B.IX. Multivariate Extensions

Olivier Scaillet

University of Geneva and Swiss Finance Institute

Outline

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Introduction

Extension of AR process

$\Rightarrow$ Vector Autoregressive (VAR).

Y_t = vector of dimension n.

$$Y_t = \mu + \Psi_1 Y_{t-1} + ... + \Psi_p Y_{t-p} + \varepsilon_t.$$

Coefficients: $\Psi_i = n \times n$ matrices.

Mean: $\mu = n \times 1$ vector.

Error term: $\varepsilon_t = n \times 1$ vector, usually taken as normally distributed.

Introduction

Example: n = 2,

$$\begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \Psi_{1,11} & \Psi_{1,12} \\ \Psi_{1,21} & \Psi_{1,22} \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \dots + \begin{pmatrix} \Psi_{p,11} & \Psi_{p,12} \\ \Psi_{p,21} & \Psi_{p,22} \end{pmatrix} \begin{pmatrix} Y_{1,t-p} \\ Y_{2,t-p} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix}$$

$$\Rightarrow \begin{cases} Y_{1,t} = \mu_1 + \Psi_{1,11} Y_{1,t-1} + \Psi_{1,12} Y_{2,t-1} + \\ \quad \dots + \Psi_{p,11} Y_{1,t-p} + \Psi_{p,12} Y_{2,t-p} + \varepsilon_{1,t}, \\ Y_{2,t} = \mu_2 + \Psi_{1,21} Y_{1,t-1} + \Psi_{1,22} Y_{2,t-1} + \\ \quad \dots + \Psi_{p,21} Y_{1,t-1} + \Psi_{p,22} Y_{2,t-1} + \varepsilon_{2,t} \end{cases}$$

VAR has the form of a system of stacked linear regressions with the same set of explanatory variables in each regression.

VAR Estimation

Maximum likelihood estimation under normality assumption will be numerically identical to estimation by OLS equation by equation.

$\Rightarrow$ Seemingly Unrelated REgressions (SURE) method.

Thus VAR are extremely easy to estimate since it only requires a regression software.

If we do not use the same set of explanatory variables, estimation by OLS equation by equation will be biased.

Order of a VAR

The order of a VAR can be obtained by *likelihood ratio (LR) tests*.

Hypothesis of p_0 lags against p_1 lags with $p_1 > p_0$:

1. Estimate by OLS both models and derive the estimated residuals.
2. Compute the covariance matrices of residuals and the loglikelihoods

$$L(\hat{\Omega}_i) = (-Tn/2)\ln(2\pi) + (T/2)\ln\left(\left|\hat{\Omega}_i^{-1}\right|\right) - (Tn/2).$$

LR test statistic:

$$2\left(L(\hat{\Omega}_1) - L(\hat{\Omega}_0)\right) \sim \chi^2(n^2(p_1 - p_0)).$$

Goodness-of-Fit Criteria

We should balance good model fit and complexity (overfitting).

Choose the order p to minimise one of the following criteria:

- Akaike Information Criterion (AIC) - Tends to overestimate p:

$$C(p) = T \ln |\Omega| + 2(n^2 p).$$

- Bayesian Information Criterion (BIC):

$$C(p) = T \ln |\Omega| + (n^2 p) \ln(T).$$

- Hannan-Quin criterion(HQ):

$$C(p) = T \ln |\Omega| + 2(n^2 p) \ln(\ln(T)).$$

Granger Causality

It helps to see whether a variable can help to forecast another one.

Bivariate example:

$$\begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \Psi_{1,11} & \Psi_{1,12} \\ \Psi_{1,21} & \Psi_{1,22} \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{1,t} \end{pmatrix}$$

We want to test whether Y_2 does not cause Y_1, i.e. $\psi_{1,12} = 0$.
We need to regress both the general model

$$Y_{1,t} = \mu_1 + \psi_{1,11} Y_{1,t-1} + \psi_{1,12} Y_{2,t-1} + \varepsilon_{1,t}$$

and the restricted model

$$Y_{1,t} = \mu_1 + \psi_{1,11} Y_{1,t-1} + \eta_{1,t}$$

and compare the sum of squared residuals

$$\frac{T(SSR(\eta) - SSR(\varepsilon))}{SSR(\varepsilon)} \sim \chi^2(1).$$

Difficulties

Lots of parameters to estimate, as illustrated by the following examples:

- VAR(1):

$$Y_t = \mu + \Psi_1 Y_{t-1} + \varepsilon_t$$

where $V[\varepsilon_t] = \Sigma$.

$\Rightarrow$ Total number of parameters in this model is given by

$$n + n^2 + \frac{n(n+1)}{2}.$$

So, 5 assets = 45 parameters!

Difficulties

- Multivariate GARCH(1,1):

$$\mathrm{vech}\ (H_t) = \mathrm{A} + B\ \mathrm{vech}\ (H_{t-1}) + C\ \mathrm{vech}\ \left(\varepsilon_{t-1}\varepsilon_{t-1}^{'}\right)$$

where

- vech stacks the lower-triangular part of a squared matrix (nxn) into a n(n + 1)/2 Ã 1 vector.
- A = vector of size $n(n+1)/2$. $B, C = (n(n+1)/2) \times (n(n+1)/2)$matrices.

$\Rightarrow$ Total number of parameters:

$$\frac{n(n+1)}{2} + \frac{n^2(n+1)^2}{4} + \frac{n^2(n+1)^2}{4}.$$

5 assets = 465 parameters!!

Difficulties

- Multivariate stochastic volatility:

$$\begin{aligned} Y_t =& \mu + \exp(H_t/2)\varepsilon_t, V[\varepsilon_t] = \Sigma_\varepsilon \\ H_t =& BH_{t-1} + \eta_t, V[\eta_t] = \Sigma_\eta \end{aligned}$$

$\mu = n \times 1$ vector, $B = n \times n$ matrix

$\Rightarrow$ Number of parameters:

$$n + n^2 + \frac{n(n+1)}{2} + \frac{n(n+1)}{2}$$

5 assets = 60 parameters!

Possible Solutions

Solutions to inflation of parameters:

- Impose some restrictions:
  - Diagonality on parameter matrices.
  - Symmetry on parameter matrices, etc