

# TIME-VARYING RISK PREMIUM IN LARGE CROSS-SECTIONAL EQUITY DATASETS

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## Abstract

We develop an econometric methodology to infer the path of risk premia from a large unbalanced panel of individual stock returns. We estimate the time-varying risk premia implied by conditional linear asset pricing models where the conditioning includes both instruments common to all assets and asset specific instruments. The estimator uses simple weighted two-pass cross-sectional regressions, and we show its consistency and asymptotic normality under increasing cross-sectional and time series dimensions. We address consistent estimation of the asymptotic variance by hard thresholding, and testing for asset pricing restrictions induced by the no-arbitrage assumption. We derive the restrictions given by a continuum of assets in a multi-period economy under an approximate factor structure robust to asset repackaging. The empirical analysis on returns for about ten thousands US stocks from July 1964 to December 2009 shows that risk premia are large and volatile in crisis periods. They exhibit large positive and negative strays from time-invariant estimates, follow the macroeconomic cycles, and do not match risk premia estimates on standard sets of portfolios. The asset pricing restrictions are rejected for a conditional four-factor model capturing market, size, value and momentum effects.

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# 1 Introduction

Risk premia measure financial compensation asked by investors for bearing systematic risk. Financial and macroeconomic variables influence risk. Conditional linear factor models aim at capturing their time-varying influence in a simple setting (see e.g. Shanken (1990), Cochrane (1996), Ferson and Schadt (1996), Ferson and Harvey (1991, 1999), Lettau and Ludvigson (2001), Petkova and Zhang (2005)). Time variation in risk biases time-invariant estimates of alphas and betas, and therefore asset pricing test conclusions (Jagannathan and Wang (1996), Lewellen and Nagel (2006), Boguth et al. (2011)). Ghysels (1998) discusses the pros and cons of modeling time-varying betas.

The workhorse to estimate equity risk premia in a linear multi-factor setting is the two-pass cross-sectional regression method developed by Black, Jensen, and Scholes (1972) and Fama and MacBeth (1973). A series of papers address its large and finite sample properties for linear factor models with time-invariant coefficients, see e.g. Shanken (1985, 1992), Jagannathan and Wang (1998), Shanken and Zhou (2007), Kan, Robotti, and Shanken (2013), and the review paper of Jagannathan, Skoulakis, and Wang (2009). The literature has not yet formally addressed statistical inference for equity risk premia in conditional linear factor models despite its empirical relevance.

In this paper, we study how we can infer the time-varying behaviour of equity risk premia from large stock returns databases under conditional linear factor models. Our approach is inspired by the recent trend in macro-econometrics and forecasting methods trying to extract cross-sectional and time-series information simultaneously from large panels (see e.g. Stock and Watson (2002a,b), Bai (2003, 2009), Bai and Ng (2002, 2006), Forni et al. (2000, 2004, 2005), Pesaran (2006)). Ludvigson and Ng (2007, 2009) exemplify this promising route when studying bond risk premia. Connor, Hagmann, and Linton (2012) show that large cross-sections exploit data more efficiently in a semiparametric characteristic-based factor model of stock returns. The theoretical framework underlying the Arbitrage Pricing Theory (APT) also inspires our approach relying on individual stocks returns. In this setting, approximate factor structures with nondiagonal error covariance matrices (Chamberlain and Rothschild (1983, CR)) answer the potential empirical mismatch of exact factor structures with diagonal error covariance matrices underlying the original APT of Ross (1976). Under weak cross-sectional dependence among idiosyncratic error terms, such approximate factor models generate no-arbitrage restrictions in large economies where the number of assets grows to

infinity. Our paper develops an econometric methodology tailored to the APT framework. Indeed, we let the number of assets grow to infinity mimicking the large economies of financial theory.

The potential loss of information and bias induced by grouping stocks to build portfolios in asset pricing tests further motivate our approach (e.g. Litzenberger and Ramaswamy (1979), Lo and MacKinlay (1990), Berk (2000), Conrad, Cooper, and Kaul (2003), Phalippou (2007)). Avramov and Chordia (2006) have already shown that empirical findings given by conditional factor models about anomalies differ a lot when considering single securities instead of portfolios. Ang, Liu, and Schwarz (2008) argue that we lose a lot of efficiency when only considering portfolios as base assets, instead of individual stocks, to estimate equity risk premia in models with time-invariant coefficients. In our approach, the large cross-section of stock returns helps to get accurate estimation of the equity risk premia even if we get noisy time-series estimates of the factor loadings (the betas). Besides, when running asset-pricing tests, Lewellen, Nagel, and Shanken (2010) advocate working with a large number of assets instead of working with a small number of portfolios exhibiting a tight factor structure. The former gives us a higher hurdle to meet in judging model explanation based on cross-sectional  $R^2$ .

Our theoretical contributions are threefold. First, we derive no-arbitrage restrictions in a multi-period economy (Hansen and Richard (1987)) under an approximate factor structure (Chamberlain and Rothschild (1983)) with a continuum of assets. We explicitly show the relationship between the ruling out of asymptotic arbitrage opportunities and an empirically testable restriction for large economies in a conditional setting. We also formalize the sampling scheme so that observed assets are random draws from an underlying population (Andrews (2005)). Such a construction is close to the setting advocated by Al-Najjar (1995, 1998, 1999a) in a static framework with an exact factor structure. He discusses several key advantages of using a continuum economy in arbitrage pricing and risk decomposition. A key advantage is robustness of factor structures to asset repackaging (Al-Najjar (1999b)). Second, we derive a new weighted two-pass cross-sectional estimator of the path over time of the risk premia from large unbalanced panels of excess returns. We study its large sample properties in conditional linear factor models where the conditioning includes instruments common to all assets and asset specific instruments. The factor modeling permits conditional heteroskedasticity and cross-sectional dependence in the error terms (see Petersen (2008) for stressing the importance of residual dependence when computing standard errors in finance panel data). We derive con-

sistency and asymptotic normality of our estimators by letting the time dimension  $T$  and the cross-section dimension  $n$  grow to infinity simultaneously, and not sequentially. We relate the results to bias-corrected estimation (Hahn and Kuersteiner (2002), Hahn and Newey (2004)) accounting for the well-known incidental parameter problem in the panel literature (Neyman and Scott (1948)). We derive all properties for unbalanced panels to avoid the survivorship bias inherent to studies restricted to balanced subsets of available stock return databases (Brown, Goetzmann, and Ross (1995)). The two-pass regression approach is simple and particularly easy to implement in an unbalanced setting. This explains our choice over more efficient, but numerically intractable, one-pass ML/GMM estimators or generalized least-squares estimators. When  $n$  is of the order of a couple of thousands assets, numerical optimization on a large parameter set or numerical inversion of a large weighting matrix is too challenging and unstable to benefit in practice from the theoretical efficiency gains, unless imposing strong ad hoc structural restrictions. Third, we provide a test of the asset pricing restrictions for the conditional factor model underlying the estimation. The test exploits the asymptotic distribution of a weighted sum of squared residuals of the second-pass cross-sectional regression (see Lewellen, Nagel, and Shanken (2010), Kan, Robotti, and Shanken (2013) for a related approach in models with time-invariant coefficients and asymptotics with fixed  $n$ ). The test statistic relies on consistent estimation of large-dimensional sparse covariance matrices by hard thresholding (Bickel and Levina (2008), El Karoui (2008), Fan, Liao, and Mincheva (2011)). As a by-product, our approach permits inference for the cost of equity on individual stocks, in a time-varying setting (Fama and French (1997)). We know from standard textbooks in corporate finance that cost of equity = risk free rate + factor loadings  $\times$  factor risk premia. It is part of the cost of capital and is a central piece for evaluating investment projects by company managers.

For our empirical contributions, we consider the Center for Research in Security Prices (CRSP) database and take the Compustat database to match firm characteristics. The merged dataset comprises about ten thousands stocks with monthly returns from July 1964 to December 2009. We look at factor models popular in the empirical finance literature to explain monthly equity returns. They differ by the choice of the factors. The first model is the CAPM (Sharpe (1964), Lintner (1965)) using the excess market return as the single factor. Then, we consider the three-factor model of Fama and French (1993) based on two additional factors capturing the book-to-market and size effects, and a four-factor extension including a momentum factor

(Jegadeesh and Titman (1993), Carhart (1997)). We study time-invariant and time-varying versions of the factor models (Ferson and Schadt (1996), Ferson and Harvey (1999)). For the latter, we use both macrovariables and firm characteristics as instruments. The estimated paths show that the risk premia are large and volatile in crisis periods, e.g., the oil crisis in 1973-1974, the market crash in October 1987, and the recent financial crisis. Furthermore, the time-varying risk premia estimates exhibit large positive and negative strays from time-invariant estimates, follow the macroeconomic cycles, and do not match risk premia estimates on standard sets of portfolios. The asset pricing restrictions are rejected for a conditional four-factor model capturing market, size, value and momentum effects.

The outline of the paper is as follows. In Section 2, we consider a general framework of conditional factor model for asset returns and derive the no-arbitrage pricing restrictions. We also show robustness of factor structures to asset repackaging and describe the sampling scheme. In Section 3, we introduce parametric functional specifications for the time-varying model coefficients, where the instruments generating conditional information can be either common to all stocks or stock-specific. We also present our estimation and testing approaches. Section 4 contains the empirical results. In the Appendices, we gather the technical assumptions and some proofs. We use high-level assumptions to get our results and show in the online supplementary materials that we meet all of them under a block cross-sectional dependence structure on the error terms in a time-invariant and serially i.i.d. framework. We place all omitted proofs and the Monte Carlo simulation results in the online supplementary materials. There, we also include additional empirical results, estimates of the cost of equity and robustness checks.

## **2 Conditional factor model of asset returns**

In this section, we consider a conditional linear factor model with time-varying coefficients in order to model possibly time-varying risk premia (see Connor and Korajczyk (1989) for an intertemporal competitive equilibrium version of the APT yielding time-varying risk premia and Ludvigson (2013) for a discussion within scaled consumption-based models). The setting includes models with time-invariant coefficients as a particular case. This covers the CAPM where the excess market return is the single factor.

## 2.1 Excess return generation and asset pricing restrictions

We start by describing the generating process for the excess returns before examining the implications of the absence of arbitrage opportunities in terms of model restrictions. We combine the constructions of Hansen and Richard (1987) and Andrews (2005) to define a multi-period economy with a continuum of assets having strictly stationary and ergodic return processes. We use such a formal construction to guarantee that (i) the economy is invariant to time shifts, so that we can establish all properties by working at  $t = 1$ , (ii) time series averages converge almost surely to population expectations, (iii) under a suitable sampling mechanism (see Section 2.3), cross-sectional limits exist and are invariant to reordering of the assets, (iv) the derived no-arbitrage restriction is empirically testable. This construction allows reconciling finance and econometric analysis in a coherent framework.

Let  $(\Omega, \mathcal{F}, P)$  be a probability space. The random vector  $f$  admitting values in  $\mathbb{R}^K$ , the collection of random variables  $\varepsilon(\gamma)$ ,  $\gamma \in [0, 1]$ , and the collection of random vectors  $\beta(\gamma) = (a(\gamma), b(\gamma))'$ ,  $\gamma \in [0, 1]$ , admitting values in  $\mathbb{R} \times \mathbb{R}^K$ , are defined on this probability space. We describe the dynamics by a measurable time-shift transformation  $S$  mapping  $\Omega$  into itself. If  $\omega \in \Omega$  is the state of the world at time 0, then  $S^t(\omega)$  is the state at time  $t$ , where  $S^t$  denotes the transformation  $S$  applied  $t$  times successively. We assume transformation  $S$  to be measure-preserving and ergodic (i.e., any set in  $\mathcal{F}$  invariant under  $S$  has measure either 1, or 0). In order to define the information sets, let  $\mathcal{F}_0 \subset \mathcal{F}$  be a sub sigma-field. Define  $\mathcal{F}_t = \{S^{-t}(A), A \in \mathcal{F}_0\}$ ,  $t = 1, 2, \dots$ , through the inverse mapping  $S^{-t}$  and assume that  $\mathcal{F}_1$  contains  $\mathcal{F}_0$ . Then, the filtration  $\mathcal{F}_t$ ,  $t = 1, 2, \dots$ , characterizes the flow of information available to investors. We assume that random vectors  $f$  and  $\beta(\gamma)$ , for  $\gamma \in [0, 1]$ , are measurable w.r.t.  $\mathcal{F}_0$ .

**Assumption APR.1** *The excess returns  $R_t(\gamma)$  of asset  $\gamma \in [0, 1]$  at dates  $t = 1, 2, \dots$ , satisfy the conditional linear factor model:*

$$R_t(\gamma) = a_t(\gamma) + b_t(\gamma)' f_t + \varepsilon_t(\gamma), \quad (1)$$

where the random variables  $\varepsilon_t(\gamma)$ ,  $f_t$ ,  $a_t(\gamma)$  and  $b_t(\gamma)$  are defined as  $\varepsilon_t(\gamma, \omega) = \varepsilon[\gamma, S^t(\omega)]$ ,  $f_t(\omega) = f[S^t(\omega)]$ ,  $a_t(\gamma, \omega) = a[\gamma, S^{t-1}(\omega)]$  and  $b_t(\gamma, \omega) = b[\gamma, S^{t-1}(\omega)]$ , for any  $\omega \in \Omega$  and  $\gamma \in [0, 1]$ . For any  $\gamma \in [0, 1]$ , we have  $E[\varepsilon_t(\gamma)|\mathcal{F}_{t-1}] = 0$  and  $Cov[\varepsilon_t(\gamma), f_t|\mathcal{F}_{t-1}] = 0$ .

Assumption APR.1 defines the excess return processes for an economy with a continuum of assets. Without

loss of generality, the index set is the interval  $[0, 1]$ , endowed with the Borel sigma-field  $\mathcal{B}$  and a probability distribution  $G$  that is absolutely continuous w.r.t. the Lebesgue measure. A countable collection of assets from this economy is a sequence  $(\gamma_i)$  with elements in  $[0, 1]$ . We endow the set  $\Gamma = [0, 1]^\infty$  of such sequences with the product sigma-field  $\mathcal{B}^\infty$  and the product measure  $\mu_\Gamma = G^\infty$  (see e.g. Halmos (1950), Theorem 38.B). The vector  $f_t$  gathers the values of the  $K$  observable factors at date  $t$ . In Assumption APR.1, the error terms have mean zero and are uncorrelated with the factors conditionally on information  $\mathcal{F}_{t-1}$ . This allows identification of  $a_t(\gamma)$  and  $b_t(\gamma)$  as conditional regression coefficients since the intercept  $a_t(\gamma)$  and factor sensitivities  $b_t(\gamma)$  of asset  $\gamma \in [0, 1]$  are  $\mathcal{F}_{t-1}$ -measurable. Since transformation  $S$  is measure-preserving and ergodic, all processes are strictly stationary and ergodic (Doob (1953)). Define further  $x_t = (1, f_t)'$  and  $\beta_t(\gamma) = (a_t(\gamma), b_t(\gamma))'$ , which yields the compact formulation:

$$R_t(\gamma) = \beta_t(\gamma)' x_t + \varepsilon_t(\gamma). \quad (2)$$

Let us now introduce supplementary assumptions on factor loadings and error terms to derive the main result of this section. We denote by  $L^p(\Omega, \mathcal{F}_0, P)$ ,  $p > 0$ , the space of  $p$ -th order integrable random variables that are measurable w.r.t. information  $\mathcal{F}_0$ , and by  $\|Z\|_p = (E[|Z|^p])^{1/p}$ , the associated  $L^p$ -norm.

**Assumption APR.2** (i) The mapping  $\beta : [0, 1] \times \Omega \rightarrow \mathbb{R}^{K+1}$  is measurable with respect to its bivariate argument  $(\gamma, \omega)$ . (ii) The mapping  $\beta$  is such that  $\beta(\gamma) = \sum_{j=1}^{\infty} c_j(\gamma) \eta_j$ , where the  $c_j : [0, 1] \rightarrow \mathbb{R}^{K+1}$  are measurable vector functions, and the  $\eta_j$  are orthonormal random variables in  $L^2(\Omega, \mathcal{F}_0, P)$ . Moreover,  $\eta_j \in L^4(\Omega, \mathcal{F}_0, P)$  and  $\sum_{j=1}^{\infty} \|c_j(\gamma)\| \|\eta_j\|_4 \leq M$ , for any  $\gamma \in [0, 1]$ , for a constant  $M > 0$ . (iii) The random matrix  $\int b(\gamma) b(\gamma)' dG(\gamma)$  is positive definite and such that  $\left\| \left( \int b(\gamma) b(\gamma)' dG(\gamma) \right)^{-1} \right\| \leq M$ ,  $P$ -a.s., for a constant  $M > 0$ . Moreover,  $\left\| \int b(\gamma) a(\gamma) dG(\gamma) \right\| \leq M$ ,  $P$ -a.s..

Assumption APR.2 (i) is the standard measurability condition for a stochastic process with sample space  $\Omega$  and parameter space  $[0, 1]$  (see e.g. Doob (1953)). In Assumption APR.2 (ii), function  $\beta$  maps the interval  $[0, 1]$  into the Hilbert space  $L^2(\Omega, \mathcal{F}_0, P)$ . Assumption APR.2 (ii) implies that the process  $\beta$  is a bounded aggregate process in the terminology of Al-Najjar (1995). An aggregate process admits a representation as an infinite linear combination of countable orthonormal random variables. We can interpret

the  $\eta_j$  as orthonormal shocks common to the components  $\beta_k(\gamma)$  of  $\beta(\gamma)$ , for any  $k = 1, \dots, K + 1$ , and to all assets  $\gamma \in [0, 1]$ , but with different loadings  $c_{k,j}(\gamma)$ . We can approximate an arbitrary aggregate process by a finitely generated one, for example generated by the  $p$  shocks  $\{\eta_1, \dots, \eta_p\}$ , so that we can make the norm of the difference between the two small (Al-Najjar (1995), Proposition 1). As remarked in Khan and Sun (1999, p. 555), assuming a bounded aggregate process is the same as assuming a strong measurable process because of the Pettis measurability theorem (see also Al-Najjar (1995), Proposition B.1). We use that notion of measurability to show Lemma 1 in Appendix A.2.1. Khan and Sun (1999 p. 555) emphasizes that adding a non-trivial idiosyncratic part to an aggregate process induces the possibility of non-measurability of realisations. This problem is known in the literature on the Law of Large Numbers (LLN) for a continuum of random variables (Judd (1985), Uhlig (1996)). Under Assumption APR.2 (ii), we show in the proof of Lemma 1 that the sample average  $\frac{1}{n} \sum_i \beta_k(\gamma_i) \beta_l(\gamma_i)$ , with  $k, l = 1, \dots, K + 1$ , for the cross-products of the intercept and factor loadings converges in the  $L^2(\Omega, \mathcal{F}_0, P)$  norm to the Bochner integral  $B - \int \beta_k(\gamma) \beta_l(\gamma) dG(\gamma)$  for  $\mu_\Gamma$ -a.e. sequence  $(\gamma_i) \in \Gamma$ , where  $B - \int$  refers to the definition of the Bochner integral. The summability condition on  $L^4$ -norms  $\|\eta_j\|_4$  in Assumption APR.2 (ii) is needed because we work with squares of the components of  $\beta$ , instead of  $\beta$  itself. Under Assumption APR.2 (i), we show that the Bochner integral  $B - \int \beta_k(\gamma) \beta_l(\gamma) dG(\gamma)$  is equal to the  $\omega$ -wise Lebesgue integral  $\int \beta_k(\gamma) \beta_l(\gamma) dG(\gamma)$ ,  $P$ -a.s. In a time-invariant model, the coefficient vector  $\beta$  is independent of argument  $\omega$ , and Assumptions APR.2 (i)-(ii) boil down to boundedness and Lebesgue measurability of  $\beta$  w.r.t. argument  $\gamma$ , as assumed in an unconditional economy by Al Najjar (1998, p. 237). The a.s. convergence of  $\frac{1}{n} \sum_i \beta_k(\gamma_i) \beta_l(\gamma_i)$  in Lemma 1 is then simply a consequence of the strong LLN. As soon as we depart from a time-invariant setting, we have to be sure that we can make sense of the limit of the sample average of the cross-products of the intercept and factor loadings by avoiding potential non-measurability issues. This explains why we need something more restrictive than simple boundedness and Lebesgue measurability for  $\beta$  in a time-varying framework. Assumptions APR.2 (i)-(ii) highlight key technical difficulties in the extension from a time-invariant framework to a time-varying framework. Ruling out a non-trivial idiosyncratic part in Assumption APR.2 (ii) is a limitation of our setting even if we work with an infinite linear combination of common shocks. Since transformation  $S$  is measure-preserving, Assumption APR.2 (iii) implies that the matrix  $\int b_t(\gamma) b_t(\gamma)' dG(\gamma)$  is positive definite,  $P$ -a.s., for any date  $t = 1, 2, \dots$ , i.e., the non-degeneracy



in the factor loadings across assets.

In Assumption APR.3 below, we impose an approximate factor structure for the conditional distribution of the error terms given  $\mathcal{F}_{t-1}$  in almost any countable collection of assets. More precisely, for any sequence  $(\gamma_i)$  in  $[0, 1]$ , let  $\Sigma_{\varepsilon,t,n}$  denote the  $n \times n$  conditional variance-covariance matrix of the error vector  $[\varepsilon_t(\gamma_1), \dots, \varepsilon_t(\gamma_n)]'$  given  $\mathcal{F}_{t-1}$ , for  $n \in \mathbb{N}$ .

**Assumption APR.3** *There exists a set  $\mathcal{J} \subset \left\{ (\gamma_i) \in \Gamma : n^{-1} \text{eig}_{\max}(\Sigma_{\varepsilon,t,n}) \xrightarrow{L^2} 0, \text{ as } n \rightarrow \infty \right\}$  with  $\mu_{\Gamma}(\mathcal{J}) = 1$ , where  $\text{eig}_{\max}(\Sigma_{\varepsilon,t,n})$  denotes the largest eigenvalue of matrix  $\Sigma_{\varepsilon,t,n}$ .*

CR (p. 1294) use a sequence of variance-covariance matrices of the error terms that have uniformly bounded eigenvalues. Assumption APR.3 is weaker than boundedness of the largest eigenvalue almost surely, i.e.,  $\sup_{n \geq 1} \text{eig}_{\max}(\Sigma_{\varepsilon,t,n}) \leq M$ ,  $P$ -a.s., for a.e.  $(\gamma_i)$  and a constant  $M > 0$ . This is useful for the checks of Appendix 6 under a block cross-sectional dependence structure. In the proof of Lemma 14, we give an explicit (nontrivial) example of a set  $\mathcal{J}$  satisfying Assumption APR.3. A sufficient condition for measurability of the set  $\left\{ (\gamma_i) \in \Gamma : n^{-1} \text{eig}_{\max}(\Sigma_{\varepsilon,t,n}) \xrightarrow{L^2} 0, \text{ as } n \rightarrow \infty \right\}$  is the joint measurability of the conditional covariance function  $E[\varepsilon_t(\gamma_1) \varepsilon_t(\gamma_2) | \mathcal{F}_{t-1}]$  w.r.t.  $\gamma_1, \gamma_2 \in [0, 1]$  and  $\omega \in \Omega$  (see Al-Najjar (1998, 1999a) in the unconditional case). As discussed in Khan and Sun (1999), such an assumption for the error term might be restrictive, and may constitute a key difficulty in handling non-trivial idiosyncratic uncertainty. This explains why we work with a subset  $\mathcal{J}$  in APR.3, since measurability of a subset does not require the measurability of the superset. The superset in Assumption APR.3 is measurable w.r.t. the completion of measure  $\mu_{\Gamma}$ .

Absence of asymptotic arbitrage opportunities generates asset pricing restrictions in large economies (Ross (1976), CR). We define asymptotic arbitrage opportunities in terms of sequences of portfolios  $p_n$ ,  $n \in \mathbb{N}$ . We characterize portfolio  $p_n$  by the share  $\alpha_{0,n}$  invested in the riskfree asset and the shares  $\alpha_{i,n}$  invested in the selected risky assets  $\gamma_i$ , for  $i = 1, \dots, n$ . The shares are measurable w.r.t.  $\mathcal{F}_0$ . Then,  $C(p_n) = \sum_{i=0}^n \alpha_{i,n}$  is the portfolio cost at  $t = 0$ , and  $p_n = C(p_n)R_0 + \sum_{i=1}^n \alpha_{i,n}R_1(\gamma_i)$  is the portfolio payoff at  $t = 1$ , where  $R_0$  denotes the riskfree gross return measurable w.r.t.  $\mathcal{F}_0$ . We can work with  $t = 1$  because of stationarity.

**Assumption APR.4** *There are no asymptotic arbitrage opportunities in the economy, that is, there exists no portfolio sequence  $(p_n)$  such that  $\lim_{n \rightarrow \infty} P[C(p_n) = 0, p_n \geq 0] = 1$  and  $\liminf_{n \rightarrow \infty} P[p_n > 0] > 0$ .*

Assumption APR.4 excludes portfolios that approximate arbitrage opportunities when the number of included assets increases. Arbitrage opportunities are investments with zero cost and non-negative payoff in each state of the world, and with positive payoff in some states of the world. Hansen and Richard (1987), Definition 2.4, p. 594, uses non-positive cost, i.e.,  $C(p_n) \leq 0$ , in their definition of arbitrage opportunities. A weaker assumption of absence of arbitrage opportunities is sufficient for a linear pricing context. It is weaker since we rule out asymptotic arbitrage opportunities based on a more restrictive concept of arbitrage (the set of sequences  $(p_n)$  with a cost exactly zero is a subset of the set of sequences with a non-positive cost). The next proposition gives the asset pricing restriction.

**Proposition 1** *Under Assumptions APR.1-APR.4, for any date  $t = 1, 2, \dots$  there exists a unique random vector  $\nu_t \in \mathbb{R}^K$  such that  $\nu_t$  is  $\mathcal{F}_{t-1}$ -measurable and:*

$$a_t(\gamma) = b_t(\gamma)' \nu_t, \quad (3)$$

for almost all  $\gamma \in [0, 1]$ ,  $P$ -a.s.

We can rewrite the asset pricing restriction as

$$E[R_t(\gamma) | \mathcal{F}_{t-1}] = b_t(\gamma)' \lambda_t, \quad (4)$$

for almost all  $\gamma \in [0, 1]$ , where  $\lambda_t = \nu_t + E[f_t | \mathcal{F}_{t-1}]$  is the vector of the conditional risk premia. In an unconditional factor model, the sigma-field  $\mathcal{F}_0$  is trivial, and the coefficient functions  $a$  and  $b$  as well as the vectors  $\nu$  and  $\lambda$  are time-invariant, with  $\lambda = \nu + E[f_t]$ . In the CAPM, we have  $K = 1$  and  $\nu = 0$ . When a factor  $f_{k,t}$  is a portfolio excess return, we also have  $\nu_{k,t} = 0$ ,  $k = 1, \dots, K$ .

Equation (4) for a strict factor structure in an unconditional economy (static case with time-invariant coefficients) becomes  $E[R_t] = b(\gamma)' \lambda$ . Such a restriction is derived by Al-Najjar (1998) in his Proposition 2 under the definition of arbitrage used in CR. Through Proposition 1, we extend the latter equality to (4) under an approximate factor structure in a conditional economy (dynamic case with time-varying coefficients) with a definition of arbitrage analogous to Hansen and Richard (1987). Proposition 1 differs from CR Theorem 3 in terms of the returns generating framework, the definition of asymptotic arbitrage opportunities,

and the derived asset pricing restriction. Specifically, we consider a multi-period economy with conditional information and time-varying coefficients as opposed to the single period unconditional economy in CR. We prefer the definition underlying Assumption APR.4 since it corresponds to the definition of arbitrage that is standard in dynamic asset pricing theory (e.g., Duffie (2001)). As pointed out by Hansen and Richard (1987), Ross (1978) has already chosen that type of definition. It also eases the proof based on new arguments. However, in Appendix 5 in the supplementary material, we derive the link between the no-arbitrage conditions in Assumptions A.1 i) and ii) of CR, written conditionally on the information  $\mathcal{F}_0$  and for almost every countable collection of assets, and the asset pricing restriction (3) valid for the continuum of assets. In a time-invariant setting, CR derive the pricing restriction  $\sum_{i=1}^{\infty} \left( a(\gamma_i) - b(\gamma_i)' \nu \right)^2 < \infty$ , for some  $\nu \in \mathbb{R}^K$  and for a given sequence  $(\gamma_i)$ , while we derive the restriction (3), for almost all  $\gamma \in [0, 1]$ . In Appendix 5, we consider the completion  $\bar{\mu}_\Gamma$  of measure  $\mu_\Gamma$  (see e.g. Halmos (1950), Theorem 13.B) to assign measure zero to negligible sets. We show that the set of sequences  $(\gamma_i)$  such that, for any date  $t = 1, 2, \dots$ ,  $\inf_{\nu \in \mathbb{R}^K} \sum_{i=1}^{\infty} \left( a_t(\gamma_i) - b_t(\gamma_i)' \nu \right)^2 < \infty$ ,  $P$ -a.s., has measure 1 under  $\bar{\mu}_\Gamma$ , when the asset pricing restriction (3) holds, and measure 0, otherwise. In other words, validity of the summability condition in CR for a countable collection of assets without validity of the asset pricing restriction (3) is an impossible event. In a time-invariant setting, the zero-one property for the  $\bar{\mu}_\Gamma$ -measure of the set  $\inf_{\nu \in \mathbb{R}^K} \sum_{i=1}^{\infty} \left( a(\gamma_i) - b(\gamma_i)' \nu \right)^2 < \infty$  in  $\Gamma$  is a consequence of the Kolmogorov zero-one law (see e.g. Billingsley (1995)). The restriction in Proposition 1 is testable with large equity datasets and large sample sizes (Section 3.5). Therefore, we are not affected by the Shanken (1982) critique, namely the problem that finiteness of the sum  $\sum_{i=1}^{\infty} \left( a(\gamma_i) - b(\gamma_i)' \nu \right)^2$  for a given countable economy as in CR cannot be tested empirically.

## 2.2 Robustness of factor structures to asset repackaging

In this section, we prove the invariance of the approximate factor structure to asset repackaging. This concern about robustness is already put forward in Shanken (1982). We follow Al-Najjar (1999b), Section 3.1, and define a repackaging by means of sequences of measurable functions  $w_k : [0, 1] \rightarrow [-1, 1]$  and  $G_k : [0, 1] \rightarrow [0, 1]$  for  $k = 1, 2, \dots$ . Functions  $G_k$  are bijective. The idea is that any asset  $\gamma \in [0, 1]$  is cut in (at most countably many) slices, and after reshuffling the  $k$ th slice with relative weight  $w_k(\gamma)$  is assigned

to the new asset  $G_k(\gamma)$ . Then, the excess returns in the repackaged economy are:

$$\tilde{R}_t(\gamma) = \sum_k w_k[G_k^{-1}(\gamma)]R_t[G_k^{-1}(\gamma)]. \quad (5)$$

The functions  $w_k(\cdot)$  are such that  $\sum_k w_k(\gamma) = 1$ , for any  $\gamma \in [0, 1]$ , since they correspond to weights, and are bounded:  $|w_k(\gamma)| \leq \bar{w}_k$ , for any  $\gamma$  and  $k$ , with  $\sum_k \bar{w}_k < \infty$ . In addition, the functions  $G_k(\cdot)$  are measure-preserving to guarantee that “no risk that is idiosyncratic to a negligible subset of assets in the original economy can be blown up to have aggregate effects in the new economy” (Al-Najjar (1999b)).

From (2) and (5), the excess returns in the repackaged economy satisfy the conditional linear factor model:

$$\tilde{R}_t(\gamma) = \tilde{\beta}_t(\gamma)'x_t + \tilde{\varepsilon}_t(\gamma), \quad (6)$$

where  $\tilde{\beta}_t(\gamma) = \sum_k w_k[G_k^{-1}(\gamma)]\beta_t[G_k^{-1}(\gamma)]$  and  $\tilde{\varepsilon}_t(\gamma) = \sum_k w_k[G_k^{-1}(\gamma)]\varepsilon_t[G_k^{-1}(\gamma)]$ . The vector of factors is unchanged, and asset repackaging has no effect on the validity of Assumption APR.1. The following proposition shows that the condition in Assumption APR.3 on the conditional variance-covariance matrix of the error terms also holds in the new economy if it holds before repackaging.

**Proposition 2** *Let  $E[\varepsilon_t(\gamma)^2|\mathcal{F}_{t-1}]$  be bounded uniformly in  $\gamma \in [0, 1]$  and  $\omega \in \Omega$ . If the condition APR.3 holds for the original economy (2), the condition APR.3 with  $\tilde{\varepsilon}$  substituted for  $\varepsilon$  holds for the repackaged economy (6).*

The next section describes how we get the data from sampling the continuum of assets.

### 2.3 Sampling scheme

We estimate the risk premia from a sample of observations on returns and factors for  $n$  assets and  $T$  dates. In available databases, we do not observe asset returns for all firms at all dates. We account for the unbalanced nature of the panel through a collection of indicator variables  $I(\gamma)$ ,  $\gamma \in [0, 1]$ , and define  $I_t(\gamma, \omega) = I[\gamma, S^t(\omega)]$ . Then  $I_t(\gamma) = 1$  if the return of asset  $\gamma$  is observable by the econometrician at date  $t$ , and 0 otherwise (Connor and Korajczyk (1987)). To keep the factor structure linear, we assume a missing-at-random design (Rubin (1976)), that is, independence between unobservability and return generation.

**Assumption SC.1** *The random variables  $I_t(\gamma)$ ,  $\gamma \in [0, 1]$ , are independent of  $\varepsilon_t(\gamma)$ ,  $\gamma \in [0, 1]$ , and any variable in  $\mathcal{F}_t$ .*

Another design would require an explicit modeling of the link between the unobservability mechanism and the return process of the continuum of assets (Heckman (1979)); this would yield a nonlinear factor structure.

Assets are randomly drawn from the population according to the probability distribution  $G$  on  $[0, 1]$ .

**Assumption SC.2** *The random variables  $\gamma_i$ ,  $i = 1, \dots, n$ , are i.i.d. indices, independent of  $\varepsilon_t(\gamma)$ ,  $I_t(\gamma)$ ,  $\gamma \in [0, 1]$  and  $\mathcal{F}_t$ , each with distribution  $G$ .*

For any  $n, T \in \mathbb{N}$ , the excess returns are  $R_{i,t} = R_t(\gamma_i)$  and the observability indicators are  $I_{i,t} = I_t(\gamma_i)$ , for  $i = 1, \dots, n$ , and  $t = 1, \dots, T$ . The excess return  $R_{i,t}$  is observed if and only if  $I_{i,t} = 1$ . Similarly, let  $\beta_{i,t} = \beta_t(\gamma_i) = (a_{i,t}, b'_{i,t})'$  be the characteristics,  $\varepsilon_{i,t} = \varepsilon_t(\gamma_i)$  the error terms, and  $\sigma_{ij,t} = E[\varepsilon_{i,t}\varepsilon_{j,t}|\mathcal{F}_t, \gamma_i, \gamma_j]$  the conditional variances and covariances of the errors of the assets in the sample. By random sampling, we get a random coefficient panel model (e.g. Hsiao (2003), Chapter 6). The characteristic  $\beta_{i,t}$  of asset  $i$  at time  $t$  is random even conditionally on  $\mathcal{F}_{t-1}$ . It is potentially correlated with the error terms  $\varepsilon_{i,t}$ , the observability indicators  $I_{i,t}$ , and the conditional variances  $\sigma_{ii,t}$ , conditionally on  $\mathcal{F}_{t-1}$ , through the random index  $\gamma_i$ . If the coefficients  $a_{i,t}$ s and  $b_{i,t}$ s were treated as given parameters in the cross-section of assets at time  $t$ , and not as realizations of random variables, invoking cross-sectional LLNs and CLTs as in some assumptions and parts of the proofs would have no sense. Moreover, cross-sectional limits would be dependent on the selected ordering of the assets. Instead, our assumptions and results do not rely on a specific ordering of assets. Random elements  $(\beta'_{i,t}, \sigma_{ii,t}, \varepsilon_{i,t}, I_{i,t})'$ ,  $i = 1, \dots, n$ , are exchangeable (Andrews (2005)). Hence, assets randomly drawn from the population have ex-ante the same features. However, given a specific realization of the indices in the sample, assets have ex-post heterogeneous features.

### 3 Two-pass approach for time-varying risk premium

In the setting of Section 2, we do not follow rolling short-window regression approaches to account for time-variation (Fama and French (1997), Lewellen and Nagel (2006)); see also Cosemans et al. (2011) for a recent proposal of a hybrid method to estimate a one-factor time-varying beta model based on Bayesian

methodology and shrinkage. We favor a structural econometric framework taking into account the no-arbitrage restrictions to conduct formal frequentist inference for multi-factor models in large cross-sectional equity datasets. Besides, a five-year window of monthly data yields a very short time-series panel for which asymptotics with fixed  $T$  and large  $n$  are better suited, but keeping  $T$  fixed impedes consistent estimation of the risk premia (Shanken (1992)).

### 3.1 Functional specification of model coefficients

To have a workable version of Equations (1) and (3), we further specify the conditioning information and how the model coefficients depend on it via a functional specification. The conditioning information is such that instruments  $Z \in \mathbb{R}^p$  and  $Z(\gamma) \in \mathbb{R}^q$ , for  $\gamma \in [0, 1]$ , are  $\mathcal{F}_0$ -measurable. Then, the information  $\mathcal{F}_{t-1}$  contains  $Z_{t-1}$  and  $Z_{t-1}(\gamma)$ , for  $\gamma \in [0, 1]$ , where we define  $Z_t(\omega) = Z[S^t(\omega)]$  and  $Z_t(\gamma, \omega) = Z[\gamma, S^t(\omega)]$ , and denote  $Z_t = \{Z_t, Z_{t-1}, \dots\}$ . The lagged instruments  $Z_{t-1}$  are common to all stocks. They may include the constant and past observations of the factors and some additional variables such as macroeconomic variables. The lagged instruments  $Z_{t-1}(\gamma)$  are specific to stock  $\gamma$ . They may include past observations of firm characteristics and stock returns. To end up with a linear regression model, we specify in Assumption FS.1 that the vector of factor loadings  $b_t(\gamma)$  is a linear function of lagged instruments  $Z_{t-1}$  (Shanken (1990), Ferson and Harvey (1991)) and  $Z_{t-1}(\gamma)$  (Avramov and Chordia (2006)).

**Assumption FS.1** *The factor loadings are such that  $b_t(\gamma) = B(\gamma)Z_{t-1} + C(\gamma)Z_{t-1}(\gamma)$ , where  $B(\gamma) \in \mathbb{R}^{K \times p}$  and  $C(\gamma) \in \mathbb{R}^{K \times q}$ , for any  $\gamma \in [0, 1]$  and  $t = 1, 2, \dots$*

We can account for nonlinearities by including powers of some explanatory variables among the lagged instruments.

**Assumption FS.2** *(i) The risk premia vector is such that  $\lambda_t = \Lambda Z_{t-1}$ , where  $\Lambda \in \mathbb{R}^{K \times p}$ , for any  $t$ . (ii) We have  $E[f_t | \mathcal{F}_{t-1}] = F Z_{t-1}$ , where  $F \in \mathbb{R}^{K \times p}$ , for any  $t$ .*

In Assumption FS.2 (i), we specify that the vector of risk premia is a linear function of lagged instruments  $Z_{t-1}$  (Cochrane (1996), Jagannathan and Wang (1996)). Since  $f_t$  is a subvector of  $Z_t$ , Assumption FS.2 (ii) is satisfied if the conditional expectation of  $Z_t$  given the information  $\mathcal{F}_{t-1}$  depends on  $Z_{t-1}$  only and is

linear, as, for instance, in an exogenous Vector Autoregressive (VAR) model of order 1. Under the functional specifications in Assumptions FS.1 and FS.2, the asset pricing restriction (3) implies that the intercept  $a_t(\gamma)$  is a quadratic form in lagged instruments  $Z_{t-1}$  and  $Z_{t-1}(\gamma)$ , namely:

$$a_t(\gamma) = Z'_{t-1} B(\gamma)' (\Lambda - F) Z_{t-1} + Z_{t-1}(\gamma)' C(\gamma)' (\Lambda - F) Z_{t-1}. \quad (7)$$

This shows that assuming a priori linearity of  $a_t(\gamma)$  in the lagged instruments  $Z_{t-1}$  and  $Z_{t-1}(\gamma)$  is in general not compatible with linearity of  $b_t(\gamma)$  and  $E[f_t|Z_{t-1}]$ .

The sampling scheme is defined in Section 2.3, and we use the additional notation  $B_i = B(\gamma_i)$ ,  $C_i = C(\gamma_i)$  and  $Z_{i,t-1} = Z_{t-1}(\gamma_i)$ . In particular, we allow for potential correlation between parameters  $B_i$ ,  $C_i$  and asset specific instruments  $Z_{i,t-1}$  via the random index  $\gamma_i$ . Then, the conditional factor model (1) with asset pricing restriction (7) written for the sample observations becomes

$$R_{i,t} = Z'_{t-1} B'_i (\Lambda - F) Z_{t-1} + Z'_{i,t-1} C'_i (\Lambda - F) Z_{t-1} + Z'_{t-1} B'_i f_t + Z'_{i,t-1} C'_i f_t + \varepsilon_{i,t}, \quad (8)$$

which is nonlinear in the common parameters  $\Lambda$ ,  $F$  and the asset-specific parameters  $B_i$ ,  $C_i$ .

In order to implement the two-pass estimation methodology in our conditional context, we rewrite model (8) as a model that is linear in transformed parameters and new regressors. The regressors include  $x_{2,i,t} = \left( f'_t \otimes Z'_{t-1}, f'_t \otimes Z'_{i,t-1} \right)' \in \mathbb{R}^{d_2}$  with  $d_2 = K(p + q)$ . The first components with common instruments take the interpretation of scaled factors (Cochrane (2005)), while the second components do not since they depend on  $i$ . The regressors also include the predetermined variables  $x_{1,i,t} = \left( \text{vech}[X_t]', Z'_{t-1} \otimes Z'_{i,t-1} \right)' \in \mathbb{R}^{d_1}$  with  $d_1 = p(p + 1)/2 + pq$ , where the symmetric matrix  $X_t = [X_{t,k,l}] \in \mathbb{R}^{p \times p}$  is such that  $X_{t,k,l} = Z_{t-1,k}^2$ , if  $k = l$ , and  $X_{t,k,l} = 2Z_{t-1,k}Z_{t-1,l}$ , otherwise,  $k, l = 1, \dots, p$ . The vector-half operator  $\text{vech}[\cdot]$  stacks the elements of the lower triangular part of a  $p \times p$  matrix as a  $p(p + 1)/2 \times 1$  vector (see Chapter 2 in Magnus and Neudecker (2007, MN) for properties of this matrix tool). Then, we can express model (8) through appropriate redefinitions of the regressors and loadings (see Appendix A.3.1):

$$R_{i,t} = \beta'_i x_{i,t} + \varepsilon_{i,t}, \quad (9)$$

where  $x_{i,t} = (x'_{1,i,t}, x'_{2,i,t})'$  has dimension  $d = d_1 + d_2$ , and  $\beta_i = (\beta'_{1,i}, \beta'_{2,i})'$  is such that

$$\begin{aligned}\beta_{1,i} &= \left( (N_p [(\Lambda - F)' \otimes I_p] \text{vec} [B'_i])', ([(\Lambda - F)' \otimes I_q] \text{vec} [C'_i])' \right)', \quad N_p = \frac{1}{2} D_p^+ (W_p + I_{p^2}), \\ \beta_{2,i} &= \left( \text{vec} [B'_i]', \text{vec} [C'_i]' \right)'.\end{aligned}\quad (10)$$

The vector operator  $\text{vec}[\cdot]$  stacks the elements of an  $m \times n$  matrix as a  $mn \times 1$  vector. The matrix  $D_p^+$  is the  $p(p+1)/2 \times p^2$  Moore-Penrose inverse of the duplication matrix  $D_p$ , such that  $\text{vech}[A] = D_p^+ \text{vec}[A]$  for any matrix  $A \in \mathbb{R}^{p \times p}$  (see Chapter 3 in MN). The commutation matrix  $W_{p,q}$  is such that  $\text{vec}[A'] = W_{p,q} \text{vec}[A]$ , for any matrix  $A \in \mathbb{R}^{p \times q}$ , and  $W_p := W_{p,p}$ . When  $Z_t = 1$  and  $Z_{i,t} = 0$ , we have  $p = 1$  and  $q = 0$ , and the model in (9) reduces to a factor model with time-invariant coefficients and regressor  $x_t$  common across assets (scaled factors).

In Equations (10), the  $d_1 \times 1$  vector  $\beta_{1,i}$  is a linear transformation of the  $d_2 \times 1$  vector  $\beta_{2,i}$ . This clarifies that the asset pricing restriction (7) implies a constraint on the distribution of the random vector  $\beta_i$  via its support. The coefficients of the linear transformation depend on matrix  $\Lambda - F$ . For the purpose of estimating the loading coefficients of the risk premia in matrix  $\Lambda$ , we rewrite the parameter restrictions as (see Appendix A.3.2):

$$\beta_{1,i} = \beta_{3,i} \nu, \quad \nu = \text{vec} [\Lambda' - F'], \quad \beta_{3,i} = \left( [N_p (B'_i \otimes I_p)]', [W_{p,q} (C'_i \otimes I_p)]' \right)'. \quad (11)$$

Furthermore, we can relate the  $d_1 \times Kp$  matrix  $\beta_{3,i}$  to the vector  $\beta_{2,i}$  (see Appendix A.3.3):

$$\text{vec} [\beta'_{3,i}] = J_a \beta_{2,i}, \quad (12)$$

where the  $d_1 p K \times d_2$  block-diagonal matrix of constants  $J_a$  is given by  $J_a = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$  with diagonal blocks  $J_1 = W_{p(p+1)/2, pK} (I_K \otimes [(I_p \otimes N_p) (W_p \otimes I_p) (I_p \otimes \text{vec} [I_p])])$  and  $J_2 = W_{pq, pK} (I_K \otimes [(I_p \otimes W_{p,q}) (W_{p,q} \otimes I_p) (I_q \otimes \text{vec} [I_p])])$ . The link (12) is instrumental in deriving the asymptotic results. In the time-invariant setting,  $\beta_{1,i} = a_i$ ,  $\beta_{2,i} = \beta_{3,i} = b_i$ , and the matrix  $J$  is equal to  $I_K$ . Hence, Equations (11) and (12) in the time-varying case are the counterparts of restriction  $a_i = b'_i \nu$  in the time-invariant case.



### 3.2 Two-pass estimation of time-varying risk premium

We consider a two-pass approach (Fama and MacBeth (1973), Black, Jensen, and Scholes (1972)) building on Equations (9) and (11).

**First Pass:** The first pass consists in computing time-series OLS estimators  $\hat{\beta}_i = (\hat{\beta}'_{1,i}, \hat{\beta}'_{2,i})' = \hat{Q}_{x,i}^{-1} \frac{1}{T_i} \sum_t I_{i,t} x_{i,t} R_{i,t}$ , for  $i = 1, \dots, n$ , where  $\hat{Q}_{x,i} = \frac{1}{T_i} \sum_t I_{i,t} x_{i,t} x'_{i,t}$ . In available panels, the random sample size  $T_i$  for asset  $i$  can be small, and the inversion of matrix  $\hat{Q}_{x,i}$  can be numerically unstable. This can yield unreliable estimates of  $\beta_i$ . To address this, we introduce a trimming device:  $\mathbf{1}_i^X = \mathbf{1} \{CN(\hat{Q}_{x,i}) \leq \chi_{1,T}, \tau_{i,T} \leq \chi_{2,T}\}$ , where  $CN(\hat{Q}_{x,i}) = \sqrt{eig_{\max}(\hat{Q}_{x,i}) / eig_{\min}(\hat{Q}_{x,i})}$  and  $eig_{\min}(\hat{Q}_{x,i})$  denote the condition number and the smallest eigenvalue of matrix  $\hat{Q}_{x,i}$ , respectively,  $\tau_{i,T} = T/T_i$ , and the two sequences  $\chi_{1,T} > 0$  and  $\chi_{2,T} > 0$  diverge asymptotically. The first trimming condition  $\{CN(\hat{Q}_{x,i}) \leq \chi_{1,T}\}$  keeps in the cross-section only assets for which the time series regression is not too badly conditioned. A too large value of  $CN(\hat{Q}_{x,i})$  indicates multicollinearity problems and ill-conditioning (Belsley, Kuh, and Welsch (2004), Greene (2008)). The second trimming condition  $\{\tau_{i,T} \leq \chi_{2,T}\}$  keeps in the cross-section only assets for which the time series is not too short. We also use both trimming conditions in the proofs of the asymptotic results.

**Second Pass:** The second pass consists in computing a cross-sectional estimator of  $\nu$  by regressing the  $\hat{\beta}_{1,i}$  on the  $\hat{\beta}_{3,i}$  keeping non-trimmed assets only. We use a multivariate WLS approach. The weights are estimates of  $w_i = (diag[v_i])^{-1}$ , where the matrices  $v_i$  are the asymptotic variances of the standardized errors  $\sqrt{T}(\hat{\beta}_{1,i} - \hat{\beta}_{3,i}\nu)$  in the cross-sectional regression for large  $T$ . We have  $v_i = \tau_i C'_\nu Q_{x,i}^{-1} S_{ii} Q_{x,i}^{-1} C_\nu$ , where  $Q_{x,i} = E[x_{i,t} x'_{i,t} | \gamma_i]$ ,  $S_{ii} = \text{plim}_{T \rightarrow \infty} \frac{1}{T} \sum_t \sigma_{ii,t} x_{i,t} x'_{i,t} = E[\varepsilon_{i,t}^2 x_{i,t} x'_{i,t} | \gamma_i]$ ,  $\tau_i = \text{plim}_{T \rightarrow \infty} \tau_{i,T} = E[I_{i,t} | \gamma_i]^{-1}$ ,  $C_\nu = (E'_1 - (I_{d_1} \otimes \nu') J_a E'_2)'$ , with  $E_1 = (I_{d_1} : 0_{d_1 \times d_2})'$ ,  $E_2 = (0_{d_2 \times d_1} : I_{d_2})'$ . We use the estimates  $\hat{v}_i = \tau_{i,T} C'_{\hat{\nu}_1} \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} C_{\hat{\nu}_1}$ , where  $\hat{S}_{ii} = \frac{1}{T_i} \sum_t I_{i,t} \hat{\varepsilon}_{i,t}^2 x_{i,t} x'_{i,t}$ ,  $\hat{\varepsilon}_{i,t} = R_{i,t} - \hat{\beta}'_{i,1} x_{i,t}$  and  $C_{\hat{\nu}_1} = (E'_1 - (I_{d_1} \otimes \hat{\nu}'_1) J_a E'_2)'$ . To estimate  $C_\nu$ , we use the multivariate OLS estimator  $\hat{\nu}_1 = \left( \sum_i \mathbf{1}_i^X \hat{\beta}'_{3,i} \hat{\beta}_{3,i} \right)^{-1} \sum_i \mathbf{1}_i^X \hat{\beta}'_{3,i} \hat{\beta}_{1,i}$ , i.e., a first-step estimator with unit weights. The WLS estimator is:

$$\hat{\nu} = \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i \hat{\beta}'_{3,i} \hat{w}_i \hat{\beta}_{1,i}, \quad (13)$$

where  $\hat{Q}_{\beta_3} = \frac{1}{n} \sum_i \hat{\beta}'_{3,i} \hat{w}_i \hat{\beta}_{3,i}$  and  $\hat{w}_i = \mathbf{1}_i^X (\text{diag}[\hat{v}_i])^{-1}$ . Weighting accounts for the statistical precision of the first-pass estimates and includes trimming. The final estimator of the risk premia is  $\hat{\lambda}_t = \hat{\Lambda} Z_{t-1}$ , where we deduce  $\hat{\Lambda}$  from the relationship  $\text{vec}[\hat{\Lambda}'] = \hat{\nu} + \text{vec}[\hat{F}']$  with the estimator  $\hat{F}$  obtained by a SUR regression of factors  $f_t$  on lagged instruments  $Z_{t-1}$ :  $\hat{F} = \sum_t f_t Z'_{t-1} \left( \sum_t Z_{t-1} Z'_{t-1} \right)^{-1}$ .

In the time-invariant case, the estimator of the risk premia vector simplifies to

$$\hat{\lambda} = \hat{\nu} + \frac{1}{T} \sum_t f_t, \quad \hat{\nu} = \hat{Q}_b^{-1} \frac{1}{n} \sum_i \hat{w}_i \hat{b}_i \hat{a}_i, \quad (14)$$

where  $\hat{Q}_b = \frac{1}{n} \sum_i \hat{w}_i \hat{b}_i \hat{b}'_i$  and  $(\hat{a}_i, \hat{b}'_i)' = \hat{Q}_{x,i}^{-1} \frac{1}{T_i} \sum_t I_{i,t} x_t R_{i,t}$ . Hence, the model coefficients  $a_i$  and  $b_i$  are estimated by time series OLS regression, and the estimate of the risk premium vector is obtained by cross-sectional WLS regression of the  $\hat{a}_i$ s on the  $\hat{b}_i$ s augmented by the factor mean. Moreover, under conditional homoskedasticity  $\sigma_{ii,t} = \sigma_{ii}$  and a balanced panel  $\tau_{i,T} = 1$ , we have  $v_i = c'_\nu Q_x^{-1} c_\nu \sigma_{ii}$ , where  $c_\nu = (1, -\nu)'$  and  $Q_x = E[x_t x'_t]$ . Then,  $v_i$  is directly proportional to  $\sigma_{ii}$ , and we can simply pick the weights as  $\hat{w}_i = \hat{\sigma}_{ii}^{-1}$ , where  $\hat{\sigma}_{ii} = \frac{1}{T} \sum_t \hat{\varepsilon}_{i,t}^2$  (Shanken (1992)). In the time-invariant case, we can avoid the trimming on the condition number if we substitute  $\hat{Q}_x = \frac{1}{T} \sum_t x_t x'_t$  for  $\hat{Q}_{x,i}$  in the first-pass estimator definition. However, this increases the asymptotic variance of the bias corrected estimator of  $\nu$ , and does not extend to the time-varying case. Starting from the asset pricing restriction  $E[R_{i,t}] = b'_i \lambda$  in the time-invariant case, another estimator of  $\lambda$  is  $\bar{\lambda} = \hat{Q}_b^{-1} \frac{1}{n} \sum_i \hat{w}_i \hat{b}_i \bar{R}_i$ , where  $\bar{R}_i = \frac{1}{T_i} \sum_t I_{i,t} R_{i,t}$ . This estimator is numerically equivalent to  $\hat{\lambda}$  in the balanced case, where  $I_{i,t} = 1$  for all  $i$  and  $t$ . In the unbalanced case, it is equal to  $\bar{\lambda} = \hat{\nu} + \hat{Q}_b^{-1} \frac{1}{n} \sum_i \hat{w}_i \hat{b}_i \hat{b}'_i \bar{f}_i$ , where  $\bar{f}_i = \frac{1}{T_i} \sum_t I_{i,t} f_t$ . Estimator  $\bar{\lambda}$  is often studied by the literature (see, e.g., Shanken (1992), Kandel and Stambaugh (1995), Jagannathan and Wang (1998)), and is also consistent. Estimating  $E[f_t]$  with a simple average of the observed factor instead of a weighted average based on estimated betas simplifies the form of the asymptotic distribution in the unbalanced case. This explains our preference for  $\hat{\lambda}$  over  $\bar{\lambda}$ .

### 3.3 Asymptotic properties of risk premium estimation

We derive the asymptotic properties under assumptions on the conditional distribution of the error terms.

**Assumption A.1** *There exists a positive constant  $M$  such that for all  $n, T$ :*

- a)  $E[\varepsilon_{i,t} | \{\varepsilon_{j,t-1}, \gamma_j, j = 1, \dots, n\}, \mathcal{F}_t] = 0$ , with  $\varepsilon_{j,t} = \{\varepsilon_{j,t}, \varepsilon_{j,t-1}, \dots\}$ ; b)  $\frac{1}{M} \leq \sigma_{ii,t} \leq M$ ,  $i = 1, \dots, n$ ;  
c)  $E\left[\frac{1}{n} \sum_{i,j} E\left[|\sigma_{ij,t}|^2 | \gamma_i, \gamma_j\right]^{1/2}\right] \leq M$ , where  $\sigma_{ij,t} = E[\varepsilon_{i,t}\varepsilon_{j,t} | \mathcal{F}_t, \gamma_i, \gamma_j]$ .

Assumption A.1 allows for a martingale difference sequence for the error terms (part a)) including potential conditional heteroskedasticity (part b)) as well as weak cross-sectional dependence (part c)). In particular, Assumption A.1 c) is the same as Assumption C.3 in Bai and Ng (2002), except that we have an expectation w.r.t. the random draws of assets. More general error structures are possible but complicate consistent estimation of the asymptotic variances of the estimators (see Section 3.4).

Proposition 3 summarizes consistency of estimators  $\hat{\nu}$  and  $\hat{\Lambda}$  under the double asymptotics  $n, T \rightarrow \infty$ .

**Proposition 3** *Under Assumptions APR.1-APR.4, SC.1-SC.2, FS.1-FS.2, A.1b) and B.1, B.4-B.6, we get*

- a)  $\|\hat{\nu} - \nu\| = o_p(1)$ , b)  $\|\hat{\Lambda} - \Lambda\| = o_p(1)$ , when  $n, T \rightarrow \infty$  such that  $n = O(T^{\bar{\gamma}})$  for  $\bar{\gamma} > 0$ .

Part b) implies  $\sup_t \|\hat{\lambda}_t - \lambda_t\| = o_p(1)$  under boundedness of process  $Z_t$  (Assumption B.4).

