

B.V. Conditional VaR and ES

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Outline

- 1 Conditional VaR
- 2 Conditional ES
- 3 Empirical Illustration
- 4 Block Bootstrap

Conditional VaR

VaR may be made time dependent.

We will condition on past values

$$P \left[-a' Y_t > VaR_t(a, \alpha) \mid a' Y_{t-1}, \dots, a' Y_{t-p} \right] = \alpha.$$

We need to estimate the conditional pdf.

Nonparametric Estimation of Conditional VaR

The conditional pdf may be estimated nonparametrically

$$\begin{aligned} \hat{f}(z | a'Y_{t-1} = y_1, \dots, a'Y_{t-p} = y_p) \\ = \frac{\frac{1}{Th^{p+1}} \sum_{t=1}^T K\left(\frac{-a'y_t - z}{h}\right) \prod_{j=1}^p K\left(\frac{a'y_{t-j} - y_j}{h}\right)}{\frac{1}{Th^p} \sum_{t=1}^T \prod_{j=1}^p K\left(\frac{a'y_{t-j} - y_j}{h}\right)} \end{aligned}$$

We then solve to find estimate $\widehat{VaR}_t(a, \alpha)$

$$\int_{\widehat{VaR}_t(a, \alpha)}^{+\infty} \hat{f}(z | a'Y_{t-1} = y_1, \dots, a'Y_{t-p} = y_p) dz = \alpha$$

Nonparametric Estimation of Conditional VaR

Remark: Using the Gaussian kernel we get

$$\int_{\widehat{VaR}_t(a,\alpha)}^{+\infty} \hat{f}(z | a'Y_{t-1} = y_1, \dots, a'Y_{t-p} = y_p) dz$$

$$= \frac{\frac{1}{Th^p} \sum_{t=1}^T \Phi\left(\frac{-a'y_t - \widehat{VaR}_t(a,\alpha)}{h}\right) \prod_{j=1}^p \varphi\left(\frac{a'y_{t-j} - y_j}{h}\right)}{\frac{1}{Th^p} \sum_{t=1}^T \prod_{j=1}^p \varphi\left(\frac{a'y_{t-j} - y_j}{h}\right)}$$

Conditional ES

ES may also be made time dependent

$$ES_t(a, \alpha) = E \left[-a' Y_t \mid -a' Y_t > VaR_t(a, \alpha), a' Y_{t-1}, \dots, a' Y_{t-p} \right]$$

Kernel estimator:

$$\begin{aligned} \widehat{ES}_t(a, \alpha) \\ = \frac{\frac{1}{Th^{p+1}} \sum_{t=1}^T (-a' y_t) \int_{\widehat{VaR}_t(a, \alpha)}^{\infty} K\left(\frac{-a' y_t - z}{h}\right) dz \prod_{j=1}^p K\left(\frac{a' y_{t-j} - y_j}{h}\right)}{\alpha \frac{1}{Th^p} \sum_{t=1}^T \prod_{j=1}^p K\left(\frac{a' y_{t-j} - y_j}{h}\right)} \end{aligned}$$

Conditional ES

Remark: In case a Gaussian kernel is used

$$\begin{aligned} \widehat{ES}_t(a, \alpha) \\ = \frac{\frac{1}{Th^p} \sum_{t=1}^T (-a' y_t) \Phi\left(\frac{-a' y_t - \hat{V} a R_t(a, \alpha)}{h}\right) \prod_{j=1}^p \varphi\left(\frac{a' y_{t-j} - y_j}{h}\right)}{\alpha \frac{1}{Th^p} \sum_{t=1}^T \prod_{j=1}^p \varphi\left(\frac{a' y_{t-j} - y_j}{h}\right)} \end{aligned}$$

Empirical Illustration

Daily Return of Stock Indices

- Sample:
 - Indices: CAC40, DAX30, S&P500, DJI, Nikkei225.
 - From 03/01/1994 to 07/07/2000, 1700 observations.
- 90% pointwise confidence bands.
- Computed by block bootstrap (block length = $T^{1/3} = 11$).

Block Bootstrap

Idea of block bootstrap:

- 1 Divide sample into blocks of given length.
- 2 Resample blocks to build new data sample of length T (concatenate the blocks) and reestimate quantities of interest (here VaR or ES).
- 3 Repeat (2) a sufficiently large number of times.
- 4 Compute quantiles of level $\alpha/2$ and $1 - \alpha/2$ (ex. 5% and 95%) of empirical distribution of estimates to build confidence intervals of level $1 - \alpha$ (ex. 90%).