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5. Abstract: We describe the basics and fundamentals of swap market models. We review the key modelling ingredients, namely the continuous time modelling of a set of forward swap rates on a tenor structure under different probability measures. We review the three main classes known as the co-terminal, co-initial and co-sliding models, and describe their use in

interest rate derivative pricing. Finally we discuss numerical implementation via model calibration and approximation of pricing formulas.

6. Main text: Swap Market Models

Black formula ([1]) is popular among practitioners as a simple tool to price European options on Libor rates, i.e., caplets and floorlets, and on swap rates, i.e., swaptions. More recently, Brace et al. [2], Miltersen et al. [10], and Jamshidian [9] provided a sound theoretical basis to that practice by introducing a general framework to consistently price interest rate options by arbitrage. These works paved the way towards a broader acceptance of the so-called market models for interest rate derivatives by the academic community since they can be recasted within the general arbitrage-free framework of [7]. These models have the advantage, over those based on the evolution of the spot interest rate, of concentrating on rates that are market observable.

The Libor market model ([2], [10]) and the co-terminal swap market model ([9]) are the two major representatives of this class. These models are built by assigning arbitrage-free dynamics on a set of forward Libor rates and of co-terminal forward swap rates, respectively. The advent of new kinds of exotic (over-the-counter) derivatives in fixed income markets has recently inspired the introduction of "hybrid" or "generalized" market models where the underlying variables constitute a mixed set comprising both Libor and swap rates simultaneously. In this context, an extensive study is provided in Galluccio et al. [4]. Without doubt the availability of a general setup to build market models with mixed sets is of great interest in the applications, e.g., to better capture the risk embedded in some complex financial derivatives.

TENOR STRUCTURE AND FORWARD SWAP RATES

We assume that we are given a pre-specified collection of reset/settlement dates $\mathrm{T} = \{T_1,...,T_M\}$, referred to as the tenor structure, with $T_j < T_k$, $1 \le j < k \le M$, and starting time $T_0 < T_1$. Let us denote the year fraction between any two consecutive dates by $\delta_j = T_j - T_{j-1}$, for $j = 1,...,M$. We write $B(t,T_j)$, $j = 1,...,M$, to denote the price at time t of a discount bond that matures at time $T_j > t$. The forward swap rate $S(t,T_j,T_k)$, with j and k satisfying $1 \le j < k \le M$, is defined through $S(t,T_j,T_k) = \dfrac{B(t,T_j) - B(t,T_k)}{G(t,T_j,T_k)}$ for all $t \in \lfloor 0, T_j \rfloor$.

Here, $G(t,T_j,T_k)$ is the price of the annuity (or level) numéraire. Swap market models are based on the continuous time modelling of $S(t,T_j,T_k)$ and, generally, assume that forward swap rates follow a multi-dimensional diffusion process. In particular, $S(t,T_j,T_k)$ is a P^{T_j,T_k}-martingale so that, under P^{T_j,T_k},

$$\frac{dS(t,T_j,T_k)}{S(t,T_j,T_k)} = \lambda(t,T_j,T_k)' dW^{T_j,T_k}(t), \qquad \forall t \in \left[0, T_j\right], \tag{1}$$

where $\lambda(t,T_j,T_k)$ is a vector valued volatility function. The probability measure P^{T_j,T_k} is equivalent to the historical probability measure P, and is called the forward swap probability measure associated with the dates T_j and T_k, or simply the forward swap measure. For every *i = 1,…,M,* the relative (or "deflated") bond $B(t,T_i)/G(t,T_i,T_k)$, $\forall t \in \lfloor 0, \min(T_i, T_{j+1}) \rfloor$, follows a local martingale process under P^{T_j,T_k}. We denote the corresponding Brownian motion under P^{T_j,T_k} by W^{T_j,T_k}. The forward Libor rate $L(t,T_j)$, *j = 1,…,M-1,* defined as $L(t,T_j) = \dfrac{B(t,T_j) - B(t,T_k)}{\delta_{j+1} B(t,T_{j+1})}$, $\forall t \in \lfloor 0, T_j \rfloor$, is itself a forward swap rate $S(t,T_j,T_k)$ corresponding to *k = j + 1,* whose volatility function is denoted by $\lambda(t,T_j)$. Accordingly, we denote by P^{T_j} the corresponding forward probability measure associated to the discount bond price $B(t,T_j)$, and by W^{T_j} a Brownian motion under P^{T_j}. Then, for every *i = 1,…,M,* the relative bond price $B(t,T_j)/(\delta_{j+1}B(t,T_{j+1}))$, $\forall t \in \lfloor 0, \min(T_i, T_{j+1}) \rfloor$, follows a local martingale under $\mathrm{P}^{T_{j+1}}$. We refer to [11] (Chapters 12-13) for further material on the theoretical side.

