s+m+1)\times 1}{\mu_z} &\equiv \begin{bmatrix} \mu_X' & \mu' & \mu_L' \end{bmatrix}', \\ \underset{d\times(s+m+1)}{\Omega} &\equiv \begin{bmatrix} \Sigma_X & \Sigma & \Sigma_L \end{bmatrix}. \end{aligned}$$

The so-called *cost functional* which measures the performance of the controls (as in Yong and Zhou, 1998) is written, in this case, as

$$\Psi(t,w,R,z) = \mathbb{E}_t\left[\int_t^{\infty} \lambda(s)\,({}_sp_x)\,e^{-\rho(s-t)}U(R(s),z(s),s)\,ds\right],$$

and the Hamiltonian of this problem is thus

$$\begin{aligned} \mathcal{H} &= \lambda(t)\,({}_tp_x)\,U(R,z,t) + J_R\left(Rr + w'(\mu - Sr) + ({}_tp_x)\,\mu_L\right) \\ &\quad + \frac{1}{2}J_{RR}\left(w'\Sigma' + ({}_tp_x)\,\Sigma_L'\right)\left(\Sigma w + ({}_tp_x)\,\Sigma_L\right) + \mu_z' J_z + \frac{1}{2}\operatorname{tr}\left(\Omega'\Omega J_{zz}\right), \end{aligned}$$

where all the variables are evaluated in t, and the value function $J(t,R,z) = \max_w \Psi(t,w,R,z)$ must solve the differential equation

$$\begin{aligned} 0 &= J_t + \lambda p U + J_R\left(Rr + w'(\mu - Sr) + p\mu_L\right) \\ &\quad + \frac{1}{2}J_{RR}\left(w'\Sigma' + p\Sigma_L'\right)\left(\Sigma w + p\Sigma_L\right) + \mu_z' J_z + \frac{1}{2}\operatorname{tr}\left(\Omega'\Omega J_{zz}\right). \end{aligned}$$

The first order conditions[4] for w on the Hamiltonian give us the optimal portfolio in the implicit form

$$w^* = \underbrace{-p\Sigma^{-1}\Sigma_L}_{w^*_{(1)}} \underbrace{-\frac{J_R}{J_{RR}}\left(\Sigma'\Sigma\right)^{-1}(\mu - Sr)}_{w^*_{(2)}} \underbrace{-\frac{1}{J_{RR}}\Sigma^{-1}\Omega J_{zR}}_{w^*_{(3)}}. \tag{21}$$

In Equation (21) we can distinguish three components of the optimal portfolio.

[4] We recall that the second order condition is verified if J is concave in R (and this condition is guaranteed by the concavity of U).

1. A preference free component ($w^*_{(1)}$) coinciding with the portfolio which minimizes the fund wealth volatility (as shown in the previous section). With respect to the usual case where the non-financial flows are uncertain in their amount but certain in their happening, here the first portfolio component is multiplied by the survival probability.

2. A speculative component ($w^*_{(2)}$) proportional to both the Sharpe ratio and the Arrow-Pratt risk aversion index (computed on the value function).

3. An hedging component ($w^*_{(3)}$) whose present is due to the presence of stochastic state variables. In fact, when the state variables are not stochastic (i.e. $\Omega = \mathbf{0}$) then this component vanishes. $w^*_{(3)}$ is proportional to the ratio between the state variables volatility and the asset price volatility. Furthermore, its weight on the optimal portfolio depends on how much the changes in the state variables affect the value function.

3.1 A separable value function

Given the implicit optimal portfolio in (21) the value function J must solve the following partial differential equation (called Hamilton-Jacobi-Bellman equation, hereafter HJB):[5]

$$\begin{aligned} 0 &= J_t + \lambda p U + \mu_z' J_z - \frac{1}{2}\frac{J_R^2}{J_{RR}}\xi'\xi - \frac{J_R}{J_{RR}}\xi'\Omega J_{zR} \\ &\quad + J_R\left(Rr + p\mu_L - p\Sigma_L'\xi\right) + \frac{1}{2}\operatorname{tr}\left(\Omega'\Omega J_{zz}\right) - \frac{1}{2}\frac{1}{J_{RR}}J_{zR}'\Omega'\Omega J_{zR}, \end{aligned}$$

where ξ is as in (3), and with the boundary (trasversality) condition

$$\lim_{t\to\infty} J\left(R,z,t\right) = 0.$$

One of the most challenging task in the stochastic dynamic control approach is to solve the HJB equation. As it can be seen in the optimal portfolio literature, the form of the value function strongly depends on the form of the utility function. In this work, we want to proceed in a kind of backward way for recovering the form of a suitable utility function allowing us to have a separable value function of the following form

$$J\left(R,z,t\right) = V\left(z,t\right)U\left(R,z,t\right), \tag{22}$$

where $V\left(z,t\right)$ is a function that must be determined. After substituting this function form into the HJB equation and dividing by U we can write the HJB