Consistency of the estimators holds under double asymptotics such that the cross-sectional size  $n$  grows not faster than a power of the time series size  $T$ . For instance, the conditions in Proposition 3 allow for  $n$  large w.r.t.  $T$  (short panel asymptotics) when  $\bar{\gamma} > 1$ . In the time-invariant setting, Shanken (1992) shows consistency of  $\hat{\nu}$  and  $\hat{\lambda}$  for a fixed  $n$  and  $T \rightarrow \infty$ . This consistency does not imply Proposition 3. Shanken (1992) (see also Litzenberger and Ramaswamy (1979)) further shows that we can estimate  $\nu$  consistently in the second pass with a modified cross-sectional estimator for a fixed  $T$  and  $n \rightarrow \infty$ . Since  $\lambda = \nu + E[f_t]$ , consistent estimation of the risk premia themselves is impossible for a fixed  $T$  (see Shanken (1992) for the same point).

Proposition 4 below gives the large-sample distributions under the double asymptotics  $n, T \rightarrow \infty$ . Let us define  $\tau_{ij,T} = T/T_{ij}$ , where  $T_{ij} = \sum_t I_{ij,t}$  and  $I_{ij,t} = I_{i,t}I_{j,t}$  for  $i, j = 1, \dots, n$ , and  $\tau_{ij} = \text{plim}_{T \rightarrow \infty} \tau_{ij,T} = E[I_{ij,t} | \gamma_i, \gamma_j]^{-1}$ . We make use of matrices  $Q_{\beta_3} = E[\beta'_{3,i} w_i \beta_{3,i}]$ ,  $Q_z = E[Z_t Z_t']$ ,  $S_{ij} = \text{plim}_{T \rightarrow \infty} \frac{1}{T} \sum_t \sigma_{ij,t} x_{i,t} x'_{j,t} = E[\varepsilon_{i,t} \varepsilon_{j,t} x_{i,t} x'_{j,t} | \gamma_i, \gamma_j]$  and  $S_{Q,ij} = Q_{x,i}^{-1} S_{ij} Q_{x,j}^{-1}$ . The following assumption describes the CLTs underlying the proof of the distributional properties.

**Assumption A.2** As  $n, T \rightarrow \infty$ , a)  $\frac{1}{\sqrt{n}} \sum_i \tau_i \left[ (Q_{x,i}^{-1} Y_{i,T}) \otimes v_{3,i} \right] \Rightarrow N(0, S_{v_3})$ , with  $Y_{i,T} = \frac{1}{\sqrt{T}} \sum_t I_{i,t} x_{i,t} \varepsilon_{i,t}$ ,  $v_{3,i} = \text{vec}[\beta'_{3,i} w_i]$  and  $S_{v_3} = \lim_{n \rightarrow \infty} E \left[ \frac{1}{n} \sum_{i,j} \frac{\tau_i \tau_j}{\tau_{ij}} S_{Q,ij} \otimes v_{3,i} v'_{3,j} \right] = a.s.- \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i,j} \frac{\tau_i \tau_j}{\tau_{ij}} [S_{Q,ij} \otimes v_{3,i} v'_{3,j}]$ ;  
b)  $\frac{1}{\sqrt{T}} \sum_t u_t \otimes Z_{t-1} \Rightarrow N(0, \Sigma_u)$ , where  $\Sigma_u = E[u_t u'_t \otimes Z_{t-1} Z'_{t-1}]$  and  $u_t = f_t - F Z_{t-1}$ .

Assumptions A.2a) and b) require the asymptotic normality of cross-sectional and time series averages of scaled error terms, and of time series averages of factor values, respectively. These CLTs hold under weak serial and cross-sectional dependencies such as temporal mixing and block dependence (see Appendix 6). In the online supplementary materials, we verify through simulations that the CLT in Assumption A.2a) holds true for a design calibrated on our data. Assumption A.2a) would be debatable if we face a power law behavior in the cross-section for the unknown multivariate products  $\tau_i \left[ \left( Q_{x,i}^{-1} Y_{i,T} \right) \otimes v_{3,i} \right]$ . Gabaix (2011) reports such an issue for firm size distribution.

**Assumption A.3** For any  $1 \leq t, s \leq T$ ,  $T \in \mathbb{N}$  and  $\gamma \in [0, 1]$ , we have  $E[\varepsilon_t(\gamma)^2 \varepsilon_s(\gamma) | \mathcal{F}_T] = 0$ .

Assumption A.3 is a symmetry condition on the conditional distribution of errors. It is used to prove that the sampling variability of the estimated weights  $\hat{w}_i$  does not impact the asymptotic distribution of estimator  $\hat{\nu}$ . Our setting differs from the standard feasible WLS framework since we have to estimate each incidental parameter  $S_{ii}$ . We can dispense with Assumption A.3 if we use OLS to estimate parameter  $\nu$ , i.e., estimator  $\hat{\nu}_1$ , or if we put a more restrictive condition on the relative rate of  $n$  w.r.t.  $T$ .

**Proposition 4** Under Assumptions APR.1-APR.4, SC.1-SC.2, FS.1-FS.2, A.1-A.3 and B.1-B.6, we have

a)  $\sqrt{nT} \left( \hat{\nu} - \nu - \frac{1}{T} \hat{B}_\nu \right) \Rightarrow N(0, \Sigma_\nu)$  where  $\hat{B}_\nu = \hat{Q}_{\beta_3}^{-1} J_b \frac{1}{n} \sum_i \tau_{i,T} \text{vec} \left[ E'_2 \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i \right]$  and  $\Sigma_\nu = \left( \text{vec} [C'_\nu] \otimes Q_{\beta_3}^{-1} \right) S_{v_3} \left( \text{vec} [C'_\nu] \otimes Q_{\beta_3}^{-1} \right)$ , with  $J_b = \left( \text{vec} [I_{d_1}]' \otimes I_{Kp} \right) (I_{d_1} \otimes J_a)$ , and  $C_{\hat{\nu}} = (E'_1 - (I_{d_1} \otimes \hat{\nu}') J_a E'_2)'$ ; b)  $\sqrt{T} \text{vec} [\hat{\Lambda}' - \Lambda'] \Rightarrow N(0, \Sigma_\Lambda)$  where  $\Sigma_\Lambda = (I_K \otimes Q_z^{-1}) \Sigma_u (I_K \otimes Q_z^{-1})$ , when  $n, T \rightarrow \infty$  such that  $n = O(T^{\bar{\gamma}})$  for  $0 < \bar{\gamma} < 3$ .

Since  $\lambda_t = \Lambda Z_{t-1} = (Z'_{t-1} \otimes I_K) W_{p,K} \text{vec} [\Lambda']$ , part b) and the properties of commutation matrices imply conditionally on  $Z_{t-1}$  that  $\sqrt{T} \left( \hat{\lambda}_t - \lambda_t \right) \Rightarrow N(0, (Z'_{t-1} Q_z^{-1} \otimes I_K) E[Z_{t-1} Z'_{t-1} \otimes u_t u'_t] (Q_z^{-1} Z_{t-1} \otimes I_K))$ .

We can simplify the asymptotic variance matrix in Proposition 4 in the time-invariant setting to  $\Sigma_\nu = Q_b^{-1} S_b Q_b^{-1}$ , where  $Q_b = E[w_i b_i b_i']$  and  $S_b = \lim_{n \rightarrow \infty} E \left[ \frac{1}{n} \sum_{i,j} w_i w_j v_{ij} b_i b_j' \right]$ , with  $v_{ij} = \frac{\tau_i \tau_j}{\tau_{ij}} c_\nu' Q_x^{-1} S_{ij} Q_x^{-1} c_\nu$ , which gives  $v_{ii} = v_i$ . We can also write the asymptotic variance matrix as a limit:

$$\Sigma_\nu = a.s.- \lim_{n \rightarrow \infty} \Sigma_{\nu,n}, \quad \Sigma_{\nu,n} := \left( \frac{1}{n} B_n' W_n B_n \right)^{-1} \frac{1}{n} B_n' W_n V_n W_n B_n \left( \frac{1}{n} B_n' W_n B_n \right)^{-1}, \quad (15)$$

where  $B_n = (b_1, \dots, b_n)'$ ,  $W_n = \text{diag}(w_1, \dots, w_n)$ , and  $V_n = [v_{ij}]_{i,j=1,\dots,n}$ . In the conditionally homoskedastic and balanced case, we further have  $c_\nu' Q_x^{-1} c_\nu = 1 + \lambda' V[f_t]^{-1} \lambda$  and  $V_n = (1 + \lambda' V[f_t]^{-1} \lambda) \Sigma_{\varepsilon,n}$ , where  $\Sigma_{\varepsilon,n} = [\sigma_{ij}]_{i,j=1,\dots,n}$ . In particular, in the CAPM, we have  $K = 1$  and  $\nu = 0$ , which implies that  $\sqrt{\lambda^2 / V[f_t]}$  is equal to the slope of the Capital Market Line  $\sqrt{E[f_t]^2 / V[f_t]}$ , i.e., the Sharpe Ratio of the market portfolio.

Proposition 4 shows that the estimator  $\hat{\nu}$  has a fast convergence rate  $\sqrt{nT}$  and features an asymptotic bias term. Both  $\hat{\beta}_{1,i}$  and  $\hat{\beta}_{3,i}$  in the definition of  $\hat{\nu}$  contain an estimation error; for  $\hat{\beta}_{3,i}$ , this is the well-known Error-In-Variable (EIV) problem. The EIV problem does not impede consistency since we let  $T$  grow to infinity. However, it induces the bias term  $\hat{B}_\nu / T$  which centers the asymptotic distribution of  $\hat{\nu}$ . The upper bound on the relative expansion rates of  $n$  and  $T$  in Proposition 4 is  $n = O(T^{\bar{\gamma}})$  for  $\bar{\gamma} < 3$ . The control of the first-pass estimation errors uniformly across assets requires that the cross-section dimension  $n$  is not too large w.r.t. the time series dimension  $T$ .

In the above setting, we study a semiparametric framework with a double asymptotic treatment and unbalanced panel under a general approximate factor structure. Ang, Liu, and Schwarz (2008) look at a maximum likelihood analysis with a single asymptotic treatment (large  $T$ ,  $n$  fixed) and balanced panel under a particular approximate Gaussian factor structure (block diagonal covariance matrix of residuals) and time-invariant coefficients. Their setting further assumes that the factors have zero mean. Such an assumption gives  $\hat{\lambda} = \hat{\nu}$  in a time-invariant setting (see (14)). Under a zero mean (or a known mean, i.e., not to be estimated), the asymptotic variance of  $\hat{\lambda}$  corresponds to the asymptotic variance  $\Sigma_\nu$  of  $\hat{\nu}$  and the rate of convergence is  $\sqrt{nT}$ . Our results for  $\Sigma_\nu$  yields their Equation (13), when their intercept parameter  $\alpha$  is assumed 0, for uncorrelated and conditionally homoskedastic idiosyncratic risks across assets. On the contrary, if we do not know the mean of the factor and need to estimate it, we have  $\hat{\lambda} = \hat{\nu} + \frac{1}{T} \sum f_t$ . The asymptotic variance of  $\hat{\lambda}$  corresponds to the asymptotic variance  $\Sigma_f$  of the sam-

ple average of the factors, and the rate of convergence is  $\sqrt{T}$ . Jagannathan and Wang (2002) is an early reference on the impact of knowing or not the mean of the factors for asymptotic analysis. With an unknown mean, only the variability of the factor drives the asymptotic distribution of  $\hat{\lambda}$ , since the estimation error  $O_p\left(1/\sqrt{T}\right)$  of the sample average  $\frac{1}{T} \sum_t f_t$  dominates the estimation error  $O_p\left(1/\sqrt{nT} + 1/T\right)$  of  $\hat{\nu}$ . This result is an oracle property for  $\hat{\lambda}$ , namely that its asymptotic distribution is the same irrespective of the knowledge of  $\nu$ . This property is in sharp difference with the single asymptotics with a fixed  $n$  and  $T \rightarrow \infty$ . In the balanced case and with homoskedastic errors for the time-invariant case, Theorem 1 of Shanken (1992) shows that the rate of convergence of  $\hat{\lambda}$  is  $\sqrt{T}$  and that its asymptotic variance is  $\Sigma_{\lambda,n} = \Sigma_f + \frac{1}{n}(1 + \lambda'V[f_t]^{-1}\lambda) \left(\frac{1}{n}B_n'W_nB_n\right)^{-1} \frac{1}{n}B_n'W_n\Sigma_{\varepsilon,n}W_nB_n \left(\frac{1}{n}B_n'W_nB_n\right)^{-1}$ , for fixed  $n$  and  $T \rightarrow \infty$ . The two components in  $\Sigma_{\lambda,n}$  come from estimation of  $E[f_t]$  and  $\nu$ , respectively. In the conditionally heteroskedastic setting with fixed  $n$ , a slight extension of Theorem 1 in Jagannathan and Wang (1998), or Theorem 3.2 in Jagannathan, Skoulakis, and Wang (2009), to the unbalanced case yields  $\Sigma_{\lambda,n} = \Sigma_f + \frac{1}{n}\Sigma_{\nu,n}$ , where  $\Sigma_{\nu,n}$  is defined in (15). Letting  $n \rightarrow \infty$  gives  $\Sigma_f$  under weak cross-sectional dependence. Thus, exploiting the full cross-section of assets improves efficiency asymptotically, and the positive definite matrix  $\Sigma_{\lambda,n} - \Sigma_f$  corresponds to the efficiency gain. Using a large number of assets instead of a small number of portfolios does help to eliminate the contribution coming from estimation of  $\nu$ .

Proposition 4 suggests exploiting the analytical bias correction  $\hat{B}_\nu/T$  and using estimator  $\hat{\nu}_B = \hat{\nu} - \frac{1}{T}\hat{B}_\nu$  instead of  $\hat{\nu}$ . In the time-invariant setting,  $\hat{\lambda}_B = \hat{\nu}_B + \frac{1}{T} \sum_t f_t$  delivers a bias-free estimator of  $\lambda$  at order  $1/T$ , which shares the same root- $T$  asymptotic distribution as  $\hat{\lambda}$ .

Finally, we can relate the results of Proposition 4 to bias-corrected estimation accounting for the well-known incidental parameter problem (Neyman and Scott (1948)) in the panel literature (see Lancaster (2000) for a review). To highlight the main idea, let us focus on the model with time-invariant coefficients. We can write the factor model under restriction  $a_i = b_i'\nu$  as  $R_{i,t} = b_i'(f_t + \nu) + \varepsilon_{i,t}$ . In the likelihood setting of Hahn and Newey (2004) (see also Hahn and Kuersteiner (2002)), the  $b_i$ s correspond to the individual fixed effects and  $\nu$  to the common parameter of interest. Available results on the fixed-effect approach tell us: (i) the ML estimator of  $\nu$  is inconsistent if  $n$  goes to infinity while  $T$  is held fixed, (ii) the ML estimator of  $\nu$  is asymptotically biased even if  $T$  grows at the same rate as  $n$ , (iii) an analytical bias correction may yield an estimator of  $\nu$  that is root- $(nT)$  asymptotically normal and centered at the truth if  $T$  grows faster than  $n^{1/3}$ .

The two-pass estimators  $\hat{\nu}$  and  $\hat{\nu}_B$  exhibit the properties (i)-(iii) as expected by analogy with unbiased estimation in large panels. This clear link with the incidental parameter literature highlights another advantage of working with  $\nu$  in the second-pass regression. Chamberlain (1992) considers a general random coefficient model nesting the factor model with time-invariant coefficients. He establishes asymptotic normality of an estimator of  $\nu$  for fixed  $T$  and balanced panel data. His estimator does not admit a closed-form and requires a numerical optimization. This leads to computational difficulties in the conditional setting. This also makes the study of his estimator under double asymptotics and cross-sectional dependence challenging. Recent advances on the incidental parameter problem in random coefficient models for fixed  $T$  are Arellano and Bonhomme (2012) and Bonhomme (2012).

### 3.4 Confidence intervals

We can use Proposition 4 to build confidence intervals by means of consistent estimation of the asymptotic variance-covariance matrices. We can check with these intervals whether the risk of a given factor  $f_{k,t}$  is not remunerated, i.e.,  $\lambda_{k,t} = 0$ , or the restriction  $\nu = 0$  holds when the factors are traded. We replace the unknown quantities  $Q_{x,i}$ ,  $Q_z$ ,  $Q_{\beta_3}$ , and  $\nu$  by their empirical counterparts  $\hat{Q}_{x,i}$ ,  $\hat{Q}_z$ ,  $\hat{Q}_{\beta_3}$ , and  $\hat{\nu}$ . We estimate  $\Sigma_u$  (or  $\Sigma_f$  in the time-invariant setting), by a standard HAC estimator  $\hat{\Sigma}_u$  such as in Newey and West (1994) or Andrews and Monahan (1992). Hence, the construction of confidence intervals with valid asymptotic coverage for components of  $\hat{\Lambda}$  is straightforward. On the contrary, getting a HAC estimator for  $\bar{\Sigma}_f$  appearing in the asymptotic distribution of  $\bar{\lambda}$  as discussed after Equation (14) is not obvious in the unbalanced case.

The construction of confidence intervals for the components of  $\hat{\nu}$  is more difficult. Indeed, the variance-covariance matrix  $\Sigma_\nu$  through  $S_{v_3}$  involves a limiting double sum over  $S_{ij}$  scaled by  $n$  and not  $n^2$ . A naive approach consists in replacing  $S_{ij}$  by any consistent estimator such as  $\hat{S}_{ij} = \frac{1}{T_{ij}} \sum_t I_{ij,t} \hat{\varepsilon}_{i,t} \hat{\varepsilon}_{j,t} x_{i,t} x'_{j,t}$ , but this does not work here. To handle this, we rely on recent proposals in the statistical literature on consistent estimation of large-dimensional sparse covariance matrices by hard thresholding (Bickel and Levina (2008), El Karoui (2008)). Fan, Liao, and Mincheva (2011) focus on the estimation of the variance-covariance matrix of the errors in large balanced panel with nonrandom time-invariant coefficients and i.i.d. disturbances.

The idea is to assume sparse contributions of the  $S_{ij}$ s to the double sum. Then, we only have to account for sufficiently large contributions in the estimation, i.e., contributions larger than a threshold vanishing asymptotically. Thresholding permits an estimation invariant to asset permutations; the absence of any natural cross-sectional ordering among the matrices  $S_{ij}$ s motivates this choice of estimator. In the following assumption, we use the notion of sparsity suggested by Bickel and Levina (2008) adapted to our framework with random coefficients.

**Assumption A.4** *There exist constants  $\bar{q}, \bar{\delta} \in [0, 1)$  such that  $\max_i \sum_j \|S_{ij}\|^{\bar{q}} = O_p(n^{\bar{\delta}})$ .*

Assumption A.4 tells us that we can neglect most cross-asset contributions  $\|S_{ij}\|$ . As sparsity increases, we can choose coefficients  $\bar{q}$  and  $\bar{\delta}$  closer to zero. Assumption A.4 does not impose sparsity of the covariance matrix of the returns themselves. Assumption A.1 c) is also a sparsity condition, which ensures that the limit matrix  $\Sigma_\nu$  is well-defined when combined with Assumption B.4. We meet both sparsity assumptions, as well as the approximate factor structure Assumption APR.3, under weak cross-sectional dependence between the error terms, for instance, under a block dependence structure (see Appendix 6). We can also check that the sparsity structure is robust to asset repackaging using the framework of Section 2.2.

As in Bickel and Levina (2008), let us introduce the thresholded estimator  $\tilde{S}_{ij} = \hat{S}_{ij} \mathbf{1} \left\{ \left\| \hat{S}_{ij} \right\| \geq \kappa \right\}$  of  $S_{ij}$ , which we refer to as  $\hat{S}_{ij}$  thresholded at  $\kappa = \kappa_{n,T}$ . We can derive an asymptotically valid confidence interval for the components of  $\hat{\nu}$  from the next proposition giving a feasible asymptotic normality result based on the estimator  $\tilde{\Sigma}_\nu = \left( \text{vec} [C'_\nu] \otimes \hat{Q}_{\beta_3}^{-1} \right) \tilde{S}_{v_3} \left( \text{vec} [C'_\nu] \otimes \hat{Q}_{\beta_3}^{-1} \right)$ , with  $\tilde{S}_{v_3} = \frac{1}{n} \sum_{i,j} \frac{\tau_{i,T} \tau_{j,T}}{\tau_{ij,T}} [\tilde{S}_{Q,ij} \otimes \hat{v}_{3,i} \hat{v}'_{3,j}]$ , and  $\tilde{S}_{Q,ij} = \hat{Q}_{x,i}^{-1} \tilde{S}_{ij} \hat{Q}_{x,j}^{-1}$ .

**Proposition 5** *Under Assumptions APR.1-APR.4, SC.1-SC.2, FS.1-FS.2, A.1-A.4, B.1-B.6, we have  $\tilde{\Sigma}_\nu^{-1/2} \sqrt{nT} \left( \hat{\nu} - \frac{1}{T} \hat{B}_\nu - \nu \right) \Rightarrow N(0, I_K)$ , when  $n, T \rightarrow \infty$  such that  $n = O(T^{\bar{\gamma}})$  for  $0 < \bar{\gamma} < \min \left\{ 3, \eta \frac{1 - \bar{q}}{2\bar{\delta}} \right\}$ , and  $\kappa = M \sqrt{\frac{\log n}{T\eta}}$  for a constant  $M > 0$  and  $\eta \in (0, 1]$  as in Assumption B.1.*

In Assumption B.1, we define constant  $\eta \in (0, 1]$  which is related to the time series dependence of processes  $(\varepsilon_{i,t})$  and  $(x_{i,t})$ . We have  $\eta = 1$ , when  $(\varepsilon_{i,t})$  and  $(x_{i,t})$  are serially i.i.d. as in Appendix 6 and Bickel and Levina (2008). The stronger the time series dependence (smaller  $\eta$ ) and the lower the sparsity



( $\bar{q}$  and  $\bar{\delta}$  closer to 1), the more restrictive the condition on the relative rate  $\bar{\gamma}$ . We cannot guarantee all thresholded blocks  $\tilde{S}_{ij}$  to be semi definite positive (sdp). However, we expect that the double summation on  $i$  and  $j$  makes  $\tilde{\Sigma}_\nu$  sdp in empirical applications. In case it is not, El Karoui (2008) discusses a few solutions based on shrinkage.

### 3.5 Tests of asset pricing restrictions

From (11), the null hypothesis underlying the asset pricing restriction (3) is

$$\mathcal{H}_0 : \text{there exists } \nu \in \mathbb{R}^{pK} \text{ such that } \beta_1(\gamma) = \beta_3(\gamma)\nu, \text{ for almost all } \gamma \in [0, 1],$$

where  $\beta_1(\gamma)$  and  $\beta_3(\gamma)$  are defined as  $\beta_{1,i}$  and  $\beta_{3,i}$  in Equations (10) and (11) replacing  $B(\gamma)$  and  $C(\gamma)$  for  $B_i$  and  $C_i$ . This null hypothesis is written on the continuum of assets. Under  $\mathcal{H}_0$ , we have  $E[(\beta_{1,i} - \beta_{3,i}\nu)'(\beta_{1,i} - \beta_{3,i}\nu)] = 0$ . Since we estimate  $\nu$  via the WLS cross-sectional regression of the estimates  $\hat{\beta}_{1,i}$  on the estimates  $\hat{\beta}_{3,i}$ , we suggest a test based on the weighted sum of squared residuals SSR of the cross-sectional regression. The weighted SSR is  $\hat{Q}_e = \frac{1}{n} \sum_i \hat{e}_i' \hat{w}_i \hat{e}_i$ , with  $\hat{e}_i = \hat{\beta}_{1,i} - \hat{\beta}_{3,i} \hat{\nu} = C_\nu' \hat{\beta}_i$ , which is an empirical counterpart of  $E[(\beta_{1,i} - \beta_{3,i}\nu)' w_i (\beta_{1,i} - \beta_{3,i}\nu)]$ .

Let us define  $S_{ii,T} = \frac{1}{T} \sum_t I_{i,t} \sigma_{ii,t} x_t x_t'$  and introduce the next assumption.

**Assumption A.5** As  $n, T \rightarrow \infty$ , we have  $\frac{1}{\sqrt{n}} \sum_i \tau_i^2 \left[ \left( Q_{x,i}^{-1} \otimes Q_{x,i}^{-1} \right) (Y_{i,T} \otimes Y_{i,T} - \text{vec}[S_{ii,T}]) \right] \otimes \text{vec}[w_i] \Rightarrow N(0, \Omega)$ , where the asymptotic variance matrix is:

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} E \left[ \frac{1}{n} \sum_{i,j} \frac{\tau_i^2 \tau_j^2}{\tau_{ij}^2} [S_{Q,ij} \otimes S_{Q,ij} + (S_{Q,ij} \otimes S_{Q,ij}) W_d] \otimes (\text{vec}[w_i] \text{vec}[w_j]') \right] \\ &= a.s.-\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i,j} \frac{\tau_i^2 \tau_j^2}{\tau_{ij}^2} [S_{Q,ij} \otimes S_{Q,ij} + (S_{Q,ij} \otimes S_{Q,ij}) W_d] \otimes (\text{vec}[w_i] \text{vec}[w_j]'). \end{aligned}$$

Assumption A.5 is a high-level CLT condition. We can prove this assumption under primitive conditions on the time series and cross-sectional dependence. For instance, we prove in Appendix 6 that Assumption A.5 holds under a cross-sectional block dependence structure for the errors. Intuitively, the expression of the variance-covariance matrix  $\Omega$  is related to the result that, for random  $d \times 1$  vectors  $Y_1$  and  $Y_2$  which

are jointly normal with covariance matrix  $S$ , we have  $Cov(Y_1 \otimes Y_1, Y_2 \otimes Y_2) = S \otimes S + (S \otimes S) W_d$  (see Section 3.1 and Chapter 3 of MN for the definition of the commutation matrix  $W_d$ ).

Let us now introduce the following statistic  $\hat{\xi}_{nT} = T\sqrt{n} \left( \hat{Q}_e - \frac{1}{T} \hat{B}_\xi \right)$ , where the recentering term simplifies to  $\hat{B}_\xi = d_1$  thanks to the weighting scheme. Under the null hypothesis  $\mathcal{H}_0$ , we prove  $\hat{\xi}_{nT} = \text{vec} [C'_\nu \otimes C'_\nu]' \frac{1}{\sqrt{n}} \sum_i \tau_i^2 \left[ \left( Q_{x,i}^{-1} \otimes Q_{x,i}^{-1} \right) (Y_{i,T} \otimes Y_{i,T} - \text{vec} [S_{ii,T}]) \right] \otimes \text{vec} [w_i] + o_p(1)$ , which implies

$\hat{\xi}_{nT} \Rightarrow N(0, \Sigma_\xi)$ , where  $\Sigma_\xi = 2 \lim_{n \rightarrow \infty} E \left[ \frac{1}{n} \sum_{i,j} \frac{\tau_i^2 \tau_j^2}{\tau_{ij}^2} \text{tr} \left[ \left( C'_\nu Q_{x,i}^{-1} S_{ij} Q_{x,j}^{-1} C_\nu \right) w_j \left( C'_\nu Q_{x,j}^{-1} S_{ji} Q_{x,i}^{-1} C_\nu \right) w_i \right] \right]$  as  $n, T \rightarrow \infty$  (see Appendix A.2.6). Then, a feasible testing procedure exploits the consistent estimator  $\tilde{\Sigma}_\xi = 2 \frac{1}{n} \sum_{i,j} \frac{\tau_{i,T}^2 \tau_{j,T}^2}{\tau_{ij,T}^2} \text{tr} \left[ \left( C'_\nu \hat{Q}_{x,i}^{-1} \tilde{S}_{ij} \hat{Q}_{x,j}^{-1} C_\nu \right) \hat{w}_j \left( C'_\nu \hat{Q}_{x,j}^{-1} \tilde{S}_{ji} \hat{Q}_{x,i}^{-1} C_\nu \right) \hat{w}_i \right]$  of the asymptotic variance  $\Sigma_\xi$ .

**Proposition 6** *Under  $\mathcal{H}_0$  and Assumptions APR.1-APR.4, SC.1-SC.2, FS.1-FS.2, A.1-A.5, and B.1-B.6, we have  $\tilde{\Sigma}_\xi^{-1/2} \hat{\xi}_{nT} \Rightarrow N(0, 1)$ , as  $n, T \rightarrow \infty$  such that  $n = O(T^{\bar{\gamma}})$  for  $0 < \bar{\gamma} < \min \left\{ 2, \eta \frac{1 - \bar{q}}{2\bar{\delta}} \right\}$ .*

In the conditionally homoskedastic case for time-invariant coefficients, the asymptotic variance of  $\hat{\xi}_{nT}$  reduces to  $\Sigma_\xi = 2 \text{ a.s.} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i,j} \frac{\tau_i \tau_j}{\tau_{ij}^2} \frac{\sigma_{ij}^2}{\sigma_{ii} \sigma_{jj}}$ . For fixed  $n$ , we can rely on the test statistic  $T\hat{Q}_e$ , which is asymptotically distributed as  $\frac{1}{n} \sum_j \text{eig}_j \chi_j^2$  for  $j = 1, \dots, (n - K)$ , where the  $\chi_j^2$  are independent chi-square variables with 1 degree of freedom, and the coefficients  $\text{eig}_j$  are the non-zero eigenvalues of matrix  $V_n^{1/2} (W_n - W_n B_n (B_n' W_n B_n)^{-1} B_n' W_n) V_n^{1/2}$  (see Kan, Robotti, and Shanken (2013)). By letting  $n$  grow, the sum of chi-square variables converges to a Gaussian variable after recentering and rescaling, which yields heuristically the result of Proposition 6. The condition on the relative expansion rate of  $n$  and  $T$  for the distributional result on the test statistic in Proposition 6 is more restrictive than the condition for feasible asymptotic normality of the estimators in Proposition 5.

