B.VI. ARCH Models

Olivier Scaillet

University of Geneva and Swiss Finance Institute

Outline

Introduction

In linear autoregressive models

$$Y_t = \mu + \omega_1 Y_{t-1} + \ldots + \omega_p Y_{t-p} + \varepsilon_t,$$

the conditional variance is constant equal to the variance σ^2 of the noise (homoscedasticity)

$$V\left[Y_t \,|\, Y_{t-1}, \ldots, \; Y_{t-p}\right] = \sigma^2$$

Empirical evidence: *volatility clustering*
Large, resp. small, movements tend to be followed by large, resp. small, movements.

ARCH Models

Idea:

Introduce a linear dependence structure so volatility depends on the past

$\Rightarrow$*autoregressive conditional heteroscedasticity (ARCH).*

Unconditional variance does not recognize that there may be predictible patterns in stock market volatility.

Financial asset prices seem to go through long periods of high volatility and long periods of low volatility (persistence).

In periods of volatile prices, investors should either exit the market or require a large premium as compensation for bearing unusually high risk.

Here we analyze models of conditional (on information available at date $t-1$) volatility.

ARCH(p) Model

ARCH model: ARCH(p)

$$Y_t = \sqrt{h_t}\varepsilon_t$$

where $h_t = V\left[Y_t \,|Y_{t-1}, \ldots,\; Y_{t-p}\right] = c + \sum_{j=1}^{p} a_j Y_{t-j}^2$ and $\varepsilon_t \sim N(0,1)$

The conditional variance is a linear function of the past squared variables.

If past values are large, resp. small, in absolute values, so will be the conditional variance (clustering).

We need to impose positivity of coefficients to guarantee positivity of conditional variance.

ARCH(p) Model

Remark:
The unconditional (marginal) variance is constant even if conditional variance is not. Indeed, from the law of total variance

$$V\left[Y_t\right] = E\left[V\left[Y_t \,\middle|\, \underline{Y_{t-1}}\right]\right] + V\left[E\left[Y_t \,\middle|\, \underline{Y_{t-1}}\right]\right].$$

where

$$\underline{Y_{t-1}} := Y_{t-1}, \ldots, Y_{t-p}.$$

We observe first that

$$E\left[V\left[Y_t \,\middle|\, \underline{Y_{t-1}}\right]\right] = E\left[c + \sum_{j=1}^{p} a_j Y_{t-j}^2\right] = c + \sum_{j=1}^{p} a_j E\left[Y_{t-j}^2\right],$$

and, moreover,

$$V\left[E\left[Y_t \,\middle|\, \underline{Y_{t-1}}\right]\right] = V\left[0\right] = 0.$$

ARCH(p) Model

On the other hand, $\forall j$

$$E\left[Y_{t-j}^2\right] = V\left[Y_{t-j}\right]$$

and by covariance stationarity

$$V\left[Y_{t-j}\right] = V\left[Y_t\right].$$

Hence, from the result on the previous slide we get

$$V\left[Y_t\right] = c + \sum_{j=1}^{p} a_j V\left[Y_t\right].$$

So, the unconditional variance is

$$V\left[Y_t\right] = \frac{c}{1 - \sum_{j=1}^{p} a_j}.$$

GARCH(p,q) Models

Problem with ARCH: When estimated on financial data, ARCH models usually involve large number of p of significant lags.

GARCH model: GARCH(p,q), i.e., Generalized ARCH

$$Y_t = \sqrt{h_t}\varepsilon_t$$

where

$$h_t = V\left[Y_t \,|\, Y_{t-1}, \ldots,\ Y_{t-p}\right] = c + \sum_{j=1}^{q} a_j Y_{t-j}^2 + \sum_{l=1}^{p} b_l h_{t-l}.$$

When estimated on financial data, usually a GARCH(1,1) gives a good fit (parsimonious model).

GARCH(p,q)

The GARCH(p,q) model can be rewritten as an ARMA model on squared observations

$$Y_t^2 = c + \sum_{j=1}^{\max(p,q)} (a_j + b_j) Y_{t-j}^2 + u_t - \sum_{l=1}^{q} b_l u_{t-l}$$

with $u_t = Y_t^2 - h_t$.

This relationship helps the identification of the orders p and q since we can use the tools used in ARMA modeling.

Remark:
We may use the GARCH specification directly on the innovation term of a given model to introduce conditional heteroscedasticity.

Examples

- ARMA-GARCH:

$$\Omega(L)Y_t = \mu + \Theta(L)\varepsilon_t$$

where ε_t follows a GARCH process $\varepsilon_t = \sqrt{h_t}\upsilon_t$ and $h_t = V\left[\varepsilon_t \left|\varepsilon_{t-1}, ..., \varepsilon_{t-p}\right.\right] = c + \sum_{j=1}^{q} a_j\varepsilon_{t-j}^2 + \sum_{l=1}^{p} b_l h_{t-l}$.

- Linear regression with GARCH errors

$$Y_t = X_t\beta + \varepsilon_t$$

where ε_t follows a GARCH process.

Detection of ARCH/GARCH Effects

a) Practical 2-step procedure

1. Specify form of conditional mean (ex. arma or linear regression) and estimate parameter.
2. Compute the residuals.
3. Analyse the ARMA properties of squared residuals i.e. computation of correlograms and partial correlograms of squared residuals and indentification of order p,q as for standard ARMA models.

Detection of ARCH/GARCH Effects

b) Formal test for ARCH(p)

1. Run a regression of squared residuals e_t^2 on $e_{t-1}^2, \ldots, e_{t-p}^2$.
2. Compute TR^2 where R^2 is the squared multiple correlation coefficient of the regression.
3. Reject the null hypothesis of no ARCH effects with lag p at level α if TR^2 is larger than quantile of level $1 - \alpha$ of chi-square distribution with p degrees of freedom.

Empirical Example

Example:
Dollar/sterling exchange rate from January 1974 and December 1994 (5192 observations)

Loss criteria for comparison:

$$L_1 = \sum_{t=1}^{T} \left(e_t^2 - \sigma_t^2\right)^2, L_2 = \sum_{t=1}^{T} \left(\ln(e_t^2/\sigma_t^2)\right)^2$$

GARCH Extensions

Modifications of GARCH(1,1) process:

Impact of news, i.e. ε_{t-1}, is symmetric on volatility due to square, thus large positive shocks will have the same impact on volatility as large negative shocks.

The model does not account for *leverage effects.*

Black (1976): tendency for stock prices to be negatively correlated with changes in stock volatility.

A firm with outstanding debt and equity typically becomes more highly leveraged when the value of the firm falls.

This raises the equity (stock price) volatility since the firm becomes more risky.

GARCH Extensions

Extensions of GARCH include

- Exponential: (EGARCH) modelling of the logarithm of conditional variance which includes absolute values of the noise.
- Threshold: (TARCH) parameter value depends on whether shock is positive (above zero) or not.
- Switching regimes: (SWARCH) another process determines whether the volatility follows one type (regime) of GARCH process or another one.