[5] For the sake of simplicity we have eliminated all the functional dependences. For a more detailed explanation of the HJB equation the reader is referred to Øksendal (2000).

in the following way

$$
\begin{aligned}
0 &= V_t + \left(\mu_z' - \frac{U_R^2}{U_{RR}U}\xi'\Omega - \frac{U_R}{U_{RR}U}U_{zR}'\Omega'\Omega + \frac{1}{U}U_z'\Omega'\Omega\right)V_z \\
&\quad + \frac{1}{2}\operatorname{tr}\left(\Omega'\Omega V_{zz}\right) + \lambda p + VA\left(z,R,t\right) - \frac{1}{2}\frac{U_R^2}{U_{RR}U}\frac{1}{V}V_z'\Omega'\Omega V_z,
\end{aligned} \tag{23}
$$

where

$$
\begin{aligned}
A\left(z,R,t\right) &\equiv \frac{U_t}{U} + \frac{1}{U}\mu_z' U_z - \frac{1}{2}\frac{U_R^2}{U_{RR}U}\xi'\xi - \frac{U_R}{U_{RR}U}\xi'\Omega U_{zR} \\
&\quad + \frac{U_R}{U}\left(Rr + p\mu_L - p\Sigma_L'\xi\right) - \frac{1}{2}\frac{1}{U_{RR}U}U_{zR}'\Omega'\Omega U_{zR} \\
&\quad + \frac{1}{2}\frac{1}{U}\operatorname{tr}\left(\Omega'\Omega U_{zz}\right).
\end{aligned}
$$

Now, since we want the function V to be independent of R, then the coefficient of V_z, the function $A\left(z,R,t\right)$, and the the ratio $\frac{U_R^2}{U_{RR}U}$ must not depend on R. Let us start from the last ratio. It is easy to demonstrate (by solving a second order ordinary differential equation) that it does not depend on R if and only if $U\left(R,z,t\right)$ belongs to the HARA (Hyperbolic Absolute Risk Aversion) family. Algebraically, U must have the following form

$$
U\left(R,z,t\right) = \frac{1}{1-\beta}\left(R + F\left(z,t\right)\right)^{1-\beta}, \tag{24}
$$

where β must be a positive constant[6] and F may depend on t and z. The economic literature generally proposes utility functions where F depends on z and can capture the so-called "habit formation". In this way we can take into account a much richer set of preferences than those we have with the simpler assumption of time separability for the utility function.

After substituting this functional form into Equation (26) we obtain the partial differential equation

$$
\begin{aligned}
0 &= V_t + \left(\mu_z' + \frac{1-\beta}{\beta}\xi'\Omega\right)V_z + \frac{1}{2}\operatorname{tr}\left(\Omega'\Omega V_{zz}\right) + \frac{1}{2}\frac{1-\beta}{\beta}\frac{1}{V}V_z'\Omega'\Omega V_z + \lambda p \\
&\quad + V\left(\frac{1}{2}\frac{1-\beta}{\beta}\xi'\xi + \left(1-\beta\right)r\right) \\
&\quad + V\frac{1-\beta}{R+F}\left(F_t + \left(\mu_z' - \xi'\Omega\right)F_z - Fr + p\mu_L - p\Sigma_L'\xi + \frac{1}{2}\operatorname{tr}\left(\Omega'\Omega F_{zz}\right)\right).
\end{aligned}
$$

For a reason that we will widely discuss in the following subsection, we now set the trasversality condition

$$
\lim_{t\to\infty} F\left(t,z\right) = 0,
$$

[6] In this way the utility function is increasing and concave in R as we want it to be.

and so we can use the Feynman-Kač theorem for writing the following solution

$$F(t,z) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\infty} ({}_s p_x)\left(\mu_L - \Sigma_L'\xi\right) e^{-\int_t^s r(\theta)d\theta} ds\right], \tag{25}$$

where the probability measure $\mathbb{Q}$ has already been defined in (4). In the next subsection, we are going to present the utility function that allows us to simplify the HJB equation.