The alternative hypothesis is

$$\mathcal{H}_1 : \inf_{\nu \in \mathbb{R}^{pK}} E [(\beta_{1,i} - \beta_{3,i}\nu)' (\beta_{1,i} - \beta_{3,i}\nu)] > 0.$$

Let us define the pseudo-true value  $\nu_\infty = \arg \inf_{\nu \in \mathbb{R}^{pK}} Q_\infty^w(\nu)$ , where  $Q_\infty^w(\nu) = E [(\beta_{1,i} - \beta_{3,i}\nu)' w_i (\beta_{1,i} - \beta_{3,i}\nu)]$  (White (1982), Gouriéroux, Monfort, and Trognon (1984)) and population errors  $e_i = \beta_{1,i} - \beta_{3,i}\nu_\infty = C'_{\nu_\infty} \beta_i$ ,  $i = 1, \dots, n$ , for all  $n$ . In the next proposition, we prove consistency of the test, namely that the

statistic  $\tilde{\Sigma}_\xi^{-1/2} \hat{\xi}_{nT}$  diverges to  $+\infty$  under the alternative hypothesis  $\mathcal{H}_1$  for large  $n$  and  $T$ . The test of the null  $\mathcal{H}_0$  against the alternative  $\mathcal{H}_1$  is a one-sided test. We also give the asymptotic distribution of estimators  $\hat{\nu}$  and  $\hat{\Lambda}$  under  $\mathcal{H}_1$ .

**Proposition 7** *Under  $\mathcal{H}_1$  and Assumptions APR.1-APR.4, SC.1-SC.2, FS.1-FS.2, A.1-A.5 and B.1-B.6, we have: a)  $\sqrt{n} \left( \hat{\nu} - \frac{1}{T} \hat{B}_\nu - \nu_\infty \right) \Rightarrow N(0, \Sigma_{\nu_\infty})$ , where  $\Sigma_{\nu_\infty} = Q_{\beta_3}^{-1} E[\beta'_{3,i} w_i e_i e_i' w_i \beta_{3,i}] Q_{\beta_3}^{-1}$ , and b)  $\sqrt{T} \text{vec} [\hat{\Lambda}' - \Lambda'_\infty] \Rightarrow N(0, \Sigma_\Lambda)$ , where  $\text{vec} [\Lambda'_\infty] = \nu_\infty + \text{vec} [F']$ , as  $n, T \rightarrow \infty$  such that  $T/n = o(1)$  and  $n = O(T^{\bar{\gamma}})$  for  $\bar{\gamma} < 3$ ; c)  $\tilde{\Sigma}_\xi^{-1/2} \hat{\xi}_{nT} \xrightarrow{p} +\infty$ , as  $n, T \rightarrow \infty$  such that  $n = O(T^{\bar{\gamma}})$  for  $0 < \bar{\gamma} < \min \left\{ 2, \eta \frac{1 - \bar{q}}{2\bar{\delta}} \right\}$ .*

Under the alternative hypothesis  $\mathcal{H}_1$ , the convergence rate of  $\hat{\nu}$  is slower than under  $\mathcal{H}_0$ , while the convergence rate of  $\hat{\Lambda}$  remains the same. The asymptotic distribution of the bias-adjusted estimator  $\hat{\nu} - \frac{1}{T} \hat{B}_\nu$  is the same as the one got from a cross-sectional regression of  $\beta_{i,1}$  on  $\beta_{i,3}$ . The condition  $T/n = o(1)$  in Propositions 7 a) and b) ensures that cross-sectional estimation of  $\nu$  has asymptotically no impact on the estimation of  $\Lambda$ .

To study the local asymptotic power, we can adopt the local alternative hypothesis  $\mathcal{H}_{1,nT} : \inf_{\nu \in \mathbb{R}^K} Q_\infty^w(\nu) = \frac{\psi}{\sqrt{nT}} > 0$ , for a constant  $\psi > 0$ . We can show that  $\hat{\xi}_{nT} \Rightarrow N(\psi, \Sigma_\xi)$ , and the test is locally asymptotically powerful. Pesaran and Yamagata (2008) consider a similar local analysis for a test of slope homogeneity in large panels.

Finally, we can derive a test for the null hypothesis when the factors come from tradable assets, i.e., are portfolio excess returns:

$$\mathcal{H}_0 : \beta_1(\gamma) = 0 \text{ for almost all } \gamma \in [0, 1],$$

against the alternative hypothesis

$$\mathcal{H}_1 : E[\beta'_{1,i} \beta_{1,i}] > 0.$$

We only have to substitute  $\hat{Q}_a = \frac{1}{n} \sum_i \hat{\beta}'_{1,i} \hat{w}_i \hat{\beta}_{1,i}$  for  $\hat{Q}_e$ , and  $E_1 = (I_{d_1} : 0_{d_1 \times d_2})'$  for  $C_{\hat{\nu}}$ . Since the constrained form of  $\beta_{1,i}$  in (11) comes from (7), we take directly into account the no-arbitrage restrictions imposed by the model specification. This gives an extension of Gibbons, Ross and Shanken (1989) to the conditional case with double asymptotics. Implementing the original Gibbons, Ross, and Shanken (1989)

test, which uses a weighting matrix corresponding to an inverted estimated large variance-covariance matrix, becomes quickly problematic. We face a large number  $nd_1$  of restrictions; each  $\beta_{1,i}$  is of dimension  $d_1 \times 1$ , and the estimated covariance matrix to invert is of dimension  $nd_1 \times nd_1$ . We expect to compensate the potential loss of power induced by a diagonal weighting via the larger number of restrictions. Our Monte Carlo simulations show that the test exhibits good power properties against both risk-based and non risk-based alternatives (e.g. MacKinlay (1995)) already for a thousand assets with a time series dimension similar to the one in the empirical analysis.

## 4 Empirical results

### 4.1 Asset pricing model and data description

Our baseline asset pricing model is a four-factor model with  $f_t = (r_{m,t}, r_{smb,t}, r_{hml,t}, r_{mom,t})'$ , where  $r_{m,t}$  is the month  $t$  excess return on CRSP NYSE/AMEX/Nasdaq value-weighted market portfolio over the risk free rate, and  $r_{smb,t}$ ,  $r_{hml,t}$  and  $r_{mom,t}$  are the month  $t$  returns on zero-investment factor-mimicking portfolios for size, book-to-market, and momentum (see Fama and French (1993), Jegadeesh and Titman (1993), Carhart (1997)). We proxy the risk free rate with the monthly 30-day T-bill beginning-of-month yield. To account for time-varying alphas, betas and risk premia, we use a conditional specification based on two common variables and a firm-level variable. We take the instruments  $Z_t = (1, Z_t^*)'$ , where bivariate vector  $Z_t^*$  includes the term spread, proxied by the difference between yields on 10-year Treasury and 3-month T-bill, and the default spread, proxied by the yield difference between Moody's Baa-rated and Aaa-rated corporate bonds. We take a scalar  $Z_{i,t}$  corresponding to the book-to-market equity of firm  $i$  as defined in logarithmic terms by Fama and French (2008). We refer to Avramov and Chordia (2006) for convincing theoretical and empirical arguments in favor of the chosen conditional specification. The vector  $x_{i,t}$  has dimension  $d = 25$ , and parsimony explains why we have not included e.g. the size of firm  $i$  as an additional stock specific instrument. We report robustness checks with other time-varying specifications in the supplementary materials.

We compute the firm characteristics from Compustat as in the appendix of Fama and French (2008). The CRSP database provides the monthly stock returns data and we exclude financial firms (Standard Industrial

Classification Codes between 6000 and 6999) as in Fama and French (2008). The dataset after matching CRSP and Compustat contents comprises  $n = 9,936$  stocks, and covers the period from July 1964 to December 2009 with  $T = 546$  months. For comparison purposes with a standard methodology for small  $n$ , we consider i) the 25 Fama-French (FF) portfolios and ii) 44 industry (Indu.) portfolios excluding four financial sectors (banking, insurance, real estate, and trading) as base assets. In those cases, the asset specific instrument  $Z_{i,t-1}$  is the book-to-market equity of the portfolio. We have checked that the reported results for the 25 FF portfolios persist for the 100 FF portfolios. We have downloaded the time series of factors, portfolio returns, and portfolio characteristics from the website of Kenneth French.

## 4.2 Estimation results for time-invariant specifications

We first present the estimates of a time-invariant specification (i.e.  $Z_t = 1$  and  $Z_{i,t} = 0$ ). We use  $\chi_{1,T} = 15$  as advocated by Greene (2008), together with  $\chi_{2,T} = 546/12$ . In the data, we have observed condition numbers as large as 1,000. The number of assets whose condition number is below 15 is 9,930 in the time-invariant four-factor model. The weights  $\hat{w}_i$  are increasing w.r.t. the market capitalisation. This finding is explained by the longer lifetimes and the smaller idiosyncratic variances of larger stocks. Thus, our weighting scheme makes sure that the estimates are not entirely driven by small stocks. To check this empirically, we also gather some results for value-weighted estimation of time-invariant and time-varying specifications in Appendix 12 of the supplementary materials. There the weights directly account for the size characteristics of the assets through their time-average market capitalisation. In the results reported for each model, we denote by  $n^\chi$  the dimension of the cross-section after trimming. We compute confidence intervals with a data-driven threshold selected by cross-validation as in Bickel and Levina (2008). Table 1 Panel A gathers the estimated annual risk premia, with the corresponding confidence intervals at 95% level, for the following time-invariant models: the four-factor model, the Fama-French model, and the CAPM. For the Fama-French model and the CAPM, the trimming level  $\chi_{1,T}$  is not binding when  $\chi_{2,T} = 546/12$ . In Table 1 Panel B we display the estimates of the components of  $\nu$ . For individual stocks, we use bias-corrected estimates for  $\lambda$  and  $\nu$ . For portfolios, we use asymptotics for fixed  $n$  and  $T \rightarrow \infty$ . The estimated risk premia for the market factor are of the same magnitude and all positive across the three universes of assets and the three models. For the four-factor model and the individual stocks, the size factor is positively remunerated

(2.86%) and it is not significantly different from zero. The value factor commands a significant negative reward (-4.60%). The momentum factor is largely remunerated (7.16%) and significantly different from zero. For the 25 FF portfolios, we observe that the size factor is not significantly positively remunerated while the value factor is significantly positively remunerated (3.02% and 4.81%). The momentum factor bears a significant positive reward (34.03%). For  $\lambda_m$ ,  $\lambda_{smb}$ , and  $\lambda_{hml}$ , we obtain similar inferential results when we consider the Fama-French model. Our point estimates of  $\lambda_m$ ,  $\lambda_{smb}$  and  $\lambda_{hml}$  for large  $n$  agree with Ang, Liu, and Schwarz (2008). Our point estimates and confidence intervals for  $\lambda_m$ ,  $\lambda_{smb}$  and  $\lambda_{hml}$  agree with the results reported by Shanken and Zhou (2007) for the 25 FF portfolios. The large, but imprecise, estimate for the momentum premium when  $n = 25$  comes from the estimate for  $\nu_{mom}$  (25.40%) that is much larger and less accurate than the estimates for  $\nu_m$ ,  $\nu_{smb}$  and  $\nu_{hml}$  (0.85%, -0.26%, 0.03%). Moreover, while the estimates of  $\nu_m$ ,  $\nu_{smb}$  and  $\nu_{hml}$  are statistically not significant for the 25 FF portfolios, the estimates of  $\nu_m$  and  $\nu_{hml}$  are statistically different from zero for individual stocks. In particular, the estimate of  $\nu_{hml}$  is large and negative. This explains the negative estimate on the value premium for individual stocks displayed in Table 1 Panel A, despite the positive time average of the value factor. Phalippou (2007) obtains a similar growth premium for portfolios built on stocks with a high institutional ownership. The results with the 44 Indu. portfolios sharply differ from those with the 25 FF portfolios. The former are more alike the results for individual stocks, in particular they yield negative estimates of coefficient  $\nu_{hml}$  and value premium  $\lambda_{hml}$  (albeit the latter not statistically significant).

The size, value and momentum factors are tradable in theory. In practice, their implementation faces transaction costs due to rebalancing and short selling. A non zero  $\nu$  might capture these market imperfections (Cremers, Petajisto, and Zitzewitz (2012)). In Appendix 8 of the online supplementary materials, we also get zero estimates with the FF portfolios, and non zero estimates with the Indu. portfolios and the individual stocks, when we use a time-invariant model with long-only factors derived from the FF methodology. Market imperfections are probably not the key drivers here (see Frazzini, Israel and Moskowitz (2012) for empirical support based on live trading data from a large institutional money manager).

A potential explanation of the discrepancies revealed in Table 1 between individual stocks and the FF portfolios is the much larger heterogeneity of the factor loadings for the former. As already discussed in Lewellen, Nagel, and Shanken (2010), the FF portfolio betas are all concentrated in the middle of the cross-

sectional distribution obtained from the individual stocks. Creating portfolios with an ad hoc methodology distorts information by shrinking the dispersion of betas. The estimation results for the momentum factor on the FF portfolios exemplify the problems related to a small number of portfolios exhibiting a tight factor structure. In Appendix 14 of the online supplementary materials, we explore another potential explanation of the discrepancy revealed in Table 1 due to the effects of model misspecification on the risk premia.

### 4.3 Estimation results for time-varying specifications

Let us now consider the time-varying four-factor specification. We use  $\chi_{1,T} = 15$  and  $\chi_{2,T} = 546/60$ . The number of assets whose condition number is below 15 is 3,909. Figure 1 plots the estimated time-varying paths of the four risk premia from the individual stocks. For comparison purpose, we also plot the time-invariant estimates and the average lambdas over time. A well-known bias coming from market-timing and volatility-timing (Jagannathan and Wang (1996), Lewellen and Nagel (2006), Boguth et al. (2011)) explains the discrepancy between the time-invariant estimate and the average over time. After trimming, we compute the risk premia on  $n^\chi = 3,900$  individual assets in the four-factor model. The observed discrepancy w.r.t. the average over time is only marginally explained by the larger size of the stock universe used for the time-invariant estimates. The risk premia for the market, size and value factors feature a counter-cyclical pattern. Indeed, these risk premia increase during economic contractions and decrease during economic booms. Gomes, Kogan, and Zhang (2003) and Zhang (2005) construct equilibrium models exhibiting a counter-cyclical behavior in size and book-to-market effects. On the contrary, the risk premium for the momentum factor is pro-cyclical. Furthermore, time-varying estimates of the value premium are often negative and take positive values mostly in recessions. Growth firms are riskier in boom times because of their in-the-money growth options; value firms are riskier in recession times because of default risk. However, empirical evidence for such an interpretation is mixed. Some papers find that distress is related to size and book-to-market effects (Griffin and Lemmon (2002), Vassalou and Xing (2004)) while other papers find the opposite (Dichev (1998), Campbell, Hilscher, and Szilagyi (2008)). Chava and Purnanandam (2010) find support for a positive relation and argue that conclusions regarding the risk-return trade-off can change significantly depending on how the expected return is measured. Gomes and Schmid (2010) and Garlappi and Yan (2011) argue that financial leverage provides a rationale for a positive relation. The time-varying estimates of the

size premium are most of the time slightly positive.

Figure 2 plots the estimated time-varying path of the four risk premia from the 25 FF portfolios. We also plot the time-invariant estimates and the average lambdas over time. The discrepancy between the time-invariant estimates and the averages over time is also observed for  $n = 25$ . The time-varying point estimates for  $\lambda_{mom,t}$  are typically smaller than the time-invariant estimate in Table 1, but both estimates are rather inaccurate. Finally, by comparing Figures 1 and 2, we observe that the patterns of risk premia look similar except for the book-to-market factor. Indeed, the risk premium for the value effect estimated from the 25 portfolios is pro-cyclical, contradicting the counter-cyclical behavior predicted by finance theory. In Appendix 10 of the online supplementary materials, we display the paths of risk premia in the Fama-French model estimated from the 25 FF portfolios. They look similar to the corresponding estimates for the four-factor model in Figure 2. In Appendix 11, we provide the time-varying paths of risk premia for the 44 Indu. portfolios. They look similar to the corresponding estimates on individual stocks. This similarity, also observed in Section 4.2 with time-invariant models, is likely linked with the relative stability of the time-varying portfolio weights for the 44 Indu. portfolios compared to the weights of the 25 FF portfolios.

The time-varying risk premia involve both the conditional expectation of the factors, via the coefficients matrix  $F$ , and the process  $\nu_t$ , via the cross-sectional parameters vector  $\nu$ . To disentangle their effects on the risk-premia estimates, we report the estimates of the components of vector  $vec[F']$ , and of vector  $\nu$  for the three universes of assets, in Table 2. As expected, the conditional means are positive for all factors, when term spread and default spread are at their historical averages. The effect of term spread on the factor conditional mean is positive for all factors, while the effect of default spread is negative for the value and momentum factors. The regression coefficients for both instruments are not statistically significant at the 5% level, but their magnitude is economically important, especially for default spread. The estimates of  $\nu$  differ sharply between individual stocks and the 25 FF portfolios, especially for the value and momentum factors, and this explains the differences in average level and dynamics in the risk premia estimates observed in Figures 1 and 2. Let us focus on the value factor. The negative estimate  $-6.1642$  of the intercept coefficient for individual stocks compensates the corresponding coefficient  $4.7772$  in  $F$ , and yields a negative value premium when default spread and term spread are at their historical averages. Similarly, the positive estimate  $3.5981$  yields a value premium that depends positively on default spread and is counter-cyclical.



The estimates of  $\nu$  on the 25 FF portfolios are smaller than for stocks, and smaller than the estimates of  $F$ , which explains the mostly positive and pro-cyclical value premium observed in Figure 2. Instead, the estimates of  $\nu$  on the 44 Indu. portfolios are rather close to the estimates for individual stocks. To speed up the computation of the standard errors of  $\hat{\nu}$  for individual stocks, we use here the numerically equivalent estimated variance-covariance matrix  $\tilde{\Sigma}_\nu = \hat{Q}_{\beta_3}^{-1} \left( \frac{1}{n} \sum_{i,j} \frac{\tau_{i,T} \tau_{j,T}}{\tau_{ij,T}} \hat{\beta}'_{3,i} \hat{w}_i C'_{\hat{\nu}} \tilde{S}_{Q,ij} C_{\hat{\nu}} \hat{w}_j \hat{\beta}_{3,j} \right) \hat{Q}_{\beta_3}^{-1}$  instead of the expression of Section 3.4 where the Kronecker products of the large dimensional matrices slow down the computations. The coefficients of  $\nu$  corresponding to the effect of default spread are statistically significant at the 5% level for the size, value and momentum factors, while the coefficients of term spread are statistically significant for the market and momentum factors. The confidence intervals for the estimates of parameters  $\nu$  on the individual stocks are narrower than the confidence intervals for parameters  $F$ , reflecting the fast root- $nT$  convergence rate of the former parameters. The confidence intervals for  $\nu$  are also narrower on the individual stocks than on portfolios.

Even if Table 2 displays some statistically significant coefficients for the effects of instruments, it is useful to test formally whether risk premia indeed fluctuate over time. Time variation can pass through either the conditional expectation of the factors, or the cross-sectional parameter  $\nu$ . We distinguish accordingly the null hypotheses  $\mathcal{H}_0^F : \text{Avec}[F'] = 0$  and  $\mathcal{H}_0^\nu : A\nu = 0$ , where the matrix  $A = I_4 \otimes \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  selects the instruments coefficients for  $K = 4$ . For individual stocks we build standard asymptotic chi-square statistics with 8 degrees of freedom based on Propositions 4 and 5. The values of the test statistics are 11.8765 for the null hypothesis  $\mathcal{H}_0^F$  (p-value 0.157), and 414.44 for the null hypothesis  $\mathcal{H}_0^\nu$  (p-value 0.000). Hence, we cannot reject time-invariance of the factors conditional expectations, but we reject time-invariance of the risk premia due to the dynamics induced by the cross-sectional parameter  $\nu$ . We also reject the time-invariance of the risk premia for the 44 Indu. portfolios. The value of the test statistic for the null hypothesis  $\mathcal{H}_0^\nu$  is 26.2561 (p-value 0.001). Instead, the value of the test statistic for the null hypothesis  $\mathcal{H}_0^\nu$  is 1.5566 (p-value 0.992) when using the 25 FF portfolios. Hence, aggregation in the 25 portfolios completely masks the time variation of the risk premia. From the point of view of risk premia dynamics, a time-varying four-factor model estimated on the 25 FF portfolios is close to a time-invariant model.

The 95% confidence intervals for  $\hat{\nu}$  in Table 2 show that all entries are not significantly different from

zero when we look at the 25 FF portfolios except for the constant related to the momentum factor. However an asymptotic chi-square test of the joint hypothesis  $\nu = 0$  suggests the nullity of all entries (p-value 0.695). The hypothesis  $\nu = 0$  implies  $\nu_t = 0$  for all  $t$ , which holds true if the asset pricing restriction (3) is valid and the factors are returns of tradable portfolios. Similarly, we do not reject the null hypothesis  $\nu = 0$  for the time-varying Fama-French model estimated on the 25 FF portfolios (p-value 0.967). On the contrary, we strongly reject the null hypothesis  $\nu = 0$  when we substitute individual stocks (p-value 0.000 for both the Fama-French model and the four-factor model) or the 44 Indu. portfolios (p-value 0.000 for both the Fama-French model and the four-factor model) for the base assets.

As a visual check, we have computed the estimated path of  $\nu_t$  for the time-varying specifications in the supplementary materials. We get estimates close to zero only for the Fama-French model estimated on the FF portfolios (see Appendix 10). These estimates are consistent with a time-invariant model and tradable factors, as already revealed by the parametric test results above. For the four-factor model estimated on the 25 FF portfolios, we see that the path for the momentum factor is not centered around zero and is very imprecise (see Appendix 9). A first explanation might be the misspecification induced by the ad hoc portfolio aggregation based on size and value sorting and the time-varying specification for the momentum factor sensitivity (see also the theory in Appendix 14). A second explanation of the statistical inaccuracy might be the tight factor structure observed by Lewellen, Nagel, and Shanken (2010). The paths of  $\nu_t$  estimated from individual stocks move also a lot and are not centered on zero (see Figures in Appendix 9).

In Appendix 13 of the supplementary materials, we provide an empirical comparison between the estimated time-varying betas for the portfolios and the individual stocks. The individual stock betas and Indu. portfolio betas move over time substantially while the FF portfolio betas are more stable, albeit we find statistical evidence for their time variation. As an example, the median standard deviations range from 0.0311 to 0.1031 for the FF portfolios and from 0.3858 to 0.6275 for the individual stocks. We get a multiple of roughly 10 times uniformly across factors. The null hypothesis of time-invariant betas in the four factor model is not rejected at the 5% level for 2 out of the 25 FF portfolios, and for 6 out of the 44 Indu. portfolios. The method used in Barras, Scaillet and Wermers (2010) for multiple testing (see also Bajgrowicz and Scaillet (2012)) yields 11.67% as estimated proportion of individual stocks with time-invariant betas. This means that the time-invariant models for the individual stocks and Indu. portfolios in Section 4.2 are likely

misspecified because of the dynamics in betas and risk premia (see the theoretical arguments in Appendix 14 in the supplementary materials).

#### 4.4 Results on testing the asset pricing restrictions

As already discussed in Lewellen, Nagel, and Shanken (2010), the 25 FF portfolios have four-factor market and momentum betas close to one and zero, respectively. As depicted in Figure 1 by Lewellen, Nagel, and Shanken (2010), this empirical concentration implies that it is easy to get artificially large estimates  $\hat{\rho}^2$  of the cross-sectional  $R^2$  for three- and four-factor models. On the contrary, the observed heterogeneity in the betas coming from the individual stocks impedes this. This suggests that it is much less easy to find factors that explain the cross-sectional variation of expected excess returns on individual stocks than on portfolios. Reporting large  $\hat{\rho}^2$ , or small SSR  $\hat{Q}_e$ , when  $n$  is large, is much more impressive than when  $n$  is small.

Table 3 Panel A gathers the results for the tests of the asset pricing restrictions in factor models with time-invariant coefficients. When  $n$  is large, we prefer working with test statistics based on the SSR  $\hat{Q}_e$  instead of  $\hat{\rho}^2$  since the population  $R^2$  is not well-defined with tradable factors under the null hypothesis (its denominator is zero). For the individual stocks, we compute the test statistics  $\tilde{\Sigma}_\xi^{-1/2} \hat{\xi}_{nT}$  based on  $\hat{Q}_e$  and  $\hat{Q}_a$  as well as their associated one-side p-value. Our Monte Carlo simulations show that we need to set a stronger trimming level  $\chi_{2,T}$  to compute the test statistic than to estimate the risk premium. We use  $\chi_{2,T} = 546/240$ . For the 25 and 44 Indu. portfolios, we compute weighted test statistics (Gibbons, Ross, and Shanken (1989)) as well as their associated p-values. For individual stocks, the test statistics reject both null hypotheses  $\mathcal{H}_0 : a(\gamma) = b(\gamma)' \nu$  and  $\mathcal{H}_0 : a(\gamma) = 0$  for the three specifications at 5% level. Instead, the null hypothesis  $\mathcal{H}_0 : a(\gamma) = b(\gamma)' \nu$  is not rejected for the four-factor specification at 1% level. Similar conclusions are obtained when using the 25 FF portfolios as base assets. For the 44 Indu. portfolios, we do not reject the null hypothesis  $\mathcal{H}_0 : a(\gamma) = b(\gamma)' \nu$ , but we reject  $\mathcal{H}_0 : a(\gamma) = 0$ . Table 3 Panel B gathers the results for tests of the asset pricing restrictions in time-varying specifications. Contrary to the time-invariant case, we do not report the values of the weighted test statistics (Gibbons, Ross and Shanken (1989)) computed for portfolios because of the numerical instability in the inversion of the covariance matrix. The latter has dimension 1, 100  $\times$  1, 100 for the time-varying four-factor specification with the 44 Indu. portfolios. Instead, we report the values of the test statistics  $T\hat{Q}_e$  and  $T\hat{Q}_a$ . For individual

stocks, the test statistics reject both null hypotheses  $\mathcal{H}_0 : \beta_1(\gamma) = \beta_3(\gamma)\nu$  and  $\mathcal{H}_0 : \beta_1(\gamma) = 0$  for the three specifications at 5% level, but not for the conditional CAPM at 1% level. For the 25 FF portfolios, the two null hypotheses are not rejected under the conditional CAPM even at 5% level.

For individual stocks, the rejection of the asset pricing restriction using a time-varying multi-factor specification (at 1% level), and the non rejection under a time-invariant specification, might seem counter-intuitive. Indeed, for a given choice of the factors and instruments, the set of time-invariant specifications satisfying the no-arbitrage restriction  $a(\gamma) = b(\gamma)'\nu$ , is a strict subset of the collection of time-varying specifications with  $a_t(\gamma) = b_t(\gamma)'\nu_t$ . However, what we are testing here is whether the pseudo-true model, obtained by the projection of the DGP on a given time-varying or time-invariant specification, is compatible with no-arbitrage. The set of time-invariant models is included in the set of time-varying factor models, and it may well be the case that the projection of the DGP on the former set satisfies the no-arbitrage restrictions, while the projection on the latter does not. Therefore, the results in Table 3 for individual stocks are not incompatible with each other. A similar argument might explain why in Table 3 Panel B we fail to reject the asset pricing restriction  $\mathcal{H}_0 : \beta_1(\gamma) = \beta_3(\gamma)\nu$  under the conditional CAPM (at level 1% for individual assets, and 5% for portfolios), while this restriction is rejected under the three- and four-factor specifications.

For the time-varying Fama-French factor model estimated on the 25 FF portfolios, Table 3 Panel B shows that the time-varying intercept does not vanish. In the light of this observation, the finding of zero estimated  $\nu_t$  in Figure 8 of Appendix 10 implies that the betas do not explain the alphas cross-sectionally for the 25 FF portfolios, and not that these alphas are zero.

In Appendix 15 of the supplementary materials, we compare the cross-sectional distributions of  $\hat{\beta}'_{1,i}\hat{\beta}_{1,i}$ , the idiosyncratic risk (square root of residual variance), and the estimated time-series coefficient of determination  $\hat{\rho}_i^2$  (ratio of explained variance and total variance) for the time-invariant and time-varying four-factor model on the different asset universes. We can view those estimates as measures of limits-to-arbitrage and missing factor impact (Pontiff (2006), Lam and Wei (2011), Ang et al. (2009)). The cross-sectional average of the  $\hat{\beta}'_{1,i}\hat{w}_i\hat{\beta}_{1,i}$  is the basis for the test statistic  $T\hat{Q}_a$ . We relate these measures to characteristics such as size, book-to-market, and sample size. The 25 FF portfolios exhibit small  $\hat{\beta}'_{1,i}\hat{\beta}_{1,i}$ , small idiosyncratic risks, and large estimates  $\hat{\rho}_i^2$  compared to individual stocks as expected from the results of Sections 4.2 and 4.3. The heterogeneity is much larger for the Indu. portfolios and individual stocks. Our preliminary results

based on linear quantile regressions reveal that stocks with small size tend to yield large  $\hat{\beta}'_{1,i}\hat{\beta}_{1,i}$ , large idiosyncratic risks, and small estimated  $\hat{\rho}_i^2$ . We also find that firms with short observation periods tend to be associated with large values of both idiosyncratic and systematic risks (with a larger proportion of systematic risk to total risk), as well as small market capitalisation. Measuring and understanding limits-to-arbitrage and missing factor impact on individual stocks certainly await more work.

The analysis of the validity of the asset pricing restrictions could be completed by an analysis of correct specification of the different time-varying and time-invariant factor models. A specification test would assess whether the proposed set of linear factors captures the systematic risk component in equity returns, and whether the proposed parametric beta dynamics correctly reflects the time-variation of loadings. It clearly differs from the test of the no-arbitrage restrictions introduced above. Developing a test of correct specification of time-varying factor models with an unbalanced panel and double asymptotics is beyond the scope of the paper. We leave this interesting topic for future research.

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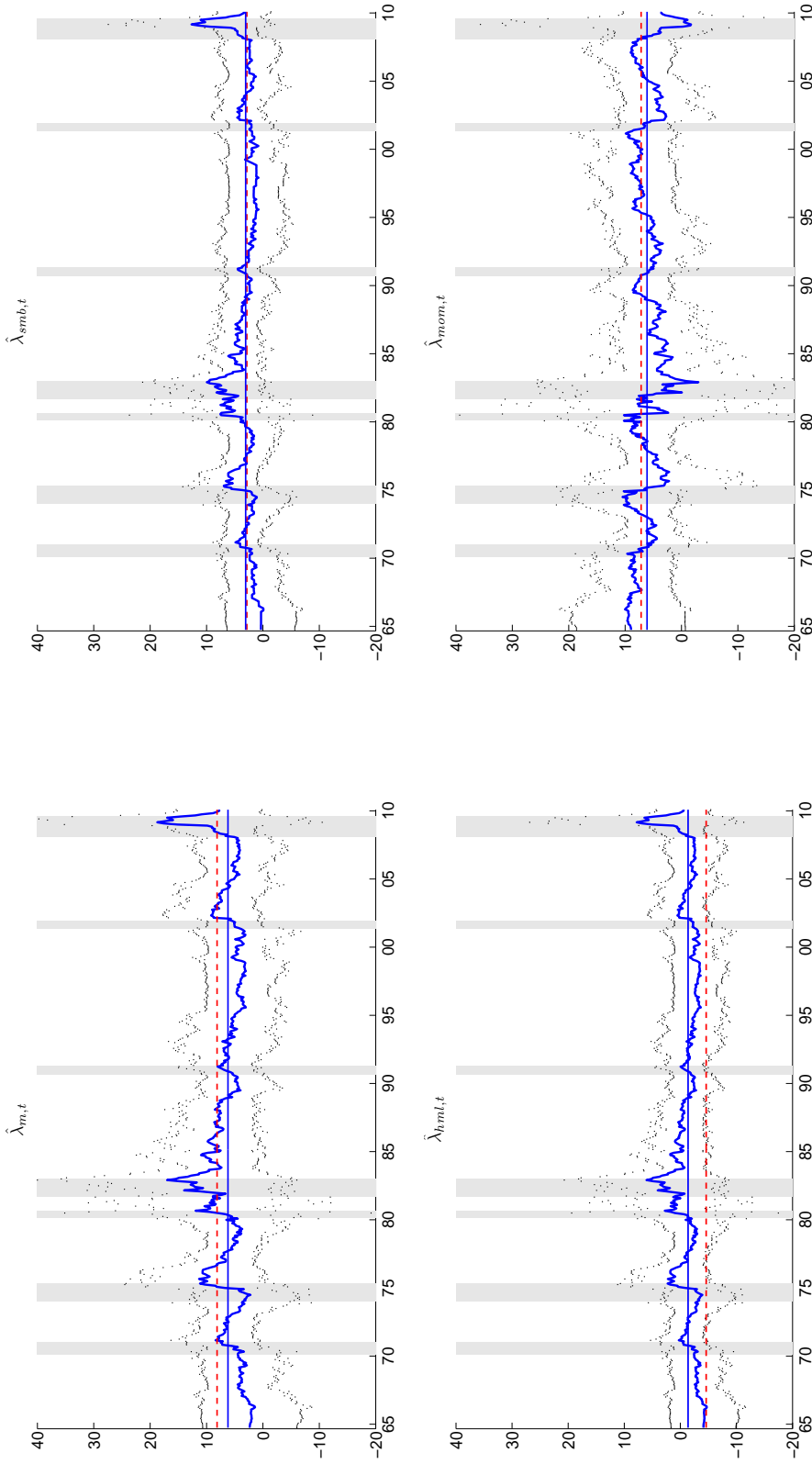
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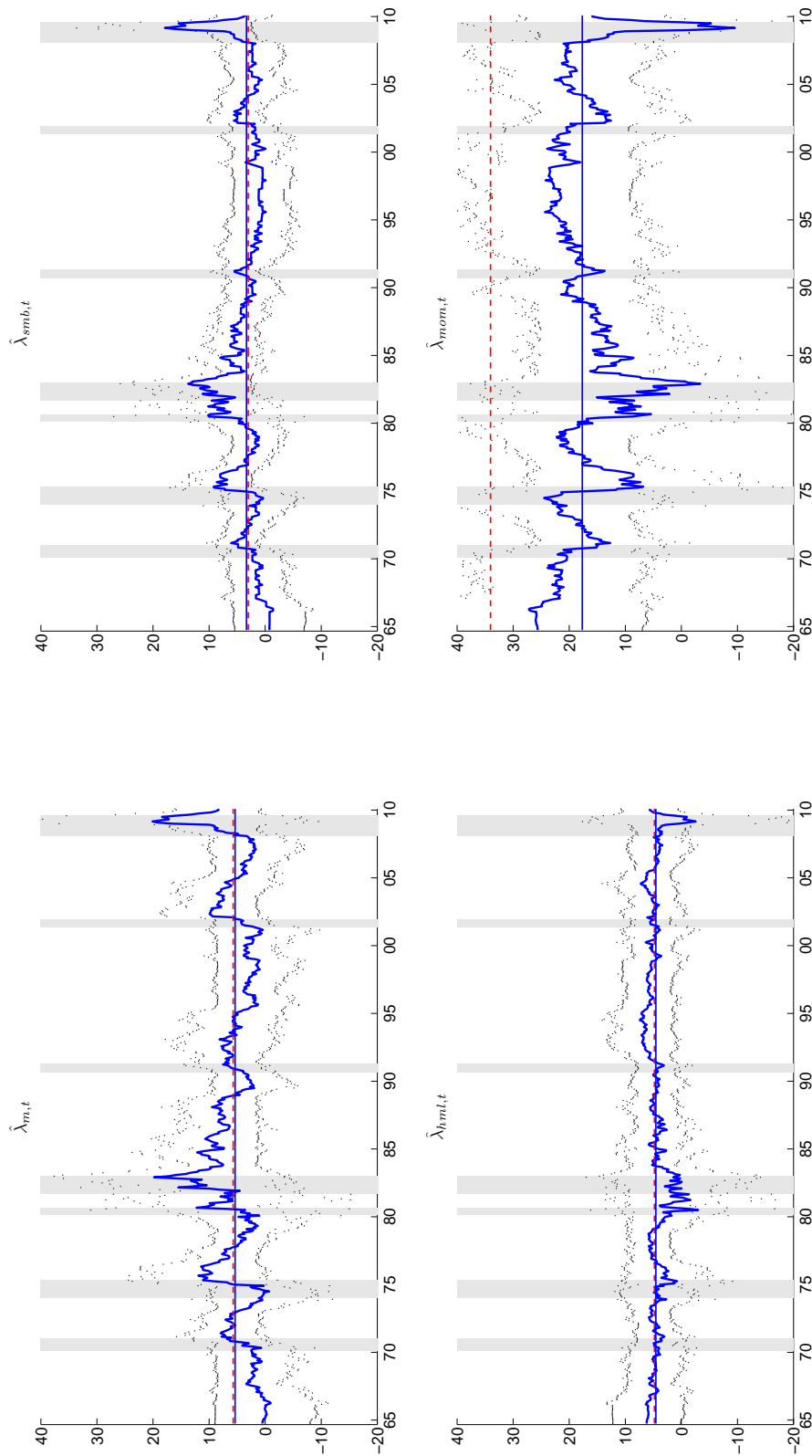
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**Figure 1: Path of estimated annualized risk premia with  $n = 9, 936$  in the four-factor model**



The figure plots the path of estimated annualized risk premia  $\hat{\lambda}_{m,t}$ ,  $\hat{\lambda}_{smb,t}$ ,  $\hat{\lambda}_{hml,t}$ , and  $\hat{\lambda}_{nom,t}$  and their pointwise confidence intervals at 95% probability level in the four-factor model. We also report the time-invariant (dashed horizontal line) and the average conditional estimate (solid horizontal line). The vertical shaded areas denote recessions determined by the National Bureau of Economic Research (NBER). The recessions start at the peak of a business cycle and end at the trough.

**Figure 2: Path of estimated annualized risk premia with  $n = 25$  in the four-factor model**



The figure plots the path of estimated annualized risk premia  $\hat{\lambda}_{m,t}$ ,  $\hat{\lambda}_{smb,t}$ ,  $\hat{\lambda}_{hml,t}$ , and  $\hat{\lambda}_{nom,t}$  and their pointwise confidence intervals at 95% probability level in the four-factor model. We use the returns of the 25 FF portfolios. We also report the time-invariant (dashed horizontal line) and the average conditional estimate (solid horizontal line). The vertical shaded areas denote recessions determined by the National Bureau of Economic Research (NBER).



**Table 1: Estimated annualized risk premia and vector  $\nu$  for the time-invariant models**

Panel A: risk premia $\lambda$		Panel B: vector $\nu$				
Stocks ( $n = 9, 936$ )	FF portfolios ( $n = 25$ )	Indu. portfolios ( $n = 44$ )	Stocks ( $n = 9, 936$ )	FF portfolios ( $n = 25$ )	Indu. portfolios ( $n = 44$ )	
Four-factor model						
$\lambda_m$	8.14 (3.26, 13.02)	5.70 (0.73, 10.67)	6.87 (1.86, 11.88)	3.29 (2.88, 3.69)	0.85 (-0.10, 1.79)	2.02 (0.90, 3.13)
$\lambda_{smb}$	2.86 (-0.50, 6.22)	3.02 (-0.48, 6.51)	-1.46 (-5.57, 2.84)	-0.41 (-0.95, 0.13)	-0.26 (-1.24, 0.72)	-4.72 (-7.40, -2.05)
$\lambda_{hml}$	-4.60 (-8.06, -1.14)	4.81 (1.21, 8.41)	-0.97 (-5.49, 3.57)	-9.38 (-10.12, -8.64)	0.03 (-0.95, 1.01)	-5.75 (-8.66, -2.84)
$\lambda_{mom}$	7.16 (2.56, 11.76)	34.03 (9.98, 58.07)	8.42 (-3.11, 19.96)	-1.47 (-2.86, -0.08)	25.40 (1.80, 49.00)	-0.20 (-10.78, 10.37)
Fama-French model						
$\lambda_m$	7.77 (2.89, 12.65)	5.04 (0.11, 9.97)	6.58 (1.60, 11.56)	2.92 (2.48, 3.35)	0.18 (-0.51, 0.87)	1.74 (0.73, 2.72)
$\lambda_{smb}$	2.64 (-0.72, 5.99)	3.00 (-0.42, 6.42)	-2.24 (-6.46, 1.98)	-0.63 (-1.11, -0.15)	-0.27 (-0.93, 0.40)	-5.51 (-8.07, -2.95)
$\lambda_{hml}$	-5.18 (-8.65, -1.72)	5.20 (1.66, 8.74)	-1.40 (-5.57, 2.95)	-9.96 (-10.62, -9.31)	0.41 (-0.32, 1.15)	-6.19 (-8.82, -3.56)
CAPM						
$\lambda_m$	7.42 (2.54, 12.31)	6.98 (1.93, 12.02)	5.95 (0.92, 10.99)	2.57 (2.17, 2.97)	2.12 (0.85, 3.40)	1.09 (-0.15, 2.35)

Panel A contains the estimated annualized risk premia, and their confidence intervals at 95% probability level, for the market ( $\lambda_m$ ), size ( $\lambda_{smb}$ ), book-to-market ( $\lambda_{hml}$ ) and momentum ( $\lambda_{mom}$ ) factors. Panel B contains the annualized estimates of the components of vector  $\nu$ , and their confidence intervals at 95% probability level, for the market ( $\nu_m$ ), size ( $\nu_{smb}$ ), book-to-market ( $\nu_{hml}$ ) and momentum ( $\nu_{mom}$ ) factors. We report the bias corrected estimates  $\hat{\lambda}_B$  and  $\hat{\nu}_B$  of  $\lambda$  and  $\nu$  for individual stocks ( $n = 9, 936$ ). The trimmed number of assets is  $n^\chi = 9, 902$  for the four-factor model and  $n^\chi = 9, 904$  for the FF model and CAPM. In order to build the confidence intervals for  $n = 9, 936$ , we use the HAC estimator  $\hat{\Sigma}_f$  defined in Section 3.4 and we compute  $\tilde{\Sigma}_\nu$  in Proposition 5. When we consider the 25 FF and 44 Indu. portfolios as base assets, we compute an estimate of the variance-covariance matrix  $\tilde{\Sigma}_{\lambda,n}$  and  $\tilde{\Sigma}_{\nu,n}$  defined in Section 3.3.

**Table 2: Estimated annualized components of  $vec[F']$  and  $\nu$  for the time-varying four-factor model**

		$vec[F']$	$\nu$ ( $n = 9,936$ )	$\nu$ ( $n = 25$ )	$\nu$ ( $n = 44$ )
<i>m</i>	<i>const</i>	4.8322 (0.2653, 9.3990)	1.3744 (0.7069, 2.0419)	0.5251 (-0.4713, 1.5216)	1.4225 (0.1000, 2.7449)
	<i>ds<sub>t-1</sub></i>	3.0353 (-2.6803, 8.7509)	-0.6032 (-1.2688, 0.0623)	-0.2916 (-1.1622, 0.5790)	0.6266 (-1.0651, 2.3182)
	<i>ts<sub>t-1</sub></i>	1.8677 (-2.8399, 6.5754)	-0.9254 (-1.5626, -0.2881)	0.0828 (-0.6666, 0.8323)	-0.9464 (-2.1876, 0.2948)
<i>smb</i>	<i>const</i>	3.2739 (0.0410, 6.5067)	-0.2130 (-0.8680, 0.4421)	0.0607 (-0.9808, 1.1122)	-4.6894 (-7.7028, -1.6760)
	<i>ds<sub>t-1</sub></i>	2.5468 (-0.5998, 5.6934)	-0.5948 (-1.1499, -0.0396)	0.4134 (-0.6139, 1.4407)	-1.3207 (-4.1131, 1.4717)
	<i>ts<sub>t-1</sub></i>	0.2855 (-2.6271, 3.1982)	-0.2157 (-0.7443, 0.3128)	-0.1966 (-0.9686, 0.5753)	-1.0377 (-3.3182, 1.2428)
<i>hml</i>	<i>const</i>	4.7772 (1.7905, 7.7639)	-6.1642 (-6.8543, -5.4741)	-0.2267 (-1.3144, 0.8611)	-3.5006 (-6.6784, -0.3229)
	<i>ds<sub>t-1</sub></i>	-1.7898 (-5.5963, 2.0167)	3.5981 (2.8995, 4.2967)	0.2187 (-1.0365, 1.4740)	4.2977 (1.6510, 6.9444)
	<i>ts<sub>t-1</sub></i>	0.8933 (-2.2598, 4.0465)	-0.4292 (-1.0043, 0.1458)	-0.0073 (-0.8766, 0.8620)	-1.3221 (-4.1853, 1.5411)
<i>mom</i>	<i>const</i>	8.6543 (-4.2482, 13.0605)	-2.5592 (-3.4153, -1.7031)	9.0179 (0.4294, 17.6064)	5.5584 (-0.6835, 11.8002)
	<i>ds<sub>t-1</sub></i>	-7.3714 (-14.6656, -0.0771)	6.0148 (5.1168, 6.9131)	1.9403 (-6.0003, 9.8808)	7.9407 (2.0813, 13.8001)
	<i>ts<sub>t-1</sub></i>	1.5804 (-2.8226, 5.9833)	-3.2960 (-4.0246, -2.5673)	-2.5080 (-9.9869, 4.9710)	-5.9833 (-10.8890, 0.4623)

The table contains the estimated annualized components of vector  $vec[F']$ , and of vector  $\nu$ , and their confidence intervals at 95% probability level, for the individual stocks ( $n = 9,936$  and  $n^x = 3,900$ ) and portfolios ( $n = 25, 44$ ). We report the bias corrected estimates  $\hat{\nu}_B$  of  $\nu$  for individual stocks. In order to build the confidence intervals for  $\nu$  for individual stocks, we use the thresholded variance-covariance matrix of Proposition 5. When we consider the 25 FF and 44 Indu. portfolios as base assets, we compute an estimate of the variance-covariance matrix  $\Sigma_{\nu,n}$  defined in Section 3.3. The default spread  $ds_{t-1}$  and the term spread  $ts_{t-1}$  are centered and standardized.

**Table 3: Test results for asset pricing restrictions**

	Stocks ( $n = 9, 936$ )	FF portfolios ( $n = 25$ )	Indu. portfolios ( $n = 44$ )	Stocks ( $n = 9, 936$ )	FF portfolios ( $n = 25$ )	Indu. portfolios ( $n = 44$ )
Panel A: Time-invariant models						
Test of the null hypothesis $\mathcal{H}_0 : a(\gamma) = b(\gamma)' \nu$						
Four-factor model						
Test statistic	$n^X = 1,400$	2.0088 (0.0223)	35.2231 (0.0267)	40.8566 (0.4327)	19.1803 (0.0000)	74.9100 (0.0000)
p-value						86.2702 (0.0001)
Fama-French model						
Test statistic	$n^X = 1,400$	2.9593 (0.0015)	83.6846 (0.0000)	43.5906 (0.3618)	28.0328 (0.0000)	87.3767 0.0000
p-value						102.0549 0.0000
CAPM						
Test statistic	$n^X = 1,400$	8.2576 (0.0000)	110.8368 (0.0000)	45.5153 (0.3678)	11.5882 (0.0000)	111.6735 (0.0000)
p-value						61.0319 (0.0453)
Panel B: Time-varying models						
Test of the null hypothesis $\mathcal{H}_0 : \beta_1(\gamma) = \beta_3(\gamma) \nu$						
Four-factor model						
Test statistic	$n^X = 1,373$	3.2514 (0.0000)	13.4815 (0.0000)	9.9586 (0.0000)	3.8683 (0.0000)	14.3080 (0.0000)
p-value						11.6767 (0.0000)
Fama-French model						
Test statistic	$n^X = 1,393$	3.1253 (0.0008)	15.7895 (0.0000)	10.8837 (0.0008)	3.8136 (0.0000)	15.9038 (0.0000)
p-value						12.0891 (0.0000)
CAPM						
Test statistic	$n^X = 1,395$	1.7322 (0.0416)	9.2934 (0.2076)	11.7524 (0.0046)	1.7381 (0.0411)	9.6680 (0.0000)
p-value						11.9020 (0.0000)

In panel A, we compute the statistics  $\tilde{\Sigma}_\xi^{-1/2} \hat{\xi}_{nT}$  based on  $\hat{Q}_e$  and  $\hat{Q}_a$  defined in Proposition 6 for the individual stocks to test the null hypotheses  $\mathcal{H}_0 : a(\gamma) = b(\gamma)' \nu$  and  $\mathcal{H}_0 : \beta_1(\gamma) = 0$ , respectively. In Panel B, we compute the statistics  $\tilde{\Sigma}_\xi^{-1/2} \hat{\xi}_{nT}$  to test the null hypotheses  $\mathcal{H}_0 : \beta_1(\gamma) = \beta_3(\gamma) \nu$  and  $\mathcal{H}_0 : \beta_1(\gamma) = 0$ . The trimming levels are  $\chi_{1,T} = 15$  and  $\chi_{2,T} = 546/240$ . Moreover, in Panel A, for  $n = 25$  and  $n = 44$ , we compute the weighted statistics  $T\hat{e}'\hat{V}^{-1}\hat{e}$  and  $T\hat{a}'\hat{V}_a^{-1}\hat{a}$  (Gibbons, Ross and Shanken (1989)), where  $\hat{e}$  and  $\hat{a}$  are  $n \times 1$  vectors with elements  $\hat{e}_i$  and  $\hat{a}_i$ , and  $\hat{V} = (\hat{v}_{ij})$  and  $\hat{V}_a = (\hat{v}_{a,ij})$  are  $n \times n$  matrices with elements  $\hat{v}_{ij} = \hat{e}'_i \hat{Q}_x^{-1} \hat{S}_{ij} \hat{Q}_x^{-1} \hat{e}_j$ , and  $\hat{v}_{a,ij} = \hat{e}'_i \hat{Q}_x^{-1} \hat{S}_{ij} \hat{Q}_x^{-1} \hat{e}_j$ . In Panel B, we compute the test statistics  $T\hat{Q}_e$  and  $T\hat{Q}_a$  for  $n = 25$  and  $n = 44$ . The table reports the p-values of the statistics in parentheses.

## Appendix 1 Regularity conditions

In this Appendix, we list and comment the additional assumptions used to derive the large sample properties of the estimators and test statistics. We use below the extended vector of common and firm-specific regressors  $x_t(\gamma) := (\text{vech}(X_t)', Z_{t-1}' \otimes Z_{t-1}(\gamma)', f_t' \otimes Z_{t-1}', f_t' \otimes Z_{t-1}(\gamma)')' \in \mathbb{R}^d$  (see Section 3.1).

**Assumption B.1** *There exist constants  $\eta, \bar{\eta} \in (0, 1]$  and  $C_1, C_2, C_3, C_4 > 0$  such that, for all  $\delta > 0$  and  $T \in \mathbb{N}$ , we have:*

$$a) \sup_{\gamma \in [0,1]} P \left[ \left\| \frac{1}{T} \sum_t (x_t(\gamma)x_t(\gamma)' - E[x_t(\gamma)x_t(\gamma)']) \right\| \geq \delta \right] \leq C_1 T \exp\{-C_2 \delta^2 T^\eta\} + C_3 \delta^{-1} \exp\{-C_4 T^{\bar{\eta}}\}.$$

*Furthermore, for all  $\delta > 0$ ,  $T \in \mathbb{N}$ , and  $1 \leq k, l, m \leq d$ , the same upper bound holds for:*

$$b) \sup_{\gamma \in [0,1]} P \left[ \left\| \frac{1}{T} \sum_t I_t(\gamma) (x_t(\gamma)x_t(\gamma)' - E[x_t(\gamma)x_t(\gamma)']) \right\| \geq \delta \right];$$

$$c) \sup_{\gamma \in [0,1]} P \left[ \left\| \frac{1}{T} \sum_t I_t(\gamma) x_t(\gamma) \varepsilon_t(\gamma) \right\| \geq \delta \right];$$

$$d) \sup_{\gamma, \tilde{\gamma} \in [0,1]} P \left[ \left| \frac{1}{T} \sum_t (I_t(\gamma)I_t(\tilde{\gamma}) - E[I_t(\gamma)I_t(\tilde{\gamma})]) \right| \geq \delta \right];$$

$$e) \sup_{\gamma, \tilde{\gamma} \in [0,1]} P \left[ \left\| \frac{1}{T} \sum_t I_t(\gamma)I_t(\tilde{\gamma}) (\varepsilon_t(\gamma)\varepsilon_t(\tilde{\gamma})x_t(\gamma)x_t(\tilde{\gamma})' - E[\varepsilon_t(\gamma)\varepsilon_t(\tilde{\gamma})x_t(\gamma)x_t(\tilde{\gamma})']) \right\| \geq \delta \right];$$

$$f) \sup_{\gamma, \tilde{\gamma} \in [0,1]} P \left[ \left| \frac{1}{T} \sum_t I_t(\gamma)I_t(\tilde{\gamma})x_{t,k}(\gamma)x_{t,l}(\gamma)x_{t,m}(\tilde{\gamma})\varepsilon_t(\gamma) \right| \geq \delta \right].$$

**Assumption B.2** *There exists a constant  $M > 0$  such that, for all  $T \in \mathbb{N}$ , we have:*

$$\sup_{\gamma \in [0,1]} E \left[ \frac{1}{T} \sum_{t_1, t_2, t_3} |\text{cov}(\varepsilon_{t_1}^2(\gamma), \varepsilon_{t_2}(\gamma)\varepsilon_{t_3}(\gamma)|\mathcal{F}_T)| \right] \leq M.$$

**Assumption B.3** *There exists a constant  $M > 0$  such that, for all  $n, T \in \mathbb{N}$ , we have for  $\eta_{i,t} = \varepsilon_{i,t}^2 - \sigma_{ii,t}$ :*

$$a) E \left[ \frac{1}{nT} \sum_{i,j} \sum_{t_1, t_2} E \left[ |\text{cov}(\varepsilon_{i,t_1}^2, \varepsilon_{j,t_2}^2 | \mathcal{F}_T, \gamma_i, \gamma_j)|^2 |\gamma_i, \gamma_j| \right]^{1/2} \right] \leq M.$$

$$b) E \left[ \frac{1}{nT^2} \sum_{i,j} \sum_{t_1, t_2, t_3, t_4} E \left[ |\text{cov}(\varepsilon_{i,t_1}\varepsilon_{i,t_2}, \varepsilon_{j,t_3}\varepsilon_{j,t_4} | \mathcal{F}_T, \gamma_i, \gamma_j)|^2 |\gamma_i, \gamma_j| \right]^{1/2} \right] \leq M;$$

$$c) E \left[ \frac{1}{nT^2} \sum_{i,j} \sum_{t_1, t_2, t_3, t_4} E \left[ |\text{cov}(\eta_{i,t_1}\varepsilon_{i,t_2}, \eta_{j,t_3}\varepsilon_{j,t_4} | \mathcal{F}_T, \gamma_i, \gamma_j)|^2 |\gamma_i, \gamma_j| \right]^{1/2} \right] \leq M;$$

$$d) E \left[ \frac{1}{nT^2} \sum_{i,j} \sum_{t_1, t_2, t_3, t_4} E \left[ |\text{cov}(\eta_{i,t_1}\eta_{i,t_2}, \eta_{j,t_3}\eta_{j,t_4} | \mathcal{F}_T, \gamma_i, \gamma_j)|^2 |\gamma_i, \gamma_j| \right]^{1/2} \right] \leq M;$$

$$\begin{aligned}
e) & E \left[ \frac{1}{nT^3} \sum_{i,j} \sum_{t_1, \dots, t_6} E \left[ |\text{cov}(\varepsilon_{i,t_1} \varepsilon_{i,t_2} \varepsilon_{i,t_3}, \varepsilon_{j,t_4} \varepsilon_{j,t_5} \varepsilon_{j,t_6} | \mathcal{F}_T, \gamma_i, \gamma_j)|^2 | \gamma_i, \gamma_j \right]^{1/2} \right] \leq M; \\
f) & E \left[ \frac{1}{nT^3} \sum_{i,j} \sum_{t_1, \dots, t_6} E \left[ |\text{cov}(\eta_{i,t_1} \varepsilon_{i,t_2} \varepsilon_{i,t_3}, \eta_{j,t_4} \varepsilon_{j,t_5} \varepsilon_{j,t_6} | \mathcal{F}_T, \gamma_i, \gamma_j)|^2 | \gamma_i, \gamma_j \right]^{1/2} \right] \leq M.
\end{aligned}$$

**Assumption B.4** a) *There exists a constant  $M > 0$  such that  $\sup_{\gamma \in [0,1]} \|x_t(\gamma)\| \leq M$ ,  $P$ -a.s.. Moreover,*  
b)  $\sup_{\gamma \in [0,1]} \|B(\gamma)\| < \infty$ ,  $\sup_{\gamma \in [0,1]} \|C(\gamma)\| < \infty$ , c)  $\inf_{\gamma \in [0,1]} E[I_t(\gamma)] > 0$ , and d)  $\inf_{\gamma \in [0,1]} \text{eig}_{\min}(E[x_t(\gamma)x_t(\gamma)']) > 0$ .

**Assumption B.5** *The trimming constants satisfy  $\chi_{1,T} = O((\log T)^{\kappa_1})$  and  $\chi_{2,T} = O((\log T)^{\kappa_2})$ , with  $\kappa_1, \kappa_2 > 0$ .*

**Assumption B.6** *There exists a constant  $M > 0$  such that  $\|E[u_t u_t' | Z_{t-1}]\| \leq M$  for all  $t$ , where  $u_t = f_t - E[f_t | \mathcal{F}_{t-1}]$ .*

Assumptions B.1 and B.2 restrict the serial dependence of the factors and the individual processes of observability indicators and error terms. Specifically, Assumption B.1 a) gives an upper bound for large-deviation probabilities of the sample average of random matrices  $x_t(\gamma)x_t(\gamma)'$ , uniformly w.r.t.  $\gamma \in [0, 1]$ . It implies that the sample moment of squared components of the regressor vector converge in probability to the corresponding population moments at a rate  $O_p(T^{-\eta/2}(\log T)^c)$ , for some  $c > 0$ . Assumptions B.1 b)-f) give similar upper bounds for large-deviation probabilities of sample averages of processes involving regressors, observability indicators and error terms, uniformly w.r.t.  $\gamma \in [0, 1]$ . We use these assumptions to prove the convergence of time series averages uniformly across assets. Assumption B.2 involves conditional covariances of products of error terms. Assumptions B.1 and B.2 are satisfied e.g. when the individual processes of the regressors, observability indicators and error terms feature mixing serial dependence, with mixing coefficients uniformly bounded w.r.t.  $\gamma \in [0, 1]$  (see e.g. Bosq (1998), Theorems 1.3 and 1.4). Assumptions B.3 a)-f) restrict both serial and cross-sectional dependence of the error terms. They involve conditional covariances between products of error terms  $\varepsilon_{i,t}$  and innovations  $\eta_{i,t} = \varepsilon_{i,t}^2 - \sigma_{ii,t}$  for different assets and dates. These assumptions can be satisfied under weak serial and cross-sectional dependence of the errors, such as temporal mixing and block dependence across assets. Assumptions B.4 a) and b) require uniform upper bounds on regressor values, model coefficients and intercepts. Assumption B.4 c) implies

that asymptotically the fraction of the time period in which an asset return is observed is bounded away from zero uniformly across assets. Assumption B.4 d) excludes asymptotically multicollinearity problems in the first-pass regression uniformly across assets. Assumptions B.4 a)-d) ease the proofs. Assumption B.5 gives an upper bound on the divergence rate of the trimming constants. The slow logarithmic divergence rate allows to control the first-pass estimation error in the second-pass regression. Assumption B.6 requires a bounded conditional variance-covariance matrix for the linear innovation  $u_t$  associated with the factor process. We use this assumption to prove that we can consistently estimate matrix  $F$  of the coefficients of the linear projection of factor  $f_t$  on variables  $Z_{t-1}$  by a SUR regression.

## Appendix 2 Proofs

### A.2.1 Proof of Proposition 1

To ease notations, we assume w.l.o.g. that the distribution  $G$  is uniform on  $[0, 1]$ . We can work at  $t = 1$  because of stationarity, and use that  $a(\gamma)$ ,  $b(\gamma)$ , for  $\gamma \in [0, 1]$ , are  $\mathcal{F}_0$ -measurable from Assumption APR.1. For a given countable collection of assets  $(\gamma_i)$  in  $\Gamma$ , we get from Assumption APR.1 that the mean vector and the variance-covariance matrix of asset excess returns  $(R_1(\gamma_1), \dots, R_1(\gamma_n))'$  conditional on  $\mathcal{F}_0$  are equal to  $\mu_n = A_n + B_n E[f_1 | \mathcal{F}_0]$  and  $\Sigma_n = B_n V[f_1 | \mathcal{F}_0] B_n' + \Sigma_{\varepsilon, 1, n}$ , for  $n \in \mathbb{N}$ , where  $A_n = [a(\gamma_1), \dots, a(\gamma_n)]'$ , and  $B_n = [b(\gamma_1), \dots, b(\gamma_n)]'$ . Let  $e_n$  be the residual of the orthogonal projection of the mean vector of asset excess returns conditional on  $\mathcal{F}_0$  onto the columns of  $B_n$ . Under Assumption APR.1, we have that  $e_n = \mu_n - B_n (B_n' B_n)^{-1} B_n' \mu_n = A_n - B_n (B_n' B_n)^{-1} B_n' A_n$ , i.e.,  $e_n$  is also equal to the residual of the orthogonal projection of  $A_n$  onto the columns of  $B_n$ . Let  $Q_n(\nu) = \frac{1}{n} \sum_i [a(\gamma_i) - b(\gamma_i)' \nu]^2$ , and let  $\nu_n = \arg \inf_{\nu \in \mathbb{R}^K} Q_n(\nu) = (B_n' B_n)^{-1} B_n' A_n$  be the associated vector of projection coefficients, so that  $Q_n(\nu_n) = \frac{1}{n} e_n' e_n$  under Assumption APR.1. Furthermore, let  $\mathcal{P}_n$  denote the set of portfolios  $p_n$  that invest in the risk-free asset and risky assets  $\gamma_1, \dots, \gamma_n$ , for  $n \in \mathbb{N}$ , with portfolio shares measurable w.r.t.  $\mathcal{F}_0$ . For portfolio  $p_n \in \mathcal{P}_n$ , the cost, the conditional expected return, and the conditional variance are given by  $C(p_n) = \alpha_{0, n} + \alpha_n' \nu_n$ ,  $E[p_n | \mathcal{F}_0] = R_0 C(p_n) + \alpha_n' \mu_n$ , and  $V[p_n | \mathcal{F}_0] = \alpha_n' \Sigma_n \alpha_n$ , where  $\nu_n = (1, \dots, 1)'$  and  $\alpha_n = (\alpha_{1, n}, \dots, \alpha_{n, n})'$ . For expository purpose, we do not make explicit the dependence of  $\mu_n$ ,  $\Sigma_n$ ,  $e_n$ , and  $\mathcal{P}_n$  on the collection of assets  $(\gamma_i)$ .

The statement of Proposition 1 is proved by contradiction. Suppose that there exists a set  $A \in \mathcal{F}_0$  with  $P(A) > 0$ , and a constant  $\underline{\delta} > 0$ , such that

$$\delta(\omega) \geq \underline{\delta} \quad (16)$$

for any  $\omega \in A$ , where  $\delta(\omega) := \inf_{\nu \in \mathbb{R}^K} Q_\infty(\omega, \nu) = Q_\infty(\omega, \nu_\infty(\omega))$ , with  $Q_\infty(\omega, \nu) = \int [a(\gamma, \omega) - b(\gamma, \omega)' \nu]^2 d\gamma$  and  $\nu_\infty = \left( \int b(\gamma) b(\gamma)' d\gamma \right)^{-1} \int b(\gamma) a(\gamma) d\gamma$ . From Assumption APR.2 (iii), we have  $\|\nu_\infty\| \leq \rho$ ,  $P$ -a.s., for the constant  $\rho = M^2$ . The next lemma, which is proved at the end of this section, gives a law of large numbers for the cross-products of the intercept and the factor loadings.

**Lemma 1** *Under Assumptions APR.2 (i)-(ii), there exists a set  $\mathcal{J}_1 \subset \Gamma$ , with measure  $\mu_\Gamma(\mathcal{J}_1) = 1$ , such that:*

$$\left\| \frac{1}{n} \sum_i \beta(\gamma_i) \beta(\gamma_i)' - \int \beta(\gamma) \beta(\gamma)' d\gamma \right\| \xrightarrow{L^2} 0, \quad (17)$$

as  $n \rightarrow \infty$ , for any sequence  $(\gamma_i)$  in  $\mathcal{J}_1$ , where  $\xrightarrow{L^2}$  denotes convergence in the  $L^2(\Omega, \mathcal{F}_0, P)$  norm.

Moreover, from Assumption APR.3, there exists a set  $\mathcal{J} \subset \Gamma$  with measure  $\mu_\Gamma(\mathcal{J}) = 1$ , such that  $n^{-1} \text{eig}_{\max}(\Sigma_{\varepsilon, 1, n}) \xrightarrow{L^2} 0$ , for any sequence  $(\gamma_i)$  in  $\mathcal{J}$ . Let us now show that an asymptotic arbitrage portfolio exists based on any sequence  $(\gamma_i)$  in  $\mathcal{J} \cap \mathcal{J}_1$ . Consider any given such sequence  $(\gamma_i)$  and a constant  $\epsilon > 0$ . Define the sequence of sets  $S_n := \left\{ \left\| \frac{1}{n} \sum_i \beta(\gamma_i) \beta(\gamma_i)' - \int \beta(\gamma) \beta(\gamma)' d\gamma \right\| \leq \epsilon \right\}$ . Equation (17) and the Chebyshev inequality imply  $P(S_n) \rightarrow 1$  as  $n \rightarrow \infty$ . Moreover, for  $\epsilon > 0$  small enough, from Assumption APR.2 (iii) we have  $\|\nu_n - \nu_\infty\| \leq \rho$ , for all  $\omega \in S_n$ . Hence, we get  $\|\nu_n\| \leq 2\rho$ , for all  $\omega \in S_n$ . From (17), we get:

$$\begin{aligned} \left| \frac{1}{n} \|e_n\|^2 - \delta \right| 1_{S_n} &= \left| \inf_{\nu \in \mathbb{R}^K} Q_n(\nu) - \inf_{\nu \in \mathbb{R}^K} Q_\infty(\nu) \right| 1_{S_n} = \left| \inf_{\nu \in \mathbb{R}^K: \|\nu\| \leq 2\rho} Q_n(\nu) - \inf_{\nu \in \mathbb{R}^K: \|\nu\| \leq 2\rho} Q_\infty(\nu) \right| 1_{S_n} \\ &\leq \sup_{\nu \in \mathbb{R}^K: \|\nu\| \leq 2\rho} |Q_n(\nu) - Q_\infty(\nu)| \\ &\leq (1 + 4\rho^2) \left\| \frac{1}{n} \sum_i \beta(\gamma_i) \beta(\gamma_i)' - \int \beta(\gamma) \beta(\gamma)' d\gamma \right\| \xrightarrow{L^2} 0. \end{aligned} \quad (18)$$

This implies that the set  $S_n^* := \left\{ \left| \frac{1}{n} \|e_n\|^2 - \delta \right| \leq \underline{\delta}/2 \right\} \cap S_n$  is such that  $P(S_n^*) \rightarrow 1$  as  $n \rightarrow \infty$ , and from Inequality (16), we get:

$$\frac{1}{n} \|e_n\|^2 1_{A \cap S_n^*} \geq 2^{-1} \underline{\delta} 1_{A \cap S_n^*}. \quad (19)$$

Define the portfolio sequence  $(q_n)$  as follows. The portfolio is based on sequence  $(\gamma_i)$  with investments  $\alpha_n = \frac{1}{\|e_n\|^2} e_n$  and  $\alpha_{0,n} = -\nu'_n \alpha_n$ , if  $\omega \in A \cap S_n^*$ , and is the zero-investment portfolio, if  $\omega \in (A \cap S_n^*)^c$ . This portfolio has a cost exactly zero, i.e.,  $C(q_n) = 0$ , as in the definition of an asymptotic arbitrage opportunity in APR.4. Let us now consider its conditional mean and variance. If  $\omega \in A \cap S_n^*$ , we have  $E[q_n | \mathcal{F}_0] = 1$  and  $V[q_n | \mathcal{F}_0] \leq \text{eig}_{\max}(\Sigma_{\varepsilon,1,n}) \|e_n\|^{-2}$  from the choice of  $\alpha_n, \alpha_{0,n}$ . Moreover,  $V[q_n | \mathcal{F}_0] = E[(q_n - E[q_n | \mathcal{F}_0])^2 | \mathcal{F}_0] \geq E[(q_n - E[q_n | \mathcal{F}_0])^2 | \mathcal{F}_0, q_n \leq 0] P[q_n \leq 0 | \mathcal{F}_0] \geq P[q_n \leq 0 | \mathcal{F}_0]$ . Hence, we get:  $P[q_n > 0 | \mathcal{F}_0] \geq 1 - V[q_n | \mathcal{F}_0] \geq 1 - \text{eig}_{\max}(\Sigma_{\varepsilon,1,n}) \|e_n\|^{-2}$ , if  $\omega \in A \cap S_n^*$ . Thus:

$$\begin{aligned} 1_A &\geq P[q_n > 0 | \mathcal{F}_0] 1_A \geq P[q_n > 0 | \mathcal{F}_0] 1_{A \cap S_n^*} \geq 1_{A \cap S_n^*} - \text{eig}_{\max}(\Sigma_{\varepsilon,1,n}) \|e_n\|^{-2} 1_{A \cap S_n^*} \\ &\geq 1_A - (1_A - 1_{A \cap S_n^*}) - n^{-1} \text{eig}_{\max}(\Sigma_{\varepsilon,1,n}) 2\underline{\delta}^{-1}, \end{aligned}$$

from Inequality (19), and then  $|P[q_n > 0 | \mathcal{F}_0] 1_A - 1_A| \leq 1_A - 1_{A \cap S_n^*} + n^{-1} \text{eig}_{\max}(\Sigma_{\varepsilon,1,n}) 2\underline{\delta}^{-1}$ . By using  $n^{-1} \text{eig}_{\max}(\Sigma_{\varepsilon,1,n}) \xrightarrow{L^2} 0$ , from Assumption APR.3, and  $P(S_n^*) \rightarrow 1$ , we get

$$P[q_n > 0 | \mathcal{F}_0] 1_A \xrightarrow{L^2} 1_A. \quad (20)$$

Then,  $E[P(q_n > 0 | \mathcal{F}_0) 1_A] \rightarrow P(A)$ . Since  $P[q_n > 0] = E[P(q_n > 0 | \mathcal{F}_0)] \geq E[P(q_n > 0 | \mathcal{F}_0) 1_A]$ , this implies  $\liminf_{n \rightarrow \infty} P[q_n > 0] \geq P(A) > 0$ . Moreover, since  $P[q_n \geq 0 | \mathcal{F}_0] 1_{A^c} = 1_{A^c}$ , and  $1_{A^c} = 1 - 1_A$ , we get  $P[q_n \geq 0 | \mathcal{F}_0] \xrightarrow{L^2} 1$  from (20). Thus,  $P[q_n \geq 0] \rightarrow 1$ . Hence, portfolio  $(q_n)$  is an asymptotic arbitrage opportunity. Since asymptotic arbitrage portfolios are ruled out by Assumption APR.4, it follows that Inequality (16) is incorrect, and we must have  $\int [a(\gamma) - b(\gamma)' \nu_\infty]^2 d\gamma < \underline{\delta}$ ,  $P$ -a.s., for any  $\underline{\delta} > 0$ , that is,  $a(\gamma) = b(\gamma)' \nu$ , with  $\nu = \nu_\infty$ , for almost all  $\gamma \in [0, 1]$ ,  $P$ -a.s. Such random vector  $\nu$  is unique by Assumption APR.2 (iii) and is  $\mathcal{F}_0$ -measurable. Then, Proposition 1 follows.

**Proof of Lemma 1:** We have seven steps to show that  $\frac{1}{n} \sum_i \beta_{k,l}^{(2)}(\gamma_i) \xrightarrow{L^2} \int \beta_{k,l}^{(2)}(\gamma) d\gamma$  as  $n \rightarrow \infty$ , where  $\beta_{k,l}^{(2)} := \beta_k \beta_l$ , for  $\mu_\Gamma$ -a.e. sequence  $(\gamma_i) \in \Gamma$  and all  $k, l = 1, \dots, K+1$ .

Step 1: For any  $k = 1, \dots, K+1$ , we show that  $\beta_k$  is a  $L^4$ -bounded process, i.e.,  $\|\beta_k(\gamma)\|_4 \leq M$ , for any  $\gamma \in [0, 1]$  and a constant  $M > 0$ . Consider the sequence of partial sums  $\beta_{k,N}(\gamma) = \sum_{j=1}^N c_{k,j}(\gamma) \eta_j$ , for any  $\gamma \in [0, 1]$ , where  $c_{k,j}$  denotes the  $k$ -th component of function  $c_j$ . From Assumption APR.2 (ii), this sequence converges to  $\beta_k(\gamma)$  in norm  $\|\cdot\|_2$  as  $N \rightarrow \infty$ . From the same assumption, the sequence



$(c_{k,j}(\gamma)\eta_j)_{j \in \mathbb{N}}$  in  $L^4(\Omega, \mathcal{F}_0, P)$  is absolutely summable. Since the space  $L^4(\Omega, \mathcal{F}_0, P)$  is complete, from Theorem III.3 in Reed and Simon (1980), sequence  $\beta_{k,N}(\gamma)$  converges in  $L^4(\Omega, \mathcal{F}_0, P)$  as  $N \rightarrow \infty$ . Since convergence in norm  $\|\cdot\|_4$  implies convergence in norm  $\|\cdot\|_2$ , and the  $L^2$  limit is unique, it follows that  $\beta_k(\gamma) \in L^4(\Omega, \mathcal{F}_0, P)$ , for any  $\gamma \in [0, 1]$ . The convergence of  $\beta_{k,N}(\gamma)$  to  $\beta_k(\gamma)$  in  $L^4(\Omega, \mathcal{F}_0, P)$  implies the convergence of the norm:  $\|\beta_{k,N}(\gamma)\|_4 \rightarrow \|\beta_k(\gamma)\|_4$  as  $N \rightarrow \infty$ . From the triangular inequality and Assumption APR.2 (ii), we have  $\|\beta_{k,N}(\gamma)\|_4 \leq \sum_{j=1}^N |c_{k,j}(\gamma)| \|\eta_j\|_4 \leq \sum_{j=1}^{\infty} \|c_j(\gamma)\| \|\eta_j\|_4 \leq M$ , for any  $\gamma$  and  $N$ . Since the inequality is maintained in the limit  $N \rightarrow \infty$ , we get  $\|\beta_k(\gamma)\|_4 \leq M$  for any  $\gamma$ .

Step 2: For any  $k, l = 1, \dots, K + 1$ , function  $\beta_{k,l}^{(2)}$  is weakly measurable, i.e., for any  $h \in L^2(\Omega, \mathcal{F}_0, P)$  the mapping  $\gamma \rightarrow H(\gamma) := E[\beta_{k,l}^{(2)}(\gamma)h]$  is measurable. Let  $\beta_{k,l,N}^{(2)} := \beta_{k,N}\beta_{l,N}$ . We have:

$$\begin{aligned} \|\beta_{k,l,N}^{(2)}(\gamma) - \beta_{k,l}^{(2)}(\gamma)\|_2 &= \|(\beta_{k,N}(\gamma) - \beta_k(\gamma))\beta_{l,N}(\gamma)\|_2 + \|\beta_k(\gamma)(\beta_{l,N}(\gamma) - \beta_l(\gamma))\|_2 \\ &\leq \|\beta_{k,N}(\gamma) - \beta_k(\gamma)\|_4 \|\beta_{l,N}(\gamma)\|_4 + \|\beta_k(\gamma)\|_4 \|\beta_{l,N}(\gamma) - \beta_l(\gamma)\|_4 \\ &\leq M (\|\beta_{k,N}(\gamma) - \beta_k(\gamma)\|_4 + \|\beta_{l,N}(\gamma) - \beta_l(\gamma)\|_4) \rightarrow 0, \end{aligned}$$

as  $N \rightarrow \infty$ , for any  $\gamma$ . Therefore,  $\beta_{k,l,N}^{(2)}(\gamma)$  converges to  $\beta_{k,l}^{(2)}(\gamma)$  in the  $L^2$  norm, for any  $\gamma$ . It follows that function  $H_N(\gamma) := E[\beta_{k,l,N}^{(2)}(\gamma)h]$  converges to  $H(\gamma)$  pointwise in  $\gamma$  as  $N \rightarrow \infty$ . Since  $H_N$  is a measurable function, and the pointwise limit of a sequence of measurable functions is measurable (see Halmos (1950), Theorem 20.A), function  $H$  is measurable.

Step 3: Function  $\beta_{k,l}^{(2)}$  has an essentially separable range. Define the set  $\mathcal{A} = \text{span}\{\eta_j\eta_k : j, k = 1, 2, \dots\}$ . Since  $\eta_j \in L^4(\Omega, \mathcal{F}_0, P)$ , we get that  $\mathcal{A}$  is a subset of  $L^2(\Omega, \mathcal{F}_0, P)$ . Random variable  $\beta_{k,l,N}^{(2)}(\gamma)$  belongs to set  $\mathcal{A}$ , for any  $N$  and  $\gamma$ . Since  $\beta_{k,l,N}^{(2)}(\gamma)$  converges to  $\beta_{k,l}^{(2)}(\gamma)$  in the  $L^2$  norm (see step 2),  $\beta_{k,l}^{(2)}(\gamma)$  is in the  $L^2$ -closure  $\bar{\mathcal{A}}$ , for any  $\gamma$ . Thus,  $\bar{\mathcal{A}}$  is the essential range of  $\beta_{k,l}^{(2)}$ . Moreover, by orthonormalizing the functions  $\{\eta_j\eta_k : j, k = 1, 2, \dots\}$ , we get a countable orthonormal basis for  $\bar{\mathcal{A}}$ . Thus,  $\bar{\mathcal{A}}$  is separable.

Step 4: By using the Pettis measurability theorem, from steps 2 and 3 it follows that process  $\beta_{k,l}^{(2)}$  is measurable (see e.g. Al-Najjar (1995), Section A.2, p. 1219, Khan and Sun (1999), p. 555).

Step 5: Since function  $\beta_{k,l}^{(2)}$  is measurable (step 4) and  $L^2$  bounded (from step 1 and the Cauchy-Schwarz inequality), it follows that  $\beta_{k,l}^{(2)}$  is Bochner integrable (see e.g. Al-Najjar (1995), Section A.3, p. 1219, Khan and Sun (1999), p. 553).

Step 6: By using Theorem 2.4 and remarks on pages 301 and 304 in Hoffman-Jorgensen (1985) (see

also Al-Najjar (1995), p. 1221), the Bochner integrability property of  $\beta_{k,l}^{(2)}$  implies:

$$\frac{1}{n} \sum_i \beta_{k,l}^{(2)}(\gamma_i) \xrightarrow{L^2} B - \int \beta_{k,l}^{(2)}(\gamma) d\gamma, \quad \text{as } n \rightarrow \infty, \text{ for } \mu_\Gamma\text{-a.e. } (\gamma_i).$$

Step 7: The conclusion follows since, from the measurability condition in Assumption APR.2 (i), the Bochner integral  $B - \int \beta_{k,l}^{(2)}(\gamma) d\gamma$  is equal to the  $\omega$ -wise Lebesgue integral  $\int \beta_{k,l}^{(2)}(\gamma) d\gamma$ ,  $P$ -a.s. Let us prove this statement. From the definition of Bochner integral, let  $(s_j)$  be a sequence of simple functions  $s_j : [0, 1] \rightarrow L^2(\Omega, \mathcal{F}_0, P)$  with

$$\int \|\beta_{k,l}^{(2)}(\gamma) - s_j(\gamma)\|_2 d\gamma \leq 2^{-j}, \quad (21)$$

for all integers  $j$ . Then:

$$B - \int \beta_{k,l}^{(2)}(\gamma) d\gamma = \lim_{j \rightarrow \infty} \int s_j(\gamma) d\gamma, \quad (22)$$

where the limit is w.r.t. the  $L^2(\Omega, \mathcal{F}_0, P)$  norm, and the integral of simple functions is defined  $\omega$ -wise. Let us now relate the r.h.s. with the Lebesgue integral of  $\beta_{k,l}^{(2)}$ . From Inequality (21), we have:

$$\begin{aligned} E \left[ \int \sum_{j=1}^{\infty} |\beta_{k,l}^{(2)}(\gamma) - s_j(\gamma)| d\gamma \right] &= \sum_{j=1}^{\infty} \int E \left[ |\beta_{k,l}^{(2)}(\gamma) - s_j(\gamma)| \right] d\gamma \\ &\leq \sum_{j=1}^{\infty} \int \|\beta_{k,l}^{(2)}(\gamma) - s_j(\gamma)\|_2 d\gamma \leq \sum_{j=1}^{\infty} 2^{-j} = 1 < \infty, \end{aligned}$$

where we use the Fubini and Lebesgue dominated convergence theorems to interchange the integral over  $\gamma$ , the expectation, and the summation over  $j$ . In particular, it follows:

$$\int \sum_{j=1}^{\infty} |\beta_{k,l}^{(2)}(\gamma) - s_j(\gamma)| d\gamma < \infty, \quad (23)$$

for all  $\omega \in \mathcal{N}$  and a set  $\mathcal{N} \subset \Omega$  with  $P(\mathcal{N}) = 1$ . Thus, we have  $|\beta_{k,l}^{(2)}(\gamma) - s_j(\gamma)| \rightarrow 0$  for  $j \rightarrow \infty$ , for a.e.  $\gamma \in [0, 1]$ , for all  $\omega \in \mathcal{N}$ . Moreover,  $|\beta_{k,l}^{(2)}(\gamma) - s_j(\gamma)|$  is dominated by the sum  $\sum_{j=1}^{\infty} |\beta_{k,l}^{(2)}(\gamma) - s_j(\gamma)|$ , which is integrable, for all  $\omega \in \mathcal{N}$ , by Inequality (23). Then, from the Lebesgue dominated convergence theorem, we get:

$$\int \beta_{k,l}^{(2)}(\gamma) d\gamma = \lim_{j \rightarrow \infty} \int s_j(\gamma) d\gamma, \quad (24)$$

where convergence is pointwise for all  $\omega \in \mathcal{N}$ . Equations (22) and (24) imply that  $\int s_j(\gamma) d\gamma$  converges in probability to  $B - \int \beta_{k,l}^{(2)}(\gamma) d\gamma$  and to  $\int \beta_{k,l}^{(2)}(\gamma) d\gamma$ , and therefore the two integrals coincide  $P$ -a.s.

## A.2.2 Proof of Proposition 2

The conditional covariance function of the error terms in the repackaged economy is given by:

$$E[\tilde{\varepsilon}(\gamma_1)\tilde{\varepsilon}(\gamma_2)|\mathcal{F}_0] = \sum_k \sum_l w_k [G_k^{-1}(\gamma_1)] w_l [G_l^{-1}(\gamma_2)] E[\varepsilon(G_k^{-1}(\gamma_1))\varepsilon(G_l^{-1}(\gamma_2))|\mathcal{F}_0]. \quad (25)$$

Let us show that the repackaged economy satisfies the condition of approximate factor structure in Assumption APR.3, namely, there exists a set  $\tilde{\mathcal{J}} \subset \left\{ (\gamma_i) \in \Gamma : n^{-1} \text{eig}_{max}(\Sigma_{\tilde{\varepsilon},1,n}) \xrightarrow{L^2} 0 \right\}$  which has  $\mu_\Gamma$ -measure 1, where  $\Sigma_{\tilde{\varepsilon},1,n}$  is the  $n \times n$  conditional variance-covariance matrix of  $[\tilde{\varepsilon}(\gamma_1), \dots, \tilde{\varepsilon}(\gamma_n)]'$ . By using Equation (25) and the Cauchy-Schwarz inequality, we prove the following Lemma.

**Lemma 2** *We have:*

$$\text{eig}_{max}(\Sigma_{\tilde{\varepsilon},1,n}) \leq \sum_k \sum_l \bar{w}_k \bar{w}_l \text{eig}_{max}(\Sigma_{\varepsilon,1,n}(G_k))^{1/2} \text{eig}_{max}(\Sigma_{\varepsilon,1,n}(G_l))^{1/2},$$

where  $\Sigma_{\varepsilon,1,n}(G_k)$  is the symmetric  $n \times n$  matrix with elements  $\text{Cov}(\varepsilon[G_k^{-1}(\gamma_i)], \varepsilon[G_k^{-1}(\gamma_j)]|\mathcal{F}_0)$ .

From Lemma 2, the triangular inequality and the Cauchy-Schwarz inequality, we get:

$$\|n^{-1} \text{eig}_{max}(\Sigma_{\tilde{\varepsilon},1,n})\|_2 \leq \sum_k \sum_l \bar{w}_k \bar{w}_l \|n^{-1} \text{eig}_{max}(\Sigma_{\varepsilon,1,n}(G_k))\|_2^{1/2} \|n^{-1} \text{eig}_{max}(\Sigma_{\varepsilon,1,n}(G_l))\|_2^{1/2}. \quad (26)$$

Now, let us define the sets  $\mathcal{J}_k = G_k^\infty(\mathcal{J})$ , for any  $k$ , where the set  $\mathcal{J}$  is defined in Assumption APR.3 for the original economy and has  $\mu_\Gamma$ -measure 1, and  $G_k^\infty$  is the mapping on  $\Gamma$  defined by  $G_k^\infty(\gamma_1, \gamma_2, \dots) = (G_k(\gamma_1), G_k(\gamma_2), \dots)$ . Then,  $\mathcal{J}_k \subset \left\{ (\gamma_i) \in \Gamma : n^{-1} \text{eig}_{max}(\Sigma_{\varepsilon,1,n}(G_k)) \xrightarrow{L^2} 0 \right\}$ . Moreover, since the mapping  $G_k$  is measure-preserving on  $[0, 1]$ , the mapping  $G_k^\infty$  is measure-preserving on  $\Gamma$ . Thus, it follows that  $\mu_\Gamma(\mathcal{J}_k) = 1$  for all  $k$ , and  $\tilde{\mathcal{J}} := \bigcap_k \mathcal{J}_k$  has  $\mu_\Gamma$  measure 1. We now prove that  $\tilde{\mathcal{J}} \subset \left\{ (\gamma_i) \in \Gamma : n^{-1} \text{eig}_{max}(\Sigma_{\tilde{\varepsilon},1,n}) \xrightarrow{L^2} 0 \right\}$ . Indeed, let us assume that  $(\gamma_i) \in \mathcal{J}_k$  for all  $k$ . Then, by taking the limit  $n \rightarrow \infty$  on both sides of (26), and interchanging the limit and the double sum, we get  $\|n^{-1} \text{eig}_{max}(\Sigma_{\tilde{\varepsilon},1,n})\|_2 \rightarrow 0$ . The limit and the double sum can be interchanged by applying the Lebesgue dominated convergence theorem. Indeed, the summands are uniformly dominated since  $n^{-1} \text{eig}_{max}(\Sigma_{\varepsilon,1,n}(G_k)) \leq \sup_{\gamma \in [0,1]} E[\varepsilon_t(\gamma)^2|\mathcal{F}_{t-1}] \leq M$ ,  $P$ -a.s., and the  $\bar{w}_k$  are summable.

