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5. Abstract: We study the pricing of Constant Maturity Swap spread options. We first discuss pricing without taking into account the presence of smiles before examining pricing with their inclusion. Further we look at the notions of implied correlation and implied normal spread volatility.

6. Main text: CMS spread options

According to recent estimates, the volume of traded contracts involving EUR-denominated options on the spread between two Constant Maturity Swap (CMS) rates amounts to 240 billions EUR in 2007 in the inter-bank market alone. This makes CMS spread options the most rapidly growing interest rate derivatives market.

The underlying of any option on CMS spread is the CMS rate. By definition, the CMS rate that fixes and settles at a generic time T (associated to a swap of maturity $T_n - T$ and starting at T) is equal to the swap rate of the associated swap. At any time prior to T the value of that rate is then formally given by $CMS(t) = E_t^{\,T}\left(S(T,T_n)\right)$, by absence of arbitrage. Here, the expectation $E_t^{\,T}(\cdot)$ is intended at time t with respect to the $T-$forward measure Q^T where the zero coupon bond $B(t,T)$ is the associated numéraire. The variable leg of a CMS swap pays a stream of CMS rates at any settlement date. In turn, the payout $f_T = f(CMS_1(T), CMS_2(T); K)$ of a European call (resp. put) option on a CMS spread expiring at time T reads as $f_T = (CMS_1(T) - CMS_2(T) - K)^+$, (resp. $f_T = (K - CMS_1(T) + CMS_2(T))^+$), where we have defined, as usual, $(\cdot)^+ = Max(\cdot,0)$.

Generally speaking, CMS spread options are simple and liquid financial instruments that allow taking a view on the future shape of the yield curve (or immunizing a portfolio against it). In the most commonly traded combination we have that $CMS_{1,2}(T) = CMS_{10Y,2Y}(T)$ (i.e., the two CMS rates are associated to a 10Y and 2Y swap, respectively). The buyer of a call (resp. put) CMS spread option will then benefit from a future curve steepening (resp. flattening) scenario. More complex option strategies involving

CMS spreads are actively traded in the over-the-counter (OTC) market, as well. They include, to name a few, digitals, barrier options, as well as Bermudan-style derivatives.

OPTION PRICING WITHOUT SMILES

The fair value at time t of the most generic call option is formally given by $C(t,T;K) = B(t,T)E_t^T\left(\alpha_1 CMS_1(T) - \alpha_2 CMS_2(T) - K\right)^+$, by arbitrage, with α_1 and α_2 constant. This expression is obviously reminiscent of an option on the spread between two assets and reduces to a simple "exchange" option when $K = 0$. In the simplest case, one assumes that each CMS rate follows a simple arithmetic or geometric Brownian motion under the relevant martingale measure. In the former case, a closed-form formula for $C(t) = C(t,T;K)$ can be given ([3]), while in the latter the price can be only expressed in integral form unless $K = 0$ in which case a closed-form formula can be exhibited ([8]). Some authors propose to use the first approach as an approximation for the second one ([3], [10]) for a generic $K \neq 0$. One must be however warned against these over-simplifications as market bid/offer spreads are relatively tight. Further risk-sensitivities are very different in the two settings with profound (negative) implications as long as portfolio replication quality is concerned.

Differently from the single asset case, the difference of two asset price processes is allowed to take negative values. Therefore, the arithmetic Brownian motion framework is generally considered as the simplest viable approach. Since, by definition, the CMS rate is a Q^T-martingale, we assume the two rates $CMS_{1,2} = X_{1,2}$ evolve according to the following Gaussian processes under Q^T : $dX_{1,2}(t) = \sigma_{1,2} dW^T{}_{1,2}(t)$ with constant volatility $\sigma_{1,2}$ and where $d\left\langle W^T{}_1, W^T{}_2 \right\rangle_t = \rho dt$ for some constant correlation coefficient ρ. In this case, it is easy to verify that the price of the option $C(t)$ is given by the modified "Bachelier" formula

$C(t) = B(t,T)\left[\sigma\sqrt{\tau}n\left(d(F_t,\tau)\right) + (F_t - K)N\left(d(F_t,\tau)\right)\right]$ where $\tau = T - t$, $F_t = \alpha_1 X_1(t) - \alpha_2 X_2(t)$, $\sigma^2 = \alpha_1{}^2\sigma_1{}^2 - 2\rho\alpha_1\alpha_2\sigma_1\sigma_2 + \alpha_2{}^2\sigma_2{}^2$ and $d(F_t,\tau) = (F_t - K)/\left(\sigma\sqrt{\tau}\right)$. Here, $n(\cdot)$ and $N(\cdot)$ stand for the standard Gaussian density and cumulative distribution function, respectively. In the lognormal case, one has to resort to a quasi-closed form formula (see [3] for a review).

The advantage of the above formula, similarly to Black-Scholes (BS) model for European options on single assets, is its simplicity. However, while in the BS case inverting the market price provides a unique "implied" volatility, here the situation is more complex. There are now three (as opposed to one) free parameters of the theory, that is ρ, σ_1, σ_2. In a perfectly liquid market one could in principle infer σ_1 and σ_2 by inverting the Bachelier formula for the two respective options on $CMS_{1,2}$, and then use the correlation as the unique free parameter of the theory. Interestingly, this indicates that buying or selling spread options is in principle equivalent to trading "implied spread correlation".