3.2 A suitable utility function

In the previous subsection we have derived a suitable form for the utility function which allows us to simplify the HJB equation for solving the dynamic optimization problem. In particular, the function $F(z,t)$ in (25) coincides with the prospective mathematical reserve. In fact, under the martingale equivalent measure, the behaviour of contributions and pensions is given by

$$dL = \left(\mu_L - \Sigma_L'\xi\right) dt + \Sigma_L' dW^{\mathbb{Q}},$$

and so the function (25) can be exactly written as in (8). This means that the utility function can be written as follows

$$U(R,z,t) = \frac{1}{1-\beta}\left(R + \Delta(z,t)\right)^{1-\beta},$$

which belongs to the HARA family and which guarantees that the investor's wealth never goes below the level of the opposite of the prospective mathematical reserve ($R(t) > -\Delta(z,t), \forall t > t_0$). Let us comment more this point which is quite important. When the pensions start being paid ($t > T$) then $-\Delta$ is simply given by the expected present value of a life annuity whose installments are the pension rates. This means that during the decumulation phase the fund manages its wealth in such a way that it never falls below the present value of all the future pensions. Instead, before the retirement date ($t < T$), the fund wealth can go beyond the present value of all the pensions by an amount given by the received contributions.

It is worth noting that the present value of all fund liabilities is always negative (i.e. $\Delta(t,z) < 0, \forall t > t_0$). So, since the form of the utility function guarantees that $R(t) > -\Delta(t), \forall t > t_0$, then we can conclude that $R(t)$ always remains positive and we can neglect the positivity constraint on the fund wealth.

We stress that we are in a fully funded framework where the demographic risk does not enter the fund optimization problem. If we had been in a "pay as you go" framework, than the hypothesis of having replicable contributions and pensions would have been much stronger, since it is quite difficult to suppose that the demographic risk can be perfectly hedged on the financial market.

Now, we want to trace a comparison between the utility function we have found in the previous subsection and the approach called "habit formation". In fact, also in those case the utility function is not separable in wealth in such a way that there exists a minimum amount of "subsistence" consumption coinciding with a weighted mean of past consumptions. In our case, instead, the time non-separability comes from taking into account the future and not the past. We guarantee that the fund wealth never goes below the expected present value of all the pensions net of the expected value of all the contributions.

Finally, we note that the sum $R + \Delta$ can be interpreted as a surplus. In fact, at each instant in time, it is given by the fund wealth, diminished by the imbalance between all the future pension liabilities. Just as an example, let us take into account the decumulation phase. During this period, the prospective mathematical reserve is always negative and just given by the present value of all the future pensions that will have to be paid. Thus, the fund wealth R is reduced by the amount of all the future fund engagements. I interpret this reduced wealth as a surplus. The same principle applies for the accumulation phase since the value of the prospective mathematical reserve remains negative for any $t > t_0$.

3.3 An explicit solution

Now, only the last step remains but, unfortunately, it is the most difficult one. We still have to solve the PDE for the function $V(z,t)$ in the following form,

$$\begin{aligned} 0 \quad = \quad & V_t + \left(\mu_z' + \frac{1-\beta}{\beta}\xi'\Omega\right) V_z + \frac{1}{2}\operatorname{tr}\left(\Omega'\Omega V_{zz}\right) \\ & + V\left((1-\beta)\, r + \frac{1}{2}\frac{1-\beta}{\beta}\xi'\xi\right) + \lambda p + \frac{1}{2}\frac{1-\beta}{\beta}\frac{1}{V} V_z'\Omega'\Omega V_z, \end{aligned} \tag{26}$$

which has been obtained by substituting the suitable utility function into (**??**). Now, the boundary (trasversality) condition is

$$\lim_{t\to\infty} V(z,t) = 0.$$

In order to eliminate the non linearity in V_z, we can do the following transformation (as suggested in Zariphopoulou, 2001)

$$V(z,t) = \Phi(z,t)^{\beta},$$

and the PDE (26) becomes

$$\begin{aligned} 0 \quad = \quad & \Phi_t + \left(\mu_z' + \frac{1-\beta}{\beta}\xi'\Omega\right) \Phi_z + \frac{1}{2}\operatorname{tr}\left(\Omega'\Omega \Phi_{zz}\right) \\ & + \Phi\left(\frac{1-\beta}{\beta} r + \frac{1}{2}\frac{1-\beta}{\beta^2}\xi'\xi\right) + \frac{1}{\beta}\lambda p \Phi^{1-\beta}, \end{aligned} \tag{27}$$

with the boundary condition

$$\lim_{t\to\infty} \Phi(z,t) = 0.$$

Accordingly, the optimal portfolio can be written as in the following proposition.