### A.2.3 Proof of Proposition 3

**a) Consistency of  $\hat{\nu}$ .** By definition of  $\hat{\nu}$ , we have:  $\hat{\nu} - \nu = \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i \hat{\beta}'_{3,i} \hat{w}_i (\hat{\beta}_{1,i} - \hat{\beta}_{3,i} \nu)$ . From Equation (12) and MN Theorem 2 p. 35, we get  $\hat{\beta}_{3,i} \nu = \text{vec}[\nu' \hat{\beta}'_{3,i}] = (I_{d_1} \otimes \nu') \text{vec}[\hat{\beta}'_{3,i}] = (I_{d_1} \otimes \nu') J_a \hat{\beta}_{2,i}$ . Moreover, by using the selection matrices  $E_1$  and  $E_2$ , we obtain  $\hat{\beta}_{1,i} - \hat{\beta}_{3,i} \nu = [E'_1 - (I_{d_1} \otimes \nu') J_a E'_2] \hat{\beta}_i = C'_\nu \hat{\beta}_i = C'_\nu (\hat{\beta}_i - \beta_i)$ , from Equation (11). It follows that

$$\hat{\nu} - \nu = \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i \hat{\beta}'_{3,i} \hat{w}_i C'_\nu (\hat{\beta}_i - \beta_i). \quad (27)$$

The consistency of  $\hat{\nu}$  follows from the next Lemma, which is proved in Section A.2.3 c) below. The notation  $I_{n,T} = O_{p,\log}(a_{n,T})$  means that  $I_{n,T}/a_{n,T}$  is bounded in probability by some power of the logarithmic term  $\log(T)$  as  $n, T \rightarrow \infty$ .

**Lemma 3** Under Assumptions A.1 b), SC.1-SC.2, B.1, B.4 and B.5, we have:

- (i)  $\sup_i \mathbf{1}_i^X \|\hat{\beta}_i - \beta_i\| = O_{p,\log}(T^{-\eta/2})$ ; (ii)  $\sup_i \|w_i\| = O(1)$ ; (iii)  $\frac{1}{n} \sum_i \|\hat{w}_i - w_i\| = o_p(1)$ ;  
(iv)  $\hat{Q}_{\beta_3} - Q_{\beta_3} = o_p(1)$ , when  $n, T \rightarrow \infty$  such that  $n = O(T^{\bar{\gamma}})$  for  $\bar{\gamma} > 0$ .

**b) Consistency of  $\hat{\Lambda}$ .** By definition of  $\hat{\Lambda}$ , we deduce  $\left\| \text{vec} [\hat{\Lambda}' - \Lambda'] \right\| \leq \|\hat{\nu} - \nu\| + \left\| \text{vec} [\hat{F}' - F'] \right\|$ . By part a),  $\|\hat{\nu} - \nu\| = o_p(1)$ . By the LLN and Assumptions B.1a), B.4a) and B.6, we have  $\frac{1}{T} \sum_t Z_{t-1} Z'_{t-1} = O_p(1)$  and  $\frac{1}{T} \sum_t u_t Z'_{t-1} = o_p(1)$ . Then, by Slutsky theorem, we get that  $\left\| \text{vec} [\hat{F}' - F'] \right\| = o_p(1)$ . The result follows.

**c) Proof of Lemma 3:** (i) We use  $\hat{\beta}_i - \beta_i = \frac{\tau_{i,T}}{\sqrt{T}} \hat{Q}_{x,i}^{-1} Y_{i,T}$  and  $\mathbf{1}_i^X \tau_{i,T} \leq \chi_{2,T}$ . Moreover, from the definition of condition number  $CN$ ,  $\|\hat{Q}_{x,i}^{-1}\|^2 = \text{Tr}(\hat{Q}_{x,i}^{-2}) = \sum_{k=1}^d \lambda_{k,i}^{-2} \leq dCN(\hat{Q}_{x,i})^4$ , where the  $\lambda_{k,i}$  are the eigenvalues of matrix  $\hat{Q}_{x,i}$  and we use  $\text{eig}_{\max}(\hat{Q}_{x,i}) \geq 1$ , which implies  $\mathbf{1}_i^X \|\hat{Q}_{x,i}^{-1}\| \leq C\chi_{1,T}^2$ . Thus,  $\sup_i \mathbf{1}_i^X \|\hat{\beta}_i - \beta_i\| = O_{p,\log}\left(T^{-1/2} \sup_i \|Y_{i,T}\|\right)$  from Assumption B.5. Now let  $\delta_T := T^{-\eta/2}(\log T)^{(1+\bar{\gamma})/(2C_2)}$ , where  $\eta, C_2 > 0$  are as in Assumption B.1 and  $\bar{\gamma} > 0$  is such that  $n = O(T^{\bar{\gamma}})$ . We have:

$$\begin{aligned} P\left[T^{-1/2} \sup_i \|Y_{i,T}\| \geq \delta_T\right] &\leq nP\left[T^{-1/2} \|Y_{i,T}\| \geq \delta_T\right] = nE\left[P\left(T^{-1/2} \|Y_{i,T}\| \geq \delta_T | \gamma_i\right)\right] \\ &\leq n \sup_{\gamma \in [0,1]} P\left[\left|\frac{1}{T} \sum_t I_t(\gamma) x_t(\gamma) \varepsilon_t(\gamma)\right| \geq \delta_T\right] \leq n(C_1 T \exp\{-C_2 \delta_T^2 T^\eta\} + C_3 \delta_T^{-1} \exp\{-C_4 T^{\bar{\eta}}\}) = O(1), \end{aligned}$$

from Assumption B.1 c). Part (i) follows. By using  $w_i = (\text{diag}[v_i])^{-1}$ ,  $\tau_i \geq 1$  and  $\text{eig}_{\min}(S_{ii}) \geq M^{-1} \text{eig}_{\min}(Q_{x,i}) \geq M^{-1} \inf_{\gamma \in [0,1]} \text{eig}_{\min}(E[x_t(\gamma)x_t(\gamma)']) > 0$  from Assumptions A.1 b) and B.4 d), part (ii) follows. Part (iii) is proved in the supplementary materials by using Assumptions B.1, B.4 and B.5. Finally, part (iv) follows from  $\hat{Q}_{\beta_3} - Q_{\beta_3} = \frac{1}{n} \sum_i (\hat{\beta}'_{3,i} \hat{w}_i \hat{\beta}_{3,i} - \beta'_{3,i} w_i \beta_{3,i}) + \frac{1}{n} \sum_i \beta'_{3,i} w_i \beta_{3,i} - Q_{\beta_3}$ , by using parts (i)-(iii) and the LLN.

#### A.2.4 Proof of Proposition 4

**a) Asymptotic normality of  $\hat{\nu}$ .** From Equation (27) and by using  $\sqrt{T}(\hat{\beta}_i - \beta_i) = \tau_{i,T} \hat{Q}_{x,i}^{-1} Y_{i,T}$ , we get

$$\begin{aligned} \sqrt{nT}(\hat{\nu} - \nu) &= \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{n}} \sum_i \tau_{i,T} \hat{\beta}'_{3,i} \hat{w}_i C'_\nu \hat{Q}_{x,i}^{-1} Y_{i,T} = \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{n}} \sum_i \tau_{i,T} \beta'_{3,i} \hat{w}_i C'_\nu \hat{Q}_{x,i}^{-1} Y_{i,T} \\ &\quad + \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{n}} \sum_i \tau_{i,T} (\hat{\beta}_{3,i} - \beta_{3,i})' \hat{w}_i C'_\nu \hat{Q}_{x,i}^{-1} Y_{i,T} =: \hat{Q}_{\beta_3}^{-1} I_{11} + I_{12}. \end{aligned}$$

To rewrite  $I_{12}$ , we use the following lemma.

**Lemma 4** *Let  $A$  be a  $m \times n$  matrix and  $b$  be a  $n \times 1$  vector. Then,  $Ab = (\text{vec}[I_n]' \otimes I_m) \text{vec}[\text{vec}[A] b']$ .*

By Lemma 4, Equation (12), and  $\sqrt{T} \text{vec}[(\hat{\beta}_{3,i} - \beta_{3,i})'] = \tau_{i,T} J_a E'_2 \hat{Q}_{x,i}^{-1} Y_{i,T}$ , we have

$$\begin{aligned} I_{12} &= \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{nT}} \sum_i \tau_{i,T}^2 (\text{vec}[I_{d_1}]' \otimes I_{Kp}) \text{vec}[J_a E'_2 \hat{Q}_{x,i}^{-1} Y_{i,T} Y'_{i,T} \hat{Q}_{x,i}^{-1} C_\nu \hat{w}_i] \\ &= \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{nT}} \sum_i \tau_{i,T}^2 J_b \text{vec}[E'_2 \hat{Q}_{x,i}^{-1} Y_{i,T} Y'_{i,T} \hat{Q}_{x,i}^{-1} C_\nu \hat{w}_i] = \sqrt{\frac{n}{T}} \hat{B}_\nu + \frac{1}{\sqrt{T}} \hat{Q}_{\beta_3}^{-1} I_{13}, \end{aligned}$$

where  $I_{13} := \frac{1}{\sqrt{n}} \sum_i \tau_{i,T}^2 J_b \text{vec}[E'_2 (\hat{Q}_{x,i}^{-1} Y_{i,T} Y'_{i,T} \hat{Q}_{x,i}^{-1} C_\nu - \tau_{i,T}^{-1} \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} C_\nu) \hat{w}_i]$ . We get:

$$\sqrt{nT} \left( \hat{\nu} - \frac{1}{T} \hat{B}_\nu - \nu \right) = \hat{Q}_{\beta_3}^{-1} I_{11} + \frac{1}{\sqrt{T}} \hat{Q}_{\beta_3}^{-1} I_{13}. \quad (28)$$

Let us first show that  $\hat{Q}_{\beta_3}^{-1} I_{11}$  in (28) is asymptotically normal. By MN Theorem 2 p. 35, we have  $I_{11} = \frac{1}{\sqrt{n}} \sum_i \tau_{i,T} [(Y'_{i,T} \hat{Q}_{x,i}^{-1}) \otimes (\beta'_{3,i} \hat{w}_i)] \text{vec}[C'_\nu]$ . We use the next Lemma, which is proved below in Subsection A.2.4 c).

**Lemma 5** *Under Assumptions A.1, A.3, SC.1-SC.2 and B.1, B.3-B.5, we have*

$I_{11} = \frac{1}{\sqrt{n}} \sum_i \tau_i [(Y'_{i,T} Q_{x,i}^{-1}) \otimes (\beta'_{3,i} w_i)] \text{vec}[C'_\nu] + o_p(1)$ , when  $n, T \rightarrow \infty$  such that  $n = O(T^{\bar{\gamma}})$  for  $\bar{\gamma} > 0$ .

Then, by the properties of the  $vec$  operator, we get

$$\hat{Q}_{\beta_3}^{-1}I_{11} = \left( vec [C'_\nu]' \otimes \hat{Q}_{\beta_3}^{-1} \right) \frac{1}{\sqrt{n}} \sum_i \tau_i vec \left[ (Y'_{i,T} Q_{x,i}^{-1}) \otimes (\beta'_{3,i} w_i) \right] + o_p(1).$$

Moreover, by using the equality  $vec \left[ (Y'_{i,T} Q_{x,i}^{-1}) \otimes (\beta'_{3,i} w_i) \right] = (Q_{x,i}^{-1} Y_{i,T}) \otimes vec [\beta'_{3,i} w_i]$  (see MN Theorem 10 p. 55), we get  $\hat{Q}_{\beta_3}^{-1}I_{11} = \left( vec [C'_\nu]' \otimes \hat{Q}_{\beta_3}^{-1} \right) \frac{1}{\sqrt{n}} \sum_i \tau_i \left[ (Q_{x,i}^{-1} Y_{i,T}) \otimes v_{3,i} \right] + o_p(1)$ . Then  $\hat{Q}_{\beta_3}^{-1}I_{11} \Rightarrow N(0, \Sigma_\nu)$  follows from Assumptions A.2 a) and Lemma 3 (iv). Let us now consider the second term in the RHS of (28), and show  $\frac{1}{\sqrt{T}}I_{13} = o_p(1)$ . We have using MN Theorem 2 p. 35:

$$\begin{aligned} I_{13} &= \frac{1}{\sqrt{n}} \sum_i \tau_{i,T}^2 J_b \left( \hat{w}_i \otimes \left[ E'_2 \hat{Q}_{x,i}^{-1} (Y_{i,T} Y'_{i,T} - S_{ii,T}) \hat{Q}_{x,i}^{-1} \right] \right) vec [C_\nu] \\ &\quad - \frac{1}{\sqrt{n}} \sum_i \tau_{i,T}^2 J_b \left( \hat{w}_i \otimes \left[ E'_2 \hat{Q}_{x,i}^{-1} \left( \tau_{i,T}^{-1} \hat{S}_{ii}^0 - S_{ii,T} \right) \hat{Q}_{x,i}^{-1} \right] \right) vec [C_\nu] \\ &\quad - \frac{1}{\sqrt{n}} \sum_i \tau_{i,T} J_b \left( \hat{w}_i \otimes \left[ E'_2 \hat{Q}_{x,i}^{-1} \left( \hat{S}_{ii} - \hat{S}_{ii}^0 \right) \hat{Q}_{x,i}^{-1} \right] \right) vec [C_\nu] \\ &\quad - \frac{1}{\sqrt{n}} \sum_i \tau_{i,T} J_b \left( \hat{w}_i \otimes \left[ E'_2 \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} \right] \right) vec [C_{\hat{\nu}} - C_\nu], \\ &=: J_b (I_{131} - I_{132} - I_{133}) vec [C_\nu] - J_b I_{134} vec [C_{\hat{\nu}} - C_\nu], \end{aligned} \tag{29}$$

where  $\hat{S}_{ii}^0 = \frac{1}{T_i} \sum_t I_{i,t} \varepsilon_{i,t}^2 x_{i,t} x'_{i,t}$  and  $S_{ii,T} = \frac{1}{T} \sum_t I_{i,t} \sigma_{ii,t} x_{i,t} x'_{i,t}$ . The various terms are bounded in the next Lemma.

**Lemma 6** Under Assumptions A.1, A.3, SC.1-SC.2, B.1-B.5, (i)  $I_{131} = O_p(1) + O_{p,\log} \left( \frac{\sqrt{n}}{T} \right)$ , (ii)  $I_{132} = O_{p,\log} \left( \frac{1}{\sqrt{T}} + \frac{\sqrt{n}}{T} \right)$ , (iii)  $I_{133} = O_{p,\log} \left( \frac{\sqrt{n}}{T} \right)$  (iv)  $I_{134} = O_{p,\log} (\sqrt{n})$  and (v)  $C_{\hat{\nu}} - C_\nu = O_{p,\log} \left( \frac{1}{\sqrt{nT}} + \frac{1}{T} \right)$ , when  $n, T \rightarrow \infty$  such that  $n = O(T^{\bar{\gamma}})$  for  $\bar{\gamma} > 0$ .

From Equation (29) and Lemma 6 we get  $\frac{1}{\sqrt{T}}I_{13} = o_p(1) + O_{p,\log} \left( \frac{\sqrt{n}}{T\sqrt{T}} \right)$ . From  $n = O(T^{\bar{\gamma}})$  with  $\bar{\gamma} < 3$ , we get  $\frac{1}{\sqrt{T}}I_{13} = o_p(1)$  and the conclusion follows.

**b) Asymptotic normality of  $vec(\hat{\Lambda}')$ .** We have  $\sqrt{T} vec [\hat{\Lambda}' - \Lambda'] = \sqrt{T} vec [\hat{F}' - F'] + \sqrt{T} (\hat{\nu} - \nu)$ . By using  $\sqrt{T} vec [\hat{F}' - F'] = \left[ I_K \otimes \left( \frac{1}{T} \sum_t Z_{t-1} Z'_{t-1} \right)^{-1} \right] \frac{1}{\sqrt{T}} \sum_t u_t \otimes Z_{t-1}$  and  $\sqrt{T} (\hat{\nu} - \nu) =$

$O_p\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{T}}\right) = o_p(1)$ , the conclusion follows from Assumption A.2b).

**c) Proof of Lemma 5:** Write:

$$\begin{aligned} I_{11} &= \frac{1}{\sqrt{n}} \sum_i \tau_{i,T} \left[ (Y'_{i,T} Q_{x,i}^{-1}) \otimes (\beta'_{3,i} \hat{w}_i) \right] \text{vec} [C'_\nu] \\ &\quad + \frac{1}{\sqrt{n}} \sum_i \tau_{i,T} \left[ (Y'_{i,T} (\hat{Q}_{x,i}^{-1} - Q_{x,i}^{-1}) \otimes (\beta'_{3,i} \hat{w}_i)) \right] \text{vec} [C'_\nu] \quad =: I_{111} \text{vec} [C'_\nu] + I_{112} \text{vec} [C'_\nu]. \end{aligned}$$

Let us decompose  $I_{111}$  as:

$$\begin{aligned} I_{111} &= \frac{1}{\sqrt{n}} \sum_i \tau_i \left[ (Y'_{i,T} Q_{x,i}^{-1}) \otimes (\beta'_{3,i} w_i) \right] + \frac{1}{\sqrt{n}} \sum_i (\mathbf{1}_i^X - 1) \tau_i \left[ (Y'_{i,T} Q_{x,i}^{-1}) \otimes (\beta'_{3,i} w_i) \right] \\ &\quad + \frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^X (\tau_{i,T} - \tau_i) \left[ (Y'_{i,T} Q_{x,i}^{-1}) \otimes (\beta'_{3,i} w_i) \right] \\ &\quad + \frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^X \tau_{i,T} \left[ (Y'_{i,T} Q_{x,i}^{-1}) \otimes (\beta'_{3,i} ((\text{diag}[\hat{v}_i])^{-1} - (\text{diag}[v_i])^{-1})) \right] \\ &=: I_{1111} + I_{1112} + I_{1113} + I_{1114}. \end{aligned}$$

Similarly, for  $I_{112}$ , we have:

$$\begin{aligned} I_{112} &= \frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^X \tau_{i,T} \left[ (Y'_{i,T} (\hat{Q}_{x,i}^{-1} - Q_{x,i}^{-1}) \otimes (\beta'_{3,i} (\text{diag}[v_i])^{-1})) \right] \\ &\quad + \frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^X \tau_{i,T} \left[ (Y'_{i,T} (\hat{Q}_{x,i}^{-1} - Q_{x,i}^{-1}) \otimes (\beta'_{3,i} ((\text{diag}[\hat{v}_i])^{-1} - (\text{diag}[v_i])^{-1}))) \right] \\ &=: I_{1121} + I_{1122}. \end{aligned}$$

The conclusion follows by proving that terms  $I_{1112}$ ,  $I_{1113}$ ,  $I_{1114}$ ,  $I_{1121}$  and  $I_{1122}$  are  $o_p(1)$ .

*Proof that  $I_{1112} = o_p(1)$ .* We use the next Lemma.

**Lemma 7** Under Assumptions SC.1-SC.2, B.1 b), d) and B.4 a), c):  $P[\mathbf{1}_i^X = 0] = O(T^{-\bar{b}})$ , for any  $\bar{b} > 0$ .

In Lemma 7, the unconditional probability  $P[\mathbf{1}_i^X = 0]$  is independent of  $i$  since the indices  $(\gamma_i)$  are i.i.d. By using the bound  $\|I_{1112}\| \leq \frac{C}{\sqrt{n}} \sum_i (1 - \mathbf{1}_i^X) \|Y_{i,T}\|$  from Assumptions B.4 b), c) and d) and Lemma 3 (ii), the bound  $\sup_i E[\|Y_{i,T}\| | x_T(\gamma_i), I_T(\gamma_i), \{\gamma_i\}] \leq C$  from Assumptions A.1 a) and b), and Lemma 7, it follows  $I_{1112} = O_p(\sqrt{n}T^{-\bar{b}})$ , for any  $\bar{b} > 0$ . Since  $n = O(T^{\bar{\gamma}})$ , with  $\bar{\gamma} > 0$ , we get  $I_{1112} = o_p(1)$ .

*Proof that  $I_{1113} = o_p(1)$ .* From Assumptions A.1 a) and B.4 a)-d), we deduce that  $E [\|I_{1113}\|^2 | \{x_T(\gamma_i), I_T(\gamma_i), \gamma_i\}] \leq \frac{C}{nT} \sum_{i,j} \sum_t \mathbf{1}_i^X \mathbf{1}_j^X |\tau_{i,T} - \tau_i| |\tau_{j,T} - \tau_j| |\sigma_{ij,t}|$ . By Cauchy-Schwarz inequality and Assumption A.1 c), we get  $E [\|I_{1113}\|^2 | \{\gamma_i\}] \leq CM \sup_{\gamma \in [0,1]} E [\mathbf{1}_i^X |\tau_{i,T} - \tau_i|^4 | \gamma_i = \gamma]^{1/2}$ . By

using  $\tau_{i,T} - \tau_i = -\tau_{i,T} \tau_i \frac{1}{T} \sum_t (I_{i,t} - E[I_{i,t} | \gamma_i])$  and  $\mathbf{1}_i^X \tau_{i,T} \leq \chi_{2,T}$ , we deduce that

$\sup_{\gamma \in [0,1]} E [\mathbf{1}_i^X |\tau_{i,T} - \tau_i|^4 | \gamma_i = \gamma] \leq C \chi_{2,T}^4 \sup_{\gamma \in [0,1]} E \left[ \left| \frac{1}{T} \sum_t (I_t(\gamma) - E[I_t(\gamma)]) \right|^4 \right] = o(1)$  from Assumption B.5 and the next Lemma.

**Lemma 8** *Under Assumption B.1 d):*  $\sup_{\gamma \in [0,1]} E \left[ \left| \frac{1}{T} \sum_t (I_t(\gamma) - E[I_t(\gamma)]) \right|^4 \right] = O(T^{-c})$ , for some  $c > 0$ .

Then,  $I_{1113} = o_p(1)$ .

*Proof that  $I_{1114} = o_p(1)$ .* From the properties of diagonal matrices

$$\text{diag}[\hat{v}_i]^{-1} - \text{diag}[v_i]^{-1} = -\text{diag}[v_i]^{-2} \text{diag}[\hat{v}_i - v_i] + \text{diag}[\hat{v}_i]^{-1} \text{diag}[v_i]^{-2} (\text{diag}[\hat{v}_i - v_i])^2,$$

we get:

$$\begin{aligned} I_{1114} &= -\frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^X \tau_{i,T} \left[ (Y'_{i,T} Q_{x,i}^{-1}) \otimes (\beta'_{3,i} \text{diag}[v_i]^{-2} \text{diag}[\hat{v}_i - v_i]) \right] \\ &\quad + \frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^X \tau_{i,T} \left[ (Y'_{i,T} Q_{x,i}^{-1}) \otimes (\beta'_{3,i} \text{diag}[\hat{v}_i]^{-1} \text{diag}[v_i]^{-2} (\text{diag}[\hat{v}_i - v_i])^2) \right] \\ &=: I_{11141} + I_{11142}. \end{aligned}$$

Let us first consider  $I_{11141}$ . We have:

$$\begin{aligned} \hat{v}_i - v_i &= \tau_{i,T} C'_{\hat{\nu}_1} \hat{Q}_{x,i}^{-1} (\hat{S}_{ii} - S_{ii}) \hat{Q}_{x,i}^{-1} C_{\hat{\nu}_1} + 2\tau_{i,T} (C_{\hat{\nu}_1} - C_\nu)' \hat{Q}_{x,i}^{-1} S_{ii} \hat{Q}_{x,i}^{-1} C_{\hat{\nu}_1} \\ &\quad + \tau_{i,T} (C_{\hat{\nu}_1} - C_\nu)' \hat{Q}_{x,i}^{-1} S_{ii} \hat{Q}_{x,i}^{-1} (C_{\hat{\nu}_1} - C_\nu) + 2\tau_{i,T} C'_\nu (\hat{Q}_{x,i}^{-1} - Q_{x,i}^{-1}) S_{ii} \hat{Q}_{x,i}^{-1} C_\nu \\ &\quad + \tau_{i,T} C'_\nu (\hat{Q}_{x,i}^{-1} - Q_{x,i}^{-1}) S_{ii} (\hat{Q}_{x,i}^{-1} - Q_{x,i}^{-1}) C_\nu + (\tau_{i,T} - \tau_i) C'_\nu Q_{x,i}^{-1} S_{ii} Q_{x,i}^{-1} C_\nu. \end{aligned} \quad (30)$$

The contributions of the first two terms to  $I_{11141}$  are:

$$\begin{aligned} I_{111411} &= -\frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^X \tau_{i,T}^2 \left[ (Y'_{i,T} Q_{x,i}^{-1}) \otimes (\beta'_{3,i} \text{diag}[v_i]^{-2} \text{diag}[C'_{\hat{\nu}_1} \hat{Q}_{x,i}^{-1} (\hat{S}_{ii} - S_{ii}) \hat{Q}_{x,i}^{-1} C_{\hat{\nu}_1}]) \right], \\ I_{111412} &= -\frac{2}{\sqrt{n}} \sum_i \mathbf{1}_i^X \tau_{i,T}^2 \left[ (Y'_{i,T} Q_{x,i}^{-1}) \otimes (\beta'_{3,i} \text{diag}[v_i]^{-2} \text{diag}[(C_{\hat{\nu}_1} - C_\nu)' \hat{Q}_{x,i}^{-1} S_{ii} \hat{Q}_{x,i}^{-1} C_{\hat{\nu}_1}]) \right]. \end{aligned}$$



