B. Dynamic (Conditional) Analysis

B.I. Stationary Processes

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Outline

Dynamic (Conditional) Analysis

We will take into account the temporal dependence in data.

Information is revealed to the market.
$\Rightarrow$influence volatility

We face periods of high and low volatility.

There is occurrence of cycles.

Classical Statistics vs Time Series Analysis

Classical statistics:

Observations are *independent* and are obtained under *identical* conditions.

Thus observations are exchangeable and can be mixed.

We have N observations of a random variable X

The information on the distribution of the random variable X increases with the sample size N.

Classical Statistics vs Time Series Analysis

Time series:
We have *one* observation $(X_1, ..., X_N)$,

a *path* or a *time series* of a random process

Information bears on the distribution of the process and will increase with N, only if there exists some stable links between the realizations at different dates

$\Rightarrow$ notion of *stationarity*

Remark:
Since the index is time, ordering matters.

Stationary Processes: Preliminaries

Random process: Sequence of variables indexed by time

Notation: $Y = \{Y_t; t \in Z\}$ for discrete time process

Property: (Kolmogorov Theorem)
The distribution of Y is fully characterized by the distribution of all N-uples $(Y_{t_1}, ..., Y_{t_N})$.

Remark:
For a fixed t, Y_t is a random variable.
Letting t vary leads to paths of the process.

Strict (Strong) Stationarity

Definition:
Y is *strictly stationary* if the distributions of

$$(Y_{t_1}, \ldots, Y_{t_N}) \text{ and } (Y_{t_1+h}, \ldots, Y_{t_N+h})$$

coincide $\forall h \in Z$, $\forall t_1, \ldots, t_n, \forall n$.

Remark:
In particular, the variables Y_i share the same distribution.
Intuitively, the graphs of the series on two time intervals of same length should have similar characteristics.

Property:
Y strictly stationary
$\Rightarrow f(Y)$ strictly stationary.

Weak (Second Order) Stationarity: Preliminaries

Notion of weak stationarity is based on second order moments.

Let L^2 be the space of all square integrable variables, i.e. variables Y such that $E\left[Y^2\right] < \infty$. This space is equipped with a norm:

$$\|Y\| = \sqrt{E\left[Y^2\right]},$$

and a scalar product: $< X, Y >= E\left[XY\right]$

Hence, $X \in L^2$ and $Y \in L^2$ are *orthogonal*, denoted by $X \perp Y$, if $E\left[XY\right] = 0$

Remark: Convergence in L^2 of $Y_N \in L^2$ (mean square convergence)

$$Y_N \overset{L^2}{\Rightarrow} Y \text{ if } \|Y_N - Y\| \rightarrow 0 \text{ when } N \rightarrow \infty$$

Weak (Second Order) Stationarity: Definition

Definition:
Y is a *second order process* if $Y_t \in L^2, \forall t \in Z$. Such a process allows to consider:

$$E[Y_t], V[Y_t], \mathrm{Cov}[Y_t, Y_{t+h}],$$

since they are all finite.

Definition:
Y is *weakly stationary* if

1. Y is a second order process.
2. $E[Y_t] = \mu, \forall t \in Z$
3. $Cov[Y_t, Y_{t+h}] = \gamma(h), \forall t \in Z, \forall h \in Z$

The function $h \to \gamma(h)$ is called the *autocovariance function* of Y. Properties (2) and (3) above imply that the mean, variance, and autocovariances are independent of t.

Weak (Second Order) Stationarity: Properties

Autocovariance of Weakly Stationary Processes:

1. $\gamma(0) = V[X_t] \geq 0$.
2. $\gamma(h) = \gamma(-h)$ (even function).
3. $|\gamma(h)| \leq \gamma(0), \forall h$.

Strict vs Second Order Stationarity

Neither type of stationarity implies the other one:

- A strictly stationary process may not belong to L^2 (infinite variance)
- On the other hand, define a process by

$$\forall t, Y_{2t} = U, Y_{2t+1} = V$$

where $U \sim N(1,1)$, $V \sim Exp(1)$

The process is weakly stationary with $\gamma(h) = 1$ or 0 depending on parity of h, but Y_t and Y_{t+1} do not share the same distribution which implies that the process is not strictly stationary.

A Gaussian process is strictly stationary if and only if it is covariance stationary, because the multivariate normal distribution is characterized entirely by its mean and covariance matrix.

Constructing Weakly Stationary Processes

Property:

Let Y be a weakly stationary process. Let $(a_i)_{i \in Z}$ be a sequence of real numbers such that $\sum_{i=-\infty}^{+\infty} |a_i| < \infty$.

Then $X_t = \sum_{i=-\infty}^{+\infty} a_i Y_{t-i}$ defines a new weakly stationary process.

Computation of moments of X:

Let $\mu = E[Y_t]$ and γ be the autocovariance function of Y. We get

1. $E[X_t] = \mu \sum_{i=-\infty}^{+\infty} a_i$
2. $\mathrm{Cov}[X_t, X_{t+h}] = \sum_{i,j=-\infty}^{+\infty} a_i a_j \gamma(h+i-j), \forall h$

If Y is noise, $\mathrm{Cov}[X_t, X_{t+h}] = \sigma^2 \sum_{i=-\infty}^{+\infty} a_i a_{i+h}$, since variance $=$ 1 and autocovariances $=$0.

In particular, a finite linear combination of variables Y_{t-i} always defines a weakly stationary process.