Proposition 6 *The optimal portfolio solving Problem (20) for a pension fund maximizing its surplus $R + \Delta$ is given by*

$$w^* = -\underbrace{p\Sigma^{-1}\Sigma_L}_{w_L^*} - \underbrace{\Sigma^{-1}\Omega\Delta_z}_{w_\Delta^*} + \underbrace{\frac{R+\Delta}{\beta}\left(\Sigma'\Sigma\right)^{-1}(\mu - Sr)}_{w_M^*} + \underbrace{\frac{R+\Delta}{\Phi}\Sigma^{-1}\Omega\Phi_z}_{w_\Phi^*}, \quad (28)$$

where the function $\Phi(z,t)$ satisfies (27).

The result stated in Proposition 6 allows us to argue that, with respect to the fund financial surplus, the optimal asset allocation implies a five fund theorem. In particular:

1. w_L^* is a preference free portfolio which minimizes the fund wealth volatility (as already shown);
2. w_Δ^* is a preference free portfolio which hedges the fund wealth from the risk given by the modification in the prospective mathematical reserve due to the changes in the state variable values;
3. w_M^* is the classical Merton's component (computed with respect to the fund financial surplus);
4. w_Φ^* is the portfolio component that depends on the mortality risk inside the term $f(t)$.

The solution of PDE (27) cannot rely on the Feynman-Kač theorem since the term containing the function $\Phi(z,t)$ are not affine in $\Phi(z,t)$. Nevertheless, there exists a case where an exact solution can be found. When the term

$$\frac{1-\beta}{\beta}r + \frac{1}{2}\frac{1-\beta}{\beta^2}\xi'\xi,$$

does not depend on the state variables z, then also $\Phi(z,t)$ is independent of z and so we have a Bernoulli's differential equation

$$0 = \Phi_t + \Phi\left(\frac{1-\beta}{\beta}r + \frac{1}{2}\frac{1-\beta}{\beta^2}\xi'\xi\right) + \frac{1}{\beta}\lambda p\Phi^{1-\beta},$$

which has only one solution satisfying the boundary condition for t tending to infinity. Since, in this case, the value of Φ does not depend on the state variables z, then the optimal portfolio component w_Φ^* vanishes and we can state what follows.

Corollary 7 *If* $\frac{1-\beta}{\beta}r + \frac{1}{2}\frac{1-\beta}{\beta^2}\xi'\xi$ *is independent of the state variables* z*, then the optimal portfolio for a pension fund solving Problem (20) is given by*

$$w^* = \underbrace{-p\Sigma^{-1}\Sigma_L}_{w_L^*} - \underbrace{\Sigma^{-1}\Omega\Delta_z}_{w_\Delta^*} + \underbrace{\frac{R+\Delta}{\beta}\left(\Sigma'\Sigma\right)^{-1}(\mu - Sr)}_{w_M^*}. \tag{29}$$

4 A simple framework

In this section we use the explicit solution found in Corollary 7 in order to present a numerical simulation of the optimal portfolio. In order to fulfill the hypothesis of Corollary 7 we take a constant riskless interest rate and geometric Brownian motions for asset prices. Furthermore, we take into account only one risky asset.[7] Thus the market structure is as follows:

$$dS = S\mu dt + S\sigma dW, \quad S(t_0) = S_0, \tag{30}$$
$$dG = rGdt, \quad G(t_0) = 1, \tag{31}$$

where μ, σ, and r are positive constant (such that $\mu > r$). In this case the market price of risk is constant and given by

$$\xi = \frac{\mu - r}{\sigma}.$$

Furthermore, let us suppose that contributions and pensions follow the processes:

$$\begin{aligned} dL_c &= \mu_c dt + \sigma_c dW, \\ dL_p &= \mu_p dt + \sigma_p dW, \end{aligned}$$

where μ_c, and μ_p are two positive constants. In order to avoid the problem of the so-called back-ground risk, we have supposed L to depend on the same risk sources than the asset prices.

This simple framework allows us to account for the two following cases:

1. a Defined Contribution system: in this case we must put $\sigma_c = 0$ since the behaviour of contributions is totally deterministic;

2. a Defined Benefit system: in this case we must put $\sigma_p = 0$ since the behaviour of pensions is totally deterministic.

We haven't specified the signs of the diffusion terms σ_c and σ_p since they include the sign of the correlation between asset returns and contributions and pensions, respectively.

[7] This only asset can be interpreted as the market price index.

We recall that the contributions are generally linked with the members' salary. In particular, when they coincide with a given percentage of the salary, the contribution volatility is given by the wage volatility. Thus, if we assume that positive shocks on firms profits positively affect wages, then σ_c must be positive. In fact, a positive dW must imply a positive dL_c.

For what concerns pensions, we recall that the characteristic of a pension fund that differentiates it with respect to a life insurance, is its link with the financial market. In particular, when we consider a DC scheme, pensions positively vary with respect to the financial market. When the financial returns increase, the fund pay higher pensions and vice versa. Thus, it seems that the positive sign for σ_p is the more appropriate.

Nevertheless, in what follows, we will always comment even the cases when either σ_p or σ_c (or both) are negative.

Finally, the unified process of pensions and contributions is

$$dL = \left(\mathbb{I}_{t<T}\mu_c - (1 - \mathbb{I}_{t<T})\,\mu_p\right) dt + \left(\mathbb{I}_{t<T}\sigma_c - (1 - \mathbb{I}_{t<T})\,\sigma_p\right) dW. \tag{32}$$

4.1 The feasible contributions and pensions

For the mortality risk we take into account the Gompertz-Makeham as in (11). Thus, the feasibility condition (10) is

$$(\mu_c^* - \sigma_c\xi) \int_{t_0}^{T} ({}_s p_x)\, e^{-r(s-t_0)} ds = \left(\mu_p^* - \sigma_p\xi\right) \int_{T}^{\infty} ({}_s p_x)\, e^{-r(s-t_0)} ds,$$

and from Corollary 4 we can immediately obtain

$$\mu_p^* = \mu_c^*\Pi + \xi\left(\sigma_p - \sigma_c\Pi\right), \tag{33}$$

where

$$\begin{aligned} \Pi &\equiv \frac{\Gamma\left(-(\phi+r)\,b, e^{\frac{t_0+x-m}{b}}\right)}{\Gamma\left(-(\phi+r)\,b, e^{\frac{T+x-m}{b}}\right)} - 1 > 0 \\ &\Rightarrow \Pi > 1, \end{aligned}$$

and both μ_c^* and μ_p^* must be positive.

Now it is easy to compare the DC and the DB cases in the following was:

1. in the DC case (i.e. $\sigma_c = 0$) the feasibility condition (33) becomes

   $$\mu_p^* = \mu_c^*\Pi + \xi\sigma_p,$$

   and we can conclude that pensions are higher (lower) when the pension risks are positively (negatively) correlated with asset prices (i.e. $\sigma_p > 0$, $\sigma_p < 0$ respectively);

2. in the DB case (i.e. $\sigma_p = 0$) the feasibility condition (33) becomes

$$\mu_p^* = \mu_c^* \Pi - \xi \sigma_c \Pi,$$

and we can conclude that pensions are higher (lower) when the contribution risks are negatively (positively) correlated with asset prices (i.e. $\sigma_c < 0$, $\sigma_c > 0$ respectively).

Furthermore, when we compare the pensions between the DC and the DB scheme, we can easily conclude what follows.

Proposition 8 *Given the risky asset, the riskless asset, the contributions and the pensions as in (30), (31), and (32) respectively, the feasible pensions paid in a DC scheme are higher (lower) than those paid in a DB scheme if and only if*

$$\sigma_p > (<) - \sigma_c \Pi.$$

Proof. Let $\mu_{p,DC}^* = \mu_p^* \big|_{\sigma_c=0}$ and $\mu_{p,DB}^* = \mu_p^* \big|_{\sigma_p=0}$, then the inequality $\mu_{p,DC}^* > (<) \, \mu_{p,DB}^*$ implies what stated in the proposition. Furthermore, what stated in the proposition can be, with suitable modifications, reduced to the inequality $\mu_{p,DC}^* > (<) \, \mu_{p,DB}^*$. ■

An immediate corollary follows.