Unfortunately the above approach relies on the assumption that CMS rates dynamics are well modelled by an arithmetic Brownian motion. In practice, this is not the case. The main reason has to do with the presence of the volatility smile rather than with the request of positivity of CMS rates. As it is well-known, in fact, a CMS rate settling at time T, and associated to a swap of length τ, can be statically replicated through a linear combination of European swaptions (of different strike) expiring at T to enter into a swap of length τ. The sum is actually infinite, i.e., it is an integral over all possible swaptions for that given maturity ([1], [5]). Because it is well known that implied swaption volatilities are different at different strikes (i.e., a volatility smile is present) it means that the swaption underlying – the forward swap rate – cannot follow a simple Gaussian process in the relevant martingale measure. Consequently, the CMS rate, viewed as a linear combination of swaptions, must evolve accordingly. Needless to say that using a model for spread options where the underlying process is inconsistent with the market available information on plain-vanilla instruments has profound consequences on the quality of the risk-management ([4]).

OPTION PRICING WITH SMILES

There are essentially three possible ways to quote CMS spread options so as to ensure partial or full consistency with the underlying (CMS) implied dynamics.

Stochastic volatility models are very popular among academics and practitioners as they provide a simple and often effective mechanism of static generation as well as dynamic smile evolution ([9]). The first approach consists of assuming that each CMS rate in the spread follows a diffusion with its own stochastic volatility. The SABR model, for instance, has become the market standard for European options on interest rates ([6]). By coupling two SABR diffusions one can easily calibrate each parameter set on the respective market-implied CMS smile. The method has however two major drawbacks. First, no known formula nor simple approximation exists on options for multivariate SABR models. Second, there are 6 independent correlations to specify and several among them are not directly observable (e.g., the correlation between the first CMS rate and the volatility of the second one). In addition, it is easy to verify that some of those parameters are fully degenerate with respect to the price of a spread option of given strike.

The second approach resorts to using arbitrage-free dynamic models for the whole yield curve dynamics, in the HJM sense ([7]). Dynamics of the spread between any two CMS rates is then inferred from dynamics of the whole curve. This second method allows pricing and risk-managing all spread options on different pairs (e.g., 10Y-2Y, 10Y-5Y, 30Y-10Y, etc.) within a unique modelling setup rather than treating them as separate problems. This offers the great advantage of measuring and aggregating correlation exposures across all pairs at once and correlation risk diversification can be achieved. Also, exotic derivatives can be priced in this framework. On the negative side, it is very difficult to reproduce the implied smile of each CMS rate unless very complex models are introduced (e.g., a multi-factor HJM model with possibly multivariate stochastic volatility).

Finally, a third possibility consists of disentangling the marginal behaviour and the dependence structure between the two CMS rates. One can infer the marginal probability density from plain-vanilla swaptions, i.e., match their respective individual smiles, and then "recombine" them via a copula-based method (see [2] and references therein) to get the bivariate distribution function. The great advantage of this approach is its simplicity and the guarantee that, by construction, the price of the spread option is, at a given time, consistent with the current swaption market. On the negative side the approach is purely static, since no simple method exists to assign a dynamics on a bivariate process such that the associated density is consistent with the chosen copula function at any time. In addition the choice of the copula itself is, to a large extent, arbitrary.

IMPLIED CORRELATION AND NORMAL SPREAD VOLATILITY

Similarly to the BS case, practitioners often prefer to measure and compare spread option prices through homogeneous quantities. For simple options, people use implied volatility. For spread options the natural equivalent is the concept of implied correlation.

Assume to price a spread option through a Gaussian copula based method. Put it simply, this amounts to infer the two CMS marginal densities from the respective swaptions market and then couple them through a Gaussian copula function. Remind that a Gaussian copula is parameterized by a single correlation ρ and that a spread option price is monotonically decreasing as a function of ρ. Therefore, given the market price of a generic call spread option $C(t,T;K)$ struck at K, and given the two marginal CMS underlying densities, it exists a unique $\rho(K)$ such that the market price is matched by a copula method with correlation $\rho(K)$. This unique number is termed *implied Copula correlation*. As for simple options, the function $\rho(K)$ displays a significant dependence on the strike. This is the correlation smile phenomenon (Fig. 1).

Interestingly, it is possible to analyse the situation from a different, albeit similar, angle. In a previous section we showed that the simplest way to price options on CMS spread consists of coupling two simple Gaussian processes. The resulting closed-form formula is of Bachelier type with a modified normal volatility given by $\sigma^2 = \alpha_1{}^2\sigma_1{}^2 - 2\rho\alpha_1\alpha_2\sigma_1\sigma_2 + \alpha_2{}^2\sigma_2{}^2$. Given the option price $C(t,T;K)$, one can then invert the Bachelier-like formula to get a unique *implied normal spread volatility* $\sigma(K)$. Once more, function $\sigma(K)$ displays a smile. This alternative approach is still very popular among some practitioners.

It must be noticed, however, that the two above smile generation methods are not equivalent. In fact, only the first one is fully consistent with the underlying swaption smile observed in the market. In addition, the former approach concentrates on correlation, while the latter on the normal spread volatility which corresponds to the covariance of the joint process. Therefore, the first method is better suited if one considers volatility and correlation markets as evolving separately so that correlation movements are partly unrelated to price changes for swaptions. On the other side, the second method assumes that correlation and volatility markets are essentially indistinguishable to the extent that only the product of volatility and correlation (i.e., the covariance) is the relevant quantity as far as risk-management is concerned.

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9. Figure caption:

The figures displays the typical pattern of the implied copula correlation smile associated to a contract on the 10Y – 2Y CMS spread. Volatility is associated to a 7x10 cap on CMS spread, starting 7 years from today and maturing 10 years from today. Source: BNP Paribas.

11. Figure 1: