A.XIII. Bootstrap Procedures

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Outline

Introduction

Asymptotic properties of estimators are valid when sample size is large, i.e. $T \to \infty$.

In small samples (finite distance), asymptotic properties may provide poor approximations of the real distribution of estimators.

In particular confidence intervals based on asymptotic normality may be too wide or too narrow and exhibit a wrong coverage probability.

We may use simulations to get a better approximation when samples are small.

Description of Bootstrap Procedures

1. Start from initial data $Y_1, ..., Y_T$ and compute the empirical cdf.
2. Generate ST independent drawings by sampling randomly in the initial data with replacement $Y_1^s, ..., Y_T^s$, $s = 1, ..., S$ (draw ST realizations of a uniform $[0, 1]$ variable and invert the empirical cdf).
3. For each simulated sample $s = 1, ..., S$ of length T, compute the estimate $\hat{\theta}^s$, for example the empirical mean $\hat{\theta}^s = \hat{m}^s = \frac{1}{T} \sum_{t=1}^{T} Y_t^s$.
4. The empirical distribution of the estimates $\hat{\theta}^s$, $s = 1, ..., S$, is a good approximation to the true distribution of the estimator in small sample (consistent as $s \to \infty$).

Pseudo random number generators are available in most softwares.

Non-Uniform Random Number Generation

Uniform random drawings are usually obtained through pseudo random generators which mimic randomness even if the recursive equation is deterministic.

The recursion requires an initial value called the seed.

Once such drawings are available, we may transform them to get drawings from nonuniform distributions.

Two methods are used, namely, the transformation method or the rejection method. Two examples of the transformation method are shown next.

Transformation Method Example I

If F is a cdf, then the equation $F(u) = \eta$ implies that $\eta \in [0, 1]$.

Hence by drawing $\tilde{\eta}$ from a uniform $[0, 1]$ distribution and using the inverse function F^{-1} of F, we will get a draw $\tilde{u} = F^{-1}(\tilde{\eta})$ coming from the distribution F.

In the case of an exponential random variable:

- pdf: $f(u) = \theta \exp(-\theta u)$
- cdf: $F(u) = 1 - \exp(-\theta u)$

$\Rightarrow$ Hence, $u = F^{-1}(\eta) = \frac{-1}{\theta} \ln(1 - \eta)$

Transformation Method Example II

Box Muller method for normal random variables:

If η_1 and η_2 are independent random variates from $U[0,1]$, then the variates

$$u_1 = (-2\ln(\eta_1))^{1/2}\cos(2\pi\eta_2),$$

$$u_2 = (-2\ln(\eta_1))^{1/2}\sin(2\pi\eta_2),$$

are independent random variates from $N(0,1)$

Once a $n \times 1$ vector u of independent random $N(0,1)$ variates is available, i.e., $u \sim N(0, Id)$, we have

$$\zeta = \mu + \Omega^{1/2}u \sim N(\mu, \Omega),$$

where $\Omega^{1/2}$ is the square root of the matrix Ω obtained by Cholesky decomposition, i.e., $\Omega^{1/2}\Omega^{1/2} = \Omega$