Corollary 9 *Given the risky asset, the riskless asset, the contributions and the pensions as in (30), (31), and (32) respectively, when feasible pensions and contributions are both positively (negatively) correlated with the asset prices, then the pensions are higher in a DC (DB) pension scheme.*

We want to stress an important point since the results stated in Proposition 8 and Corollary 9 could bring the reader towards a misinterpretation. We must recall that pensions and contributions have been chosen so that the feasibility condition (10) holds. Thus, we cannot conclude that a higher pension rate is always preferred to a lower pension rate. In fact, because of construction of feasible pairs, a risk indifferent member must be indifferent between DC and DB. In fact, the higher pension rate in the DC case when pensions are correlated with asset prices comes from the higher risk that this framework bears. Actually, since pensions move together asset prices, then they cannot be used for any hedging strategy but, instead they represent another risk that must be hedged. So, in this case, their risk is added to those of the asset prices.

In the same way, when we are in a DB framework if the contributions (which are negative cash flows) are negatively correlated with asset prices, then the total portfolio risk increases and the pension rate must be higher.

Finally, the results of Proposition 8 and Corollary 9 are strictly in line with the well known principle according to which the higher the risk the higher the return.

4.2 The prospective reserve

In order to explicitly compute the optimal portfolio in this simplified framework we have to determine the functional form of the prospective mathematical reserve as in (9). Thus, we have[8]

$$\Delta(L,t)=\int_t^{\infty}({}_sp_x)\left(\mu_L-\Sigma_L'\xi\right)e^{-r(s-t)}ds,$$

whose differential is

$$\begin{aligned}
d\Delta &= \left(-({}_tp_x)\left(\mu_L-\Sigma_L'\xi\right)+r\Delta\right)dt,\\
\Delta(t_0) &= 0.
\end{aligned}$$

Given the values of μ_L and Σ_L already defined in (16) and (17) we can follow the same steps as in Corollary 5[9] in order to find the following value of the prospective reserve

$$\begin{aligned}
\Delta(t) &= \left(\mu_c^*-\sigma_c\xi\right)be^{(\phi+r)(x-m)+rt+e^{\frac{x-m}{b}}}\Gamma\left(-(\phi+r)\,b,e^{\frac{t+x-m}{b}}\right)\\
&\quad\times\left(1-\frac{\Gamma\left(-(\phi+r)\,b,e^{\frac{t_0+x-m}{b}}\right)}{\Gamma\left(-(\phi+r)\,b,e^{\frac{\min(t,T)+x-m}{b}}\right)}\right).
\end{aligned}$$

We recall that in our simplified framework the reserve Δ does not depend on any state variable (which means that $\Delta_z=0$). Accordingly, under the martingale equivalent measure, the differential of the wealth augmented by the reserve behaves as

$$d(R+\Delta)=r(R+\Delta)\,dt+\frac{R+\Delta}{\beta}\xi'dW^{\mathbb{Q}},$$

or, alternatively

$$d(R+\Delta)=(R+\Delta)\left(r+\frac{1}{\beta}\xi'\xi\right)dt+\frac{R+\Delta}{\beta}\xi'dW.$$

So, the surplus follows a geometric Brownian motion whose solution is

$$R(t)=-\Delta(t)+R(t_0)\,e^{\left(r+\frac{1}{\beta}\xi^2-\frac{1}{2}\frac{1}{\beta^2}\xi^2\right)(t-t_0)+\frac{1}{\beta}\xi(W(t)-W(t_0))},$$

where we have the confirmation of our previous result $R(t)>-\Delta(t)$.

[8] We do not need the expected value operator in this case since the value of Δ does not contain any stochastic variable.

[9] It is sufficient to replace μ_c^* with $\left(\mu_c^*-\sigma_c\xi\right)$ and μ_p^* with $\left(\mu_p^*-\sigma_p\xi\right)$.

Table 1: Mean returns and volatilities of some stock markets (Rudolf and Ziemba, 2004)

Country	**Mean Return (%)**	**Volatility (%)**
USA	13.47	14.47
UK	9.97	17.96
Japan	3.42	25.99
EMU	10.48	15.80
Canada	5.52	18.07
Switzerland	11.56	18.17

4.3 The optimal portfolio

The optimal portfolio found in Corollary 7 can be simplified further since, in our framework, the mathematical reserve Δ does not depend on the state variables (i.e. $\Delta_z = 0$). Thus, the optimal portfolio can be divided into two parts:

1. the "classical" Merton's component given by

$$w_M^*(t)\, S(t) = \frac{R(t)}{\beta} \frac{\mu - r}{\sigma^2},$$

   whose form does not change between the accumulation and the decumulation phases (even if its value of course does);

2. a further component due to the need of keeping the fund wealth higher than the future engagements:

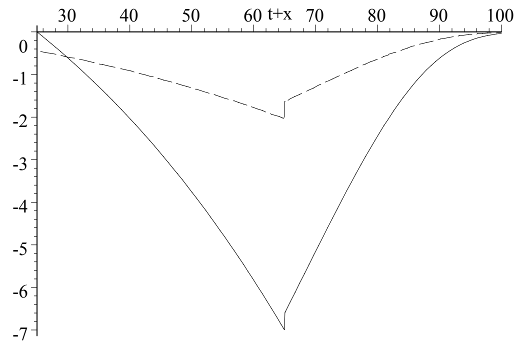
$$w_\Delta^*(t)\, S(t) = -\left({}_t p_x\right) \frac{\mathbb{I}_{t<T}\sigma_c - \left(1 - \mathbb{I}_{t<T}\right)\sigma_p}{\sigma} + \frac{\Delta(t)}{\beta} \frac{\mu - r}{\sigma^2}.$$

4.4 A numerical simulation

In order to present a numerical simulation we must choose the values for the parameters of our model. Our Table 1 reproduces Table 1 of Rudolf and Ziemba (2004).

In accordance with Rudolf and Ziemba (2004) we put $r = 0.02$, $\mu = 0.09$, and $\sigma = \sqrt{0.2}$. The parameters for the survival probability are put $(\phi, m, b) = (0, 88.18, 10.5)$ as already argued in the the previous sections. Now, we have to chose a volatility for contributions and pensions. Rudolf and Ziemba (2004) recall that the average growth rate of wages and salaries in the U.S. between January 1987 and July 2000 was 5.7% with annualized volatility of 4%. Furthermore, we assume the volatility of pension to equate the volatility of contributions (salaries). Thus, we assume $\sigma_p = \sigma_c = 0.2$. Now, we have to chose the "time"

Figure 7: Behaviour of the optimal portfolio component $w_{\Delta}^{*}(t)\,S(t)$ [DC=solid line, DB=dashed line]



parameter. We suppose that a member enters the fund when he is $x = 25$ and retires when he is 65 (i.e. $T = 40$). Finally, the risk aversion of the fund is supposed to be given by $\beta = 3$.

In this case the feasibility condition can be written as

$$\mu_p^* = \mu_c^* 4.146\,4 - 0.09849\,8, \tag{34}$$

provided that μ_c^*

$$\mu_c^* > 0.023755.$$

This means that we just have to decide either μ_p^* or μ_c^*. In particular, we want to compare the two following cases:

1. the subscriber decides for a DC pension scheme, and so we put $\mu_c^* = 1$ with $\sigma_c = 0$ (and $\sigma_p = 0.2$);

2. the subscriber decides for a DB pension scheme, and so we put $\mu_p^* = 1$ (in this case the value of μ_c^* can be easily obtained from (34)) with $\sigma_p = 0$ (and $\sigma_c = 0.2$).

These two cases are compared in Figure 7 where the solid line represent the value of the optimal portfolio component $w_{\Delta}^{*}(t)\,S(t)$ in the case of a defined contribution (DC) pension scheme, while the dashed line represents the case of a defined benefit (DB) scheme.

We can immediately see that in both cases the optimal portfolio is less risky than the Merton's one (i.e. $w_{\Delta}^{*}(t)\,S(t) < 0$). In particular, before the retirement date, the portfolio becomes less and less risky while, during the decumulation phase, the portfolio starts becoming more and more risky. This behaviour has a suitable economic interpretation. In fact, when pensions start being paid, the remaining duties for the pension fund becomes lower and lower (since the death probability increases with time). This means that the optimal portfolio can become riskier and riskier. Instead, before the retirement date, the fund engagements are still low because of the future contributions that will be received. Thus, while the expected present value of all the contributions reduces, the optimal portfolio must become less and less risky.

The main difference between the DC and the DB scheme is that in the DB case the optimal portfolio is almost always riskier. In fact, in order to provide the member with the promised benefits, the fund must undertake a riskier investment strategy.

In our framework there is no bankruptcy risk. Nevertheless, when this is the case, it could be important, for a worker, to chose a pension fund which does not undertake too risky strategies. In this case, the workers should chose for a DC pension scheme.

5 Conclusion

In this paper we have analyzed the asset allocation problem for a pension fund providing its members either with a defined contribution or with a defined benefit pension scheme. Our framework takes into account a general setting for asset prices but we assume that both contributions and pensions can be perfectly spanned on the complete financial market. Nevertheless, in spite of these simplifying assumptions, we are able to reach only an implicit solution for the optimal fund portfolio.