We first show  $I_{111412} = o_p(1)$ . For this purpose, it is enough to show that  $C_{\hat{\nu}_1} - C_\nu = O_p(T^{-c})$ , for some  $c > 0$ , and  $\frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^X \tau_{i,T}^2 \left( \hat{Q}_{x,i}^{-1} S_{ii} \hat{Q}_{x,i}^{-1} \right)_{k,l} (\beta'_{3,i} \text{diag}[v_i]^{-2})_{m,p} (Q_{x,i}^{-1} Y_{i,T})_q = O_p(\chi_{1,T}^4 \chi_{2,T}^2)$ , for any  $k, l, m, p, q$ . The first statement follows from the proof of Proposition 3 but with known weights equal to 1. To prove the second statement, we use bounds  $\mathbf{1}_i^X \tau_{i,T} \leq \chi_{2,T}$ ,  $\mathbf{1}_i^X \|\hat{Q}_{x,i}^{-1}\| \leq C \chi_{1,T}^2$  and  $\|Q_{x,i}^{-1}\| \leq C$  and Assumption A.1 c). Let us now prove that  $I_{111411} = o_p(1)$ . For this purpose, it is enough to show that

$$J_1 := \frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^X \tau_{i,T}^2 \left( \hat{Q}_{x,i}^{-1} (\hat{S}_{ii} - S_{ii}) \hat{Q}_{x,i}^{-1} \right)_{k,l} (\beta'_{3,i} \text{diag}[v_i]^{-2})_{m,p} (Q_{x,i}^{-1} Y_{i,T})_q = o_p(1), \quad (31)$$

for any  $k, l, m, p, q$ . By using  $\hat{\varepsilon}_{i,t} = \varepsilon_{i,t} - x'_{i,t} (\hat{\beta}_i - \beta_i) = \varepsilon_{i,t} - \frac{\tau_{i,T}}{\sqrt{T}} x'_{i,t} \hat{Q}_{x,i}^{-1} Y_{i,T}$ , we get:

$$\begin{aligned} \hat{S}_{ii} - S_{ii} &= \frac{1}{T_i} \sum_t I_{i,t} (\varepsilon_{it}^2 x_{i,t} x'_{i,t} - S_{ii}) + \frac{1}{T_i} \sum_t I_{i,t} (\hat{\varepsilon}_{i,t}^2 - \varepsilon_{it}^2) x_{i,t} x'_{i,t} \\ &= \frac{\tau_{i,T}}{\sqrt{T}} W_{1,i,T} + \frac{\tau_{i,T}}{\sqrt{T}} W_{2,i,T} - \frac{2\tau_{i,T}^2}{T} W_{3,i,T} \hat{Q}_{x,i}^{-1} Y_{i,T} + \frac{\tau_{i,T}^3}{T} \hat{Q}_{x,i}^{(4)} \hat{Q}_{x,i}^{-1} Y_{i,T} Y'_{i,T} \hat{Q}_{x,i}^{-1}, \end{aligned} \quad (32)$$

where  $W_{1,i,T} := \frac{1}{\sqrt{T}} \sum_t I_{i,t} x_{i,t}^2 \eta_{i,t}$ ,  $\eta_{i,t} = \varepsilon_{i,t}^2 - \sigma_{ii,t}$ ,  $W_{2,i,T} := \frac{1}{\sqrt{T}} \sum_t I_{i,t} \zeta_{i,t}$ ,  $\zeta_{i,t} := \sigma_{ii,t} x_{i,t}^2 - S_{ii}$ ,  $W_{3,i,T} := \frac{1}{\sqrt{T}} \sum_t I_{i,t} \varepsilon_{i,t} x_{i,t}^3$ ,  $\hat{Q}_{x,i}^{(4)} := \frac{1}{T} \sum_t I_{i,t} x_{i,t}^4$  and  $x_{i,t}$ ,  $v_i$  and  $Y_{i,T}$  are now treated as scalars to ease notation. Then:

$$\begin{aligned} J_1 &= \frac{1}{\sqrt{nT}} \sum_i \mathbf{1}_i^X v_i^{-2} \tau_{i,T}^3 \hat{Q}_{x,i}^{-2} W_{1,i,T} \beta_{3,i} Q_{x,i}^{-1} Y_{i,T} + \frac{1}{\sqrt{nT}} \sum_i \mathbf{1}_i^X v_i^{-2} \tau_{i,T}^3 \hat{Q}_{x,i}^{-2} W_{2,i,T} \beta_{3,i} Q_{x,i}^{-1} Y_{i,T} \\ &\quad - \frac{2}{\sqrt{nT}} \sum_i \mathbf{1}_i^X v_i^{-2} \tau_{i,T}^4 \hat{Q}_{x,i}^{-3} W_{3,i,T} \beta_{3,i} Q_{x,i}^{-1} Y_{i,T}^2 \\ &\quad + \frac{1}{\sqrt{nT}} \sum_i \mathbf{1}_i^X v_i^{-2} \tau_{i,T}^5 \hat{Q}_{x,i}^{-4} \hat{Q}_{x,i}^{(4)} \beta_{3,i} Q_{x,i}^{-1} Y_{i,T}^3 =: J_{11} + J_{12} + J_{13} + J_{14}. \end{aligned}$$

Let us consider  $J_{11}$ . We have:

$$E [J_{11} | \{x_T(\gamma_i), I_T(\gamma_i), \gamma_i\}] = \frac{1}{\sqrt{nT^3}} \sum_i \sum_{t,s} \mathbf{1}_i^X v_i^{-2} \tau_{i,T}^3 \hat{Q}_{x,i}^{-2} Q_{x,i}^{-1} \beta_{3,i} I_{i,t} I_{i,s} x_{i,t}^2 x_{i,s} E [\varepsilon_{i,t}^2 \varepsilon_{i,s} | \mathcal{F}_T, \gamma_i] = 0,$$

from Assumption A.3. Moreover, from Assumption B.4:

$$\begin{aligned} V [J_{11} | \{x_T(\gamma_i), I_T(\gamma_i), \gamma_i\}] &\leq \frac{C}{nT^3} \sum_{i,j} \sum_{t_1, t_2, t_3, t_4} \mathbf{1}_i^X \mathbf{1}_j^X \tau_{i,T}^3 \tau_{j,T}^3 \|\hat{Q}_{x,i}^{-1}\|^2 \|\hat{Q}_{x,j}^{-1}\|^2 \\ &\quad |Cov(\eta_{i,t_1} \varepsilon_{i,t_2}, \eta_{j,t_3} \varepsilon_{j,t_4} | \mathcal{F}_T, \gamma_i, \gamma_j)|. \end{aligned}$$

By using  $\mathbf{1}_i^X \|\hat{Q}_{x,i}^{-1}\| \leq C\chi_{1,T}^2$ ,  $\mathbf{1}_i^X \tau_{i,T} \leq \chi_{2,T}$ , the Law of Iterated Expectations and Assumptions B.3 c) and B.5, we get  $E[J_{11}] = 0$  and  $V[J_{11}] = o(1)$ . Thus  $J_{11} = o_p(1)$ . By similar arguments and using Assumptions A.1 c), B.3 e) and B.4, we get  $J_{12} = o_p(1)$ ,  $J_{13} = o_p(1)$  and  $J_{14} = o_p(1)$ . Hence the bound in Equation (31) follows, and  $I_{111411} = o_p(1)$ . Paralleling the detailed arguments provided above, we can show that all other remaining terms making  $I_{1114}$  are also  $o_p(1)$ .

*Proof that  $I_{1121} = o_p(1)$ . From:*

$$\hat{Q}_{x,i}^{-1} - Q_{x,i}^{-1} = -\hat{Q}_{x,i}^{-1} \left( \frac{1}{T} \sum_t I_{i,t} x_{i,t} x'_{i,t} - Q_{x,i} \right) Q_{x,i}^{-1} = -\tau_{i,T} \hat{Q}_{x,i}^{-1} W_{i,T} Q_{x,i}^{-1}, \quad (33)$$

where  $W_{i,T} := \frac{1}{T} \sum_t I_{i,t} (x_{i,t} x'_{i,t} - Q_{x,i})$ , we can write:

$$I_{1121} = -\frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^X \tau_{i,T}^2 \left[ (Y'_{i,T} \hat{Q}_{x,i}^{-1} W_{i,T} Q_{x,i}^{-1}) \otimes (\beta'_{3,i} \text{diag}[v_i]^{-1}) \right].$$

From Assumption B.4,  $\mathbf{1}_i^X \|\hat{Q}_{x,i}^{-1}\| \leq C\chi_{1,T}^2$  and  $\mathbf{1}_i^X \tau_{i,T} \leq \chi_{2,T}$ , we have:

$$E[\|I_{1121}\|^2 | \{\gamma_T(\gamma_i), I_T(\gamma_i), \gamma_i\}] \leq \frac{C\chi_{1,T}^4 \chi_{2,T}^4}{nT} \sum_{i,j} \sum_t |\sigma_{ij,t}| \|W_{i,T}\| \|W_{j,T}\|.$$

Then, from Cauchy-Schwarz inequality, we get  $E[\|I_{1121}\|^2 | \{\gamma_i\}] \leq C\chi_{1,T}^4 \chi_{2,T}^4 \frac{1}{n} \sum_{i,j} E[\sigma_{ij,t}^2 | \gamma_i, \gamma_j]^{1/2}$

$\sup_i E[\|W_{i,T}\|^4 | \gamma_i]^{1/2}$ , where

$$\sup_i E[\|W_{i,T}\|^4 | \gamma_i] \leq \sup_{\gamma \in [0,1]} E \left[ \left\| \frac{1}{T} \sum_t I_t(\gamma) (x_t(\gamma) x_t(\gamma)' - E[x_t(\gamma) x_t(\gamma)']) \right\|^4 \right] = O(T^{-c})$$

from Assumption B.1 b). Then, from Assumptions A.1 c) and B.5 it follows  $E[\|I_{1121}\|^2] = o(1)$  and thus  $I_{1121} = o_p(1)$ .

*Proof that  $I_{1122} = o_p(1)$ .* The statement follows by combining arguments similar as for  $I_{1114}$  and  $I_{1121}$ .

### A.2.5 Proof of Proposition 5

From Proposition 4, we have to show that  $\tilde{\Sigma}_\nu - \Sigma_\nu = o_p(1)$ . By  $\Sigma_\nu = \left( \text{vec}[C'_\nu]' \otimes Q_{\beta_3}^{-1} \right) S_{v_3} \left( \text{vec}[C'_\nu] \otimes Q_{\beta_3}^{-1} \right)$  and  $\tilde{\Sigma}_\nu = \left( \text{vec}[C'_\nu]' \otimes \hat{Q}_{\beta_3}^{-1} \right) \tilde{S}_{v_3} \left( \text{vec}[C'_\nu] \otimes \hat{Q}_{\beta_3}^{-1} \right)$ , where  $\tilde{S}_{v_3} = \frac{1}{n} \sum_{i,j} \frac{\tau_{i,T} \tau_{j,T}}{\tau_{ij,T}} [\tilde{S}_{Q,ij} \otimes \hat{v}_{3,i} \hat{v}'_{3,j}]$ , and

the consistency of  $\hat{v}$  and  $\hat{Q}_{\beta_3}$ , the statement follows if  $\tilde{S}_{v_3} - S_{v_3} = o_p(1)$ . The leading terms in  $\tilde{S}_{v_3} - S_{v_3}$  are given by  $I_3 := \frac{1}{n} \sum_{i,j} \frac{\tau_i \tau_j}{\tau_{ij}} \left( \tilde{S}_{Q,ij} - S_{Q,ij} \right) \otimes v_{3,i} v'_{3,j}$  and  $I_4 := \frac{1}{n} \sum_{i,j} \tau_i \tau_j (\tau_{ij,T}^{-1} - \tau_{ij}^{-1}) S_{Q,ij} \otimes v_{3,i} v'_{3,j}$ , while the other ones can be shown to be  $o_p(1)$  by arguments similar to the proofs of Propositions 3 and 4.

*Proof of  $I_3 = o_p(1)$ .* From  $\tau_i \leq M$ ,  $\tau_{ij} \geq 1$ ,  $\|v_{3,i}\| \leq M$  and  $\|Q_{x,i}^{-1}\| \leq M$ ,  $I_3 = o_p(1)$  follows if we show:  $\frac{1}{n} \sum_{i,j} \left\| \tilde{S}_{ij} - S_{ij} \right\| = o_p(1)$ . For this purpose, we introduce the following Lemmas 9 and 10 that extend results in Bickel and Levina (2008) from the i.i.d. case to the time series case including random individual effects.

**Lemma 9** *Let  $\psi_{nT} := \max_{i,j} \left\| \hat{S}_{ij} - S_{ij} \right\|$ , and  $\Psi_{nT}(\xi) := \max_{i,j} P \left[ \left\| \hat{S}_{ij} - S_{ij} \right\| \geq \xi \right]$ , for  $\xi > 0$ . Under Assumptions SC.1, SC.2, A.4,  $\frac{1}{n} \sum_{i,j} \left\| \tilde{S}_{ij} - S_{ij} \right\| = O_p \left( \psi_{nT} n^{\bar{\delta}} \kappa^{-\bar{q}} + n^{\bar{\delta}} \kappa^{1-\bar{q}} + \psi_{nT} n^2 \Psi_{nT}((1-v)\kappa) \right)$ , for any  $v \in (0, 1)$ .*

**Lemma 10** *Under Assumptions SC.1, SC.2, B.1, B.4 and B.5, if  $\kappa = M \sqrt{\frac{\log n}{T^\eta}}$  with  $M$  large, then  $n^2 \Psi_{nT}((1-v)\kappa) = O(1)$ , for any  $v \in (0, 1)$ , and  $\psi_{nT} = O_p \left( \sqrt{\frac{\log n}{T^\eta}} \right)$ , when  $n, T \rightarrow \infty$  such that  $n = O(T^{\bar{\gamma}})$  for  $\bar{\gamma} > 0$ .*

In Lemma 9, the probability  $P \left[ \left\| \hat{S}_{ij} - S_{ij} \right\| \geq \xi \right]$  is the same for all pairs  $(i, j)$  with  $i = j$ , and for all pairs with  $i \neq j$ , since this probability is marginal w.r.t. the individual random effects. From Lemmas 9 and 10, it follows  $\frac{1}{n} \sum_{i,j} \left\| \tilde{S}_{ij} - S_{ij} \right\| = O_p \left( \left( \frac{\log n}{T^\eta} \right)^{(1-\bar{q})/2} n^{\bar{\delta}} \right) = o_p(1)$ , since  $n = O(T^{\bar{\gamma}})$  with  $\bar{\gamma} < \eta \frac{1-\bar{q}}{2\bar{\delta}}$ .

*Proof of  $I_4 = o_p(1)$ .* From  $\tau_i \leq M$ ,  $\|Q_{x,i}^{-1}\| \leq M$ , and  $\|v_{3,i}\| \leq M$ , we have  $E[\|I_4\| \{\gamma_i\}] \leq C \sup_{i,j} E[|\tau_{ij,T}^{-1} - \tau_{ij}^{-1}| \gamma_i, \gamma_j] \frac{1}{n} \sum_{i,j} \|S_{ij}\|$ . By using the inequalities  $\sup_{i,j} E[|\tau_{ij,T}^{-1} - \tau_{ij}^{-1}| \gamma_i, \gamma_j] \leq \sup_{\gamma, \tilde{\gamma} \in [0,1]} E \left[ \left\| \frac{1}{T} \sum_t (I_t(\gamma) I_t(\tilde{\gamma}) - E[I_t(\gamma) I_t(\tilde{\gamma})]) \right\| \right]$  and  $\|S_{ij}\| \leq CE[|\sigma_{ij,t}| \gamma_i, \gamma_j]$ , from Assumptions A.1 c) and B.1 d) we get  $E[\|I_4\|] = o(1)$ , which implies  $I_4 = o_p(1)$ .

## A.2.6 Proof of Proposition 6

By definition of  $\hat{Q}_e$ , we get the following result:

**Lemma 11** Under  $\mathcal{H}_0$  and Assumptions APR.1-APR.4, SC.1-SC.2, FS.1-FS.2, A.1-A.3 and B.1-B.5, we have  $\hat{Q}_e = \frac{1}{n} \sum_i \left( \hat{\beta}_i - \beta_i \right)' C_{\hat{\nu}} \hat{w}_i C_{\hat{\nu}}' \left( \hat{\beta}_i - \beta_i \right) + O_{p,\log} \left( \frac{1}{nT} + \frac{1}{T^2} \right)$ , when  $n, T \rightarrow \infty$  such that  $n = O(T^{\bar{\gamma}})$  for  $\bar{\gamma} > 0$ .

From  $\sqrt{T} \left( \hat{\beta}_i - \beta_i \right) = \tau_{i,T} \hat{Q}_{x,i}^{-1} Y_{i,T}$  and the properties of the trace, we have  $\left( \hat{\beta}_i - \beta_i \right)' C_{\hat{\nu}} \hat{w}_i C_{\hat{\nu}}' \left( \hat{\beta}_i - \beta_i \right) = T^{-1} \tau_{i,T}^2 \text{tr} \left[ C_{\hat{\nu}}' \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i \right]$ . Then, by using that  $\tau_{i,T} \text{tr} \left[ C_{\hat{\nu}}' \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i \right] = \mathbf{1}_i^X d_1$ , Lemma 7, and  $n = O(T^{\bar{\gamma}})$  with  $\bar{\gamma} < 2$ , we get  $\hat{\xi}_{nT} = \frac{1}{\sqrt{n}} \sum_i \tau_{i,T}^2 \text{tr} \left[ C_{\hat{\nu}}' \hat{Q}_{x,i}^{-1} \left( Y_{i,T} Y_{i,T}' - \tau_{i,T}^{-1} \hat{S}_{ii} \right) \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i \right] + o_p(1)$ .

Similarly, as in Equation (29), let us decompose  $\hat{\xi}_{nT}$  as:

$$\begin{aligned} \hat{\xi}_{nT} &= \frac{1}{\sqrt{n}} \sum_i \tau_{i,T}^2 \text{tr} \left[ C_{\hat{\nu}}' \hat{Q}_{x,i}^{-1} \left( Y_{i,T} Y_{i,T}' - S_{ii,T} \right) \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i \right] \\ &\quad - \frac{1}{\sqrt{n}} \sum_i \tau_{i,T}^2 \text{tr} \left[ C_{\hat{\nu}}' \hat{Q}_{x,i}^{-1} \left( \tau_{i,T}^{-1} \hat{S}_{ii}^0 - S_{ii,T} \right) \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i \right] \\ &\quad - \frac{1}{\sqrt{n}} \sum_i \tau_{i,T} \text{tr} \left[ C_{\hat{\nu}}' \hat{Q}_{x,i}^{-1} \left( \hat{S}_{ii} - \hat{S}_{ii}^0 \right) \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i \right] + o_p(1). \end{aligned}$$

By similar results as in Lemma 6 (i)-(iii), and the condition  $n = O(T^{\bar{\gamma}})$  with  $\bar{\gamma} < 2$ , we get  $\hat{\xi}_{nT} = \frac{1}{\sqrt{n}} \sum_i \tau_{i,T}^2 \text{tr} \left[ C_{\hat{\nu}}' \hat{Q}_{x,i}^{-1} \left( Y_{i,T} Y_{i,T}' - S_{ii,T} \right) \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i \right] + o_p(1)$ . Now, from the properties  $\text{tr}(ABCD)$

$= \text{vec}[D]'(C' \otimes A) \text{vec}[B]$  (MN Theorem 3, p. 35) and  $\text{vec}[ABC] = (C' \otimes A) \text{vec}[B]$  for conformable matrices, we have:

$$\begin{aligned} &\text{tr} \left[ C_{\hat{\nu}}' \hat{Q}_{x,i}^{-1} \left( Y_{i,T} Y_{i,T}' - S_{ii,T} \right) \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i \right] = \text{vec}[\hat{w}_i]' \left( C_{\hat{\nu}}' \otimes C_{\hat{\nu}}' \right) \text{vec} \left[ \hat{Q}_{x,i}^{-1} \left( Y_{i,T} Y_{i,T}' - S_{ii,T} \right) \hat{Q}_{x,i}^{-1} \right] \\ &= \text{vec}[\hat{w}_i]' \left( C_{\hat{\nu}}' \otimes C_{\hat{\nu}}' \right) \left( \hat{Q}_{x,i}^{-1} \otimes \hat{Q}_{x,i}^{-1} \right) \left( Y_{i,T} \otimes Y_{i,T} - \text{vec}[S_{ii,T}] \right) \\ &= \text{vec} \left[ C_{\hat{\nu}}' \otimes C_{\hat{\nu}}' \right]' \left\{ \left[ \left( \hat{Q}_{x,i}^{-1} \otimes \hat{Q}_{x,i}^{-1} \right) \left( Y_{i,T} \otimes Y_{i,T} - \text{vec}[S_{ii,T}] \right) \right] \otimes \text{vec}[\hat{w}_i] \right\}. \end{aligned}$$

Thus, we get  $\hat{\xi}_{nT} = \text{vec} \left[ C_{\hat{\nu}}' \otimes C_{\hat{\nu}}' \right]' \frac{1}{\sqrt{n}} \sum_i \tau_{i,T}^2 \left[ \left( \hat{Q}_{x,i}^{-1} \otimes \hat{Q}_{x,i}^{-1} \right) \left( Y_{i,T} \otimes Y_{i,T} - \text{vec}[S_{ii,T}] \right) \right] \otimes \text{vec}[\hat{w}_i]$ . From Assumption A.5, we get  $\hat{\xi}_{nT} \Rightarrow N(0, \Sigma_{\xi})$ , where  $\Sigma_{\xi} = \text{vec} \left[ C_{\hat{\nu}}' \otimes C_{\hat{\nu}}' \right]' \Omega \text{vec} \left[ C_{\hat{\nu}}' \otimes C_{\hat{\nu}}' \right]$ . Now, by using that  $\text{tr}(ABCD) = \text{vec}[D]'(A \otimes C') \text{vec}[B]$ , we have:

$$\begin{aligned} &\text{vec} \left[ C_{\hat{\nu}}' \otimes C_{\hat{\nu}}' \right]' \left[ \left( S_{Q,ij} \otimes S_{Q,ij} \right) \otimes \text{vec}[\hat{w}_i] \text{vec}[\hat{w}_j]' \right] \text{vec} \left[ C_{\hat{\nu}}' \otimes C_{\hat{\nu}}' \right] \\ &= \text{tr} \left[ \left( S_{Q,ij} \otimes S_{Q,ij} \right) \left( C_{\hat{\nu}} \otimes C_{\hat{\nu}} \right) \text{vec}[\hat{w}_j] \text{vec}[\hat{w}_i]' \left( C_{\hat{\nu}}' \otimes C_{\hat{\nu}}' \right) \right] \\ &= \text{tr} \left[ \left( C_{\hat{\nu}}' S_{Q,ij} C_{\hat{\nu}} \right) \hat{w}_j \left( C_{\hat{\nu}}' S_{Q,ji} C_{\hat{\nu}} \right) \hat{w}_i \right] = \text{tr} \left[ \left( C_{\hat{\nu}}' \hat{Q}_{x,i}^{-1} S_{ij} \hat{Q}_{x,j}^{-1} C_{\hat{\nu}} \right) \hat{w}_j \left( C_{\hat{\nu}}' \hat{Q}_{x,j}^{-1} S_{ji} \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \right) \hat{w}_i \right], \end{aligned}$$

and similarly for  $\text{vec} [C'_\nu \otimes C'_\nu]' [((S_{Q,ij} \otimes S_{Q,ij})W_d) \otimes (\text{vec}[w_i]\text{vec}[w_j]')] \text{vec} [C'_\nu \otimes C'_\nu]$ . Thus, we get the asymptotic variance matrix  $\Sigma_\xi = 2 \lim_{n \rightarrow \infty} E \left[ \frac{1}{n} \sum_{i,j} \frac{\tau_i^2 \tau_j^2}{\tau_{ij}^2} \text{tr} \left[ \left( C'_\nu Q_{x,i}^{-1} S_{ij} Q_{x,j}^{-1} C_\nu \right) w_j \left( C'_\nu Q_{x,j}^{-1} S_{ji} Q_{x,i}^{-1} C_\nu \right) w_i \right] \right]$ . Finally,  $\tilde{\Sigma}_\xi = \Sigma_\xi + o_p(1)$  follows from  $\frac{1}{n} \sum_{i,j} \|\tilde{S}_{ij} - S_{ij}\| = o_p(1)$  and  $\frac{1}{n} \sum_{i,j} \|\tilde{S}_{ij} - S_{ij}\|^2 = o_p(1)$ .

## A.2.7 Proof of Proposition 7

a) **Asymptotic normality of  $\hat{\nu}$ .** By definition of  $\hat{\nu}$  and under  $\mathcal{H}_1$ , we have

$$\begin{aligned} \hat{\nu} - \nu_\infty &= \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i \hat{\beta}'_{3,i} \hat{w}_i C'_{\nu_\infty} \hat{\beta}_i = \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i \hat{\beta}'_{3,i} \hat{w}_i C'_{\nu_\infty} (\hat{\beta}_i - \beta_i) + \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i \hat{\beta}'_{3,i} \hat{w}_i e_i \quad (34) \\ &= \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i \beta'_{3,i} \hat{w}_i C'_{\nu_\infty} (\hat{\beta}_i - \beta_i) + \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i (\hat{\beta}_{3,i} - \beta_{3,i})' \hat{w}_i C'_{\nu_\infty} (\hat{\beta}_i - \beta_i) \\ &\quad + \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i \beta'_{3,i} w_i e_i + \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i \beta'_{3,i} (\hat{w}_i - w_i) e_i + \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i (\hat{\beta}_{3,i} - \beta_{3,i})' \hat{w}_i e_i. \end{aligned}$$

Equation (34) is the analogue of Equation (27). Consistency of  $\hat{\nu}$  for  $\nu_\infty$  follows as in the proof of Proposition 3 and by using  $E [\beta'_{3,i} w_i e_i] = 0$ . Thus, by using  $\sqrt{T} (\hat{\beta}_i - \beta_i) = \tau_{i,T} \hat{Q}_{x,i}^{-1} Y_{i,T}$ , Lemma 4 and similar algebraic manipulations as in the proof of Proposition 4, we get:

$$\begin{aligned} \sqrt{n} \left( \hat{\nu} - \frac{1}{T} \hat{B}_\nu - \nu_\infty \right) &= \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{nT}} \sum_i \tau_{i,T} \beta'_{3,i} \hat{w}_i C'_{\nu_\infty} \hat{Q}_{x,i}^{-1} Y_{i,T} \\ &+ \frac{1}{T} \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{n}} \sum_i \tau_{i,T}^2 J_b \text{vec} \left[ E_2' \left( \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-1} C_{\nu_\infty} - \tau_{i,T}^{-1} \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} C_\nu \right) \hat{w}_i \right] + \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{n}} \sum_i \beta'_{3,i} w_i e_i \\ &+ \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{n}} \sum_i \beta'_{3,i} (\hat{w}_i - w_i) e_i + \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{nT}} \sum_i \tau_{i,T} (\text{vec} [I_{d_1}]' \otimes I_{Kp}) \text{vec} \left[ J_a E_2' \hat{Q}_{x,i}^{-1} Y_{i,T} e_i' \hat{w}_i \right] \\ &=: I_{51} + I_{52} + I_{53} + I_{54} + I_{55}. \end{aligned}$$

The first two terms  $I_{51}$  and  $I_{52}$  are the analogue of the terms in the RHS of Equation (28) multiplied by  $1/\sqrt{T}$ . Therefore, from similar arguments as for terms  $I_{11}$  and  $I_{13}$  in the proof of Proposition 4, we get  $I_{51} = o_p(1)$  and  $I_{52} = o_p(1)$ . From Assumption SC.2 and  $E [\beta'_{3,i} w_i e_i] = 0$ , we get  $\frac{1}{\sqrt{n}} \sum_i \beta'_{3,i} w_i e_i \Rightarrow N(0, E [\beta'_{3,i} w_i e_i e_i' w_i \beta_{3,i}])$  by the CLT. Thus,  $I_{53} \Rightarrow N(0, Q_{\beta_3}^{-1} E [\beta'_{3,i} w_i e_i e_i' w_i \beta_{3,i}] Q_{\beta_3}^{-1})$ . Then, the asymptotic distribution of  $\hat{\nu}$  follows if terms  $I_{54}$  and  $I_{55}$  are  $o_p(1)$ . We have  $I_{54} = o_p(1)$  from similar arguments as for terms  $I_{1112}$  and  $I_{1114}$  in the proof of Lemma 5, and  $I_{55} = o_p(1)$  from similar arguments as for term  $I_{11}$  in the proof of Proposition 4.

**b) Asymptotic normality of  $\hat{\lambda}$ .** We have  $\sqrt{T} \text{vec} [\hat{\Lambda}' - \Lambda'_{\infty}] = \sqrt{T} (\hat{\nu} - \nu_{\infty}) + \sqrt{T} \text{vec} [\hat{F}' - F']$ . By using  $T/n = o(1)$  and  $\sqrt{T} (\hat{\nu} - \nu_{\infty}) = O_p \left( \sqrt{\frac{T}{n}} + \frac{1}{\sqrt{T}} \right) = o_p(1)$ , the conclusion follows.

**c) Consistency of the test.** By definition of  $\hat{Q}_e$ , we get the following result:

**Lemma 12** *Under  $\mathcal{H}_1$  and Assumptions APR.1-APR.4, SC.1-SC.2, FS.1-FS.2, A.1-A.3 and B.1-B.5, we have*

$$\hat{Q}_e = \frac{1}{n} \sum_i (\hat{\beta}_i - \beta_i)' C_{\hat{\nu}} \hat{w}_i C_{\hat{\nu}}' (\hat{\beta}_i - \beta_i) + \frac{1}{n} \sum_i e_i' w_i e_i + O_{p,\log} \left( \frac{1}{n} + \frac{1}{\sqrt{nT}} + \frac{1}{\sqrt{T^3}} \right), \text{ when } n, T \rightarrow \infty \text{ such that } n = O(T^{\bar{\gamma}}) \text{ for } \bar{\gamma} > 0.$$

By similar arguments as in the proof of Proposition 6 and using  $\bar{\gamma} < 2$ , we get:

$$\begin{aligned} \hat{\xi}_{nT} &= \frac{1}{\sqrt{n}} \sum_i \tau_i^2 \text{tr} \left[ C_{\hat{\nu}}' Q_{x,i}^{-1} (Y_{i,T} Y_{i,T}' - S_{ii,T}) Q_{x,i}^{-1} C_{\hat{\nu}} w_i \right] + T \frac{1}{\sqrt{n}} \sum_i e_i' w_i e_i + O_{p,\log} \left( \frac{T}{\sqrt{n}} + \sqrt{T} \right) \\ &= T \sqrt{n} E [e_i' w_i e_i] + O_p(T). \end{aligned}$$

Under  $\mathcal{H}_1$ , we have  $E [e_i' w_i e_i] > 0$ , since  $w_i$  is positive definite  $P$ -a.s., and  $e_i \neq 0$  with non-zero probability. Moreover,  $\tilde{\Sigma}_{\xi} = \Sigma_{\xi} + o_p(1)$ . Thus,  $\tilde{\Sigma}_{\xi}^{-1/2} \hat{\xi}_{nT} = T \sqrt{n} \left( \Sigma_{\xi}^{-1/2} E [e_i' w_i e_i] + o_p(1) \right)$ .