In [4] the aim is to introduce a so-called "Market Model Approach". It concerns the weakest condition under which a general specification of a model that concentrates on modelling observable forward swap rates directly yields a unique specification in all equivalent pricing measures. In this respect, the concept of *admissibility* of a set is introduced, and its theoretical and practical implications are discussed. Interestingly, the properties of these admissible sets can be best understood with the use of graph theory. This mapping allows to graphically characterize all admissible sets in a simple and intuitive way. In fact, it is possible to prove that admissible sets are topologically equivalent to a tree graph. In this way, the selection of admissible models out of a given tenor structure can be done by visual inspection, and the model construction problem is largely simplified. Further, it is possible to prove that the class of admissible market models is very large: for a given tenor structure $\mathrm{T} = \{T_1, ..., T_M\}$ comprising M dates there exist M^{M-2} admissible sets (and models). Admissible models comprise all "standard" market models ([2], [9]) as special cases. Three major subclasses denominated co-initial, co-sliding and co-terminal (according to the nature of the family of forward swap rates) can be identified. We hereby briefly discuss their respective features. Remarkably, the Libor market model is the only admissible model of co-sliding type.

CO-TERMINAL SWAP MARKET MODEL

The co-terminal swap market model dates back to [9], and is built from an admissible set of forward swap rates with different start dates $\{T_1, ..., T_{M-1}\}$ and equal maturity date T_M, so that forward swap rates satisfy (1). The model is best suited to price Bermudan swaptions (and related derivatives; see [4], [13]) where the holder has the right to enter at times $T_1, ..., T_{M-1}$ into a plain-vanilla swap maturing at T_M. In this case, the only relevant European swaptions from a pricing and hedging perspective are those expiring at $T_1,...,T_{M-1}$, and maturing at T_M. Hence, it is natural to introduce a market model where the relevant underlying set coincides with the associated co-terminal forward swap rates. In this context, apart from the aforementioned Bermudan swaptions, we mention callable cap and reverse floaters, ratchet cap floaters and LIBOR knock-in/out swaps.

CO-INITIAL SWAP MARKET MODEL

The co-initial swap market model ([4], [5]) is built from an admissible set of forward swap rates with different end dates $\{T_2,...,T_M\}$ and equal start date T_1, so that forward swap rates satisfy (1). The model is best suited to price (complex) European-style derivatives, where the holder owns the right to exercise an option at a single future date *T*. In this case the option payoff, no matter how complex, is measurable with respect to the information available at time *T*, by definition. Qualitatively speaking, a set of admissible forward swap rates sharing the same initial date *T* contains all the information needed to evaluate the payoff, the latter being a function of a set of admissible co-initial forward swap rates at that time. Hence a model market approach based on a set of co-initial forward swap rates provides a powerful tool to price and hedge a large variety of European-style derivatives including forward-start, amortizing and zero-coupon swaptions.

CO-SLIDING SWAP MARKET MODEL

In [4] it is shown that there exists a unique admissible co-sliding swap market model and that it coincides with the Libor market model ([2], [10]). The model is built from an admissible set of forward swap rates with start date T_j and end date T_{j+1} $(j = 1,..., M-1)$, so that forward swap rates satisfy (1). Non-overlapping forward swap rates of that form are indeed forward LIBOR rates. The co-sliding model is best suited to price structured CMS-linked derivatives (with possibly Bermudan features) whose pay-off function depends on a set of fixed-maturity instruments. More precisely, in a constant-maturity swap (CMS) the variable coupon that settles at a generic time T_j is linked to the value of a swap rate prevailing at that time (the latter being associated to a swap of a given maturity). In this context, the LIBOR market model provides an optimal modelling framework since (sliding) CMS rates can be easily described in terms of linear combinations of forward LIBOR rates.

NUMERICAL IMPLEMENTATION

In the applications one needs to calibrate a generic swap market model to the available prices of liquid vanilla derivatives to avoid potential arbitrage in the risk-management process. Implied model calibration is a reverse engineering procedure aimed at identifying the relevant model characteristics, such as volatility parameters, from such a set of instruments. In interest rate derivatives markets, these instruments are plain-vanilla options written on forward swap and LIBOR rates, i.e., swaptions and caplets, respectively. To achieve a fast and robust model calibration one should ideally aim at closed or quasi-closed form formulae for plain-vanilla option prices. When these are not available, good analytical approximations are called for. The accuracy of these methods is studied in several papers. They rely on the so-called "freezing" approach ([8], [12]) or, alternatively, on the "rank-one approximation" method ([2]). In turn, the specification of the instantaneous volatility function $\lambda(t,T_j,T_k)$ in (1) is generally done by introducing flexible functional forms ([3], [13]) that are meant to reproduce the observed shape of implied swaption and cap/floor volatility term structures through a low-dimensional parameterisation. One of the most appreciated features of the instantaneous volatility function among practitioners is time-stationarity. This is generally imposed to reproduce the analogous temporal evolution of the volatility term structure observed in the market. However, the constraint of perfect model stationarity is generally incompatible with the observed implied volatility market for a generic well-behaved instantaneous volatility function. This market feature forces practitioners to introduce explicit calendar-time dependent functions $\lambda(t,T_j,T_k)$ to mimic a "perturbation mode" around the time-stationary solution. Efficient simulation algorithms are also available to price exotic interest rate derivatives by Monte-Carlo methods ([6]).

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