Instead, under the simplified hypothesis that both the riskless interest rate and the market price of risk are deterministic functions, we are able to find a closed form solution for the fund optimal portfolio. We show that this optimal portfolio is always less risky that the Merton's one.

Eventually, we present a numerical simulation for a very special case (with no stochastic processes but geometric Brownian motions) and we demonstrate that, in such a framework, the optimal portfolio for a pension fund providing its members with a defined contribution pension scheme is almost always less risky than the optimal portfolio for the defined benefit case.

References

[1] Battocchio, P., Menoncin, F., 2004. Optimal Pension Management under Stochastic Interest Rates, Wages, and Inflation. Insurance: Mathematics and Economics 34, 79-95.

[2] Battocchio, P., Menoncin, F., Scaillet, O., 2003. Optimal Asset Allocation for Pension Funds Under Mortality Risk During the Accumulation and Decumulation Phases. *forthcoming on* Annals of Operations Research.

[3] Bodie, Z., Merton, R. C., Samuelson, W. F., 1992. Labor supply flexibility and portfolio choice in a life cycle model. Journal of Economics Dynamics and Control 16, 427-449.

[4] Boulier, J.-F., Huang, S. J., Taillard, G., 2001. Optimal Management Under Stochastic Interest. Insurance: Mathematics and Economics 28, 173-189.

[5] Chacko, G., Viceira, L. M., 1999. Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets. Working Paper, Harvard University.

[6] Charupat, N, Milevsky, M., 2002. Optimal asset allocation in life annuities: a note. Insurance: Mathematics and Economics 30, 199-209.

[7] Cox, J. C., Ingersoll, J. E. Jr., Ross, S. A., 1985. A Theory of the Term Structure of Interest Rates. Econometrica, 53, 385-407.

[8] Cuoco, D., 1997. Optimal Consumption and Equilibrium Prices with Portfolio Constraints and Stochastic Income. Journal of Economic Theory 72, 33-73.

[9] Deelstra, G., Grasselli, M., Koehl, P.-F., 2000. Optimal investment strategies in a CIR framework. Journal of Applied Probability 37, 1-12.

[10] El Karoui, N., Jeanblanc-Picqué, M., 1998. Optimization of consumption with labor income. Finance and Stochastics 2, 409-440.

[11] Josa-Fombellida, R., Rincón-Zapatero, J. P., 2001. Minimization of risks in pension funding by means of contributions and portfolio selection. Insurance: Mathematics and Economics 29, 35-45.

[12] Karatzas, I., Shreve, S. E., 1991. Brownian Motion and Stochastic Calculus, 2nd edition. Springer, New York.

[13] Kim, T. S., Omberg, E., 1996. Dynamic Nonmyopic Portfolio Behavior. The Review of Financial Studies 9, 141-161.

[14] Menoncin, F., 2002. Optimal portfolio and background risk: an exact and an approximated solution. Insurance: Mathematics and Economics 31, 249-265.

[15] Merton, R. C., 1969. Lifetime Portfolio Selection under Uncertainty: the Continuous-Time Case. Review of Economics and Statistics 51, 247-257.

[16] Merton, R. C., 1971. Optimum Consumption and Portfolio Rules in a Continuous-Time Model. Journal of Economic Theory 3, 373-413.

[17] Øksendal, B., 2000. Stochastic Differential Equations - An Introduction with Applications, 5th edition. Springer, Berlin.

[18] Richard S., 1975. Optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model. Journal of Financial Economics 2, 187-203.

[19] Rudolf, M., Ziemba, W. T., 2004. Intertemporal surplus Management. Jouranl of Economic Dynamics and Control 28, 975-990.

[20] Vasiček, O., 1977. An Equilibrium characterization of the Term Structure. Journal of Financial Economics 5, 177-188.

[21] Wachter, J. A., 2002. Optimal Consumption and Portfolio Allocation under Mean-Reverting Returns: An Exact Solution for Complete Markets. Journal of Financial and Quantitative Analysis 37, 63-91.

[22] Yong, J., Zhou, X. Y., 1998. Stochastic Controls - Hamiltonian Systems and HJB Equations. Springer, New York.

[23] Zariphopoulou, T., 2001. A solution approach to valuation with unhedgeable risks. Finance and Stochastics 51, 61-82.