A.VII. Life Cycle Models and CCAPM

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Outline

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2. Implications: CCAPM
3. Econometrics of the CCAPM: Nonlinear Least Squares and GMM

Life Cycle Models

Life cycle models:
We start from the *intertemporal optimization* of consumption by an economic agent.

At date t, the objective of the agent assumed to enjoy an infinite life (for mathematical tractability) is to maximize the expected utility associated with its future consumption plan

$$E_t \left[\sum_{\tau=1}^{\infty} \delta^{\tau} u(c_{t+\tau}) \right]$$

where E_t denotes the expectation conditionally to what is known by the agent up to time t.

Life Cycle Models

The parameter δ is a *discount factor* which defines the preference towards the present.

The utility function u is strictly concave which translates into *risk aversion*.

This problem makes sense if the agent has access to some financial assets which allows him to defer his consumption, i.e. invest in a financial asset in order to consume tomorrow.

Life Cycle Models

Let us consider n financial assets with random returns $Y_{i,t}$.

1. If the agent invests an additional small quantity ε in asset i at date t, he will need to reduce his consumption by that quantity. Thus, by Taylor expansion, he looses at date t a utility

$$u(c_t) - u(c_t - \varepsilon) \cong u'(c_t)\varepsilon$$

2. On the contrary at date $t+1$, by selling the asset he will be able to increase his consumption by

$$(1 + Y_{i,t+1})\varepsilon$$

Thus, by Taylor expansion he will gain at date $t+1$ a utility

$$u(c_{t+1} + (1 + Y_{i,t+1})\varepsilon) - u(c_{t+1}) \cong u'(c_{t+1})(1 + Y_{i,t+1})\varepsilon$$

Euler Equation

Discounting and taking the expectation of the utility gain at date t yields

$$\delta E_t \left[u'(c_{t+1})(1+Y_{i,t+1})\varepsilon \right].$$

If the agent behaves optimally at date t , the difference between utility gains $\delta E_t \left[u'(c_{t+1})(1+Y_{i,t+1})\varepsilon \right]$ and looses $-u'(c_t)\varepsilon$ should be zero. Hence,

$$u'(c_t) = \delta E_t \left[u'(c_{t+1})(1+Y_{i,t+1}) \right].$$

This equation is called a ***Euler equation***, and corresponds to the first order condition (FOC) of the intertemporal maximization problem of the agent.

Euler Equation

Usually the Euler equation is rewritten as

$$1 = E_t \left[M_{t+1}(1 + Y_{i,t+1}) \right],$$

where $M_{t+1} = \delta \frac{u'(c_{t+1})}{u'(c_t)}$ is called the *stochastic discount factor* (c_{t+1} is random) or *pricing kernel.*

It corresponds here to the discounted ratio of marginal utilities called the *intertemporal marginal rate of substitution.*

It is always positive since marginal utilities are.

CCAPM

CCAPM:
Assume that there exists a riskless asset whose certain return is $Y_{0,t+1}$. Then the Euler equation for this asset is

$$1 = (1 + Y_{0,t+1})E_t\left[M_{t+1}\right](*)$$

Since by definition of a covariance

$$E_t\left[M_{t+1}(1 + Y_{i,t+1})\right] = E_t\left[M_{t+1}\right]E_t\left[(1 + Y_{i,t+1})\right] + Cov_t\left[M_{t+1}, Y_{i,t+1}\right],$$

the Euler equation for a risky asset can be rewritten as

$$1 = E_t\left[M_{t+1}\right]E_t\left[(1 + Y_{i,t+1})\right] + Cov_t\left[M_{t+1}, Y_{i,t+1}\right](**)$$

CCAPM

Use of (*) and (**) gives

$$E_t\left[Y_{i,t+1}\right] - Y_{0,t+1} = -(1 + Y_{0,t+1})Cov_t\left[M_{t+1}, Y_{i,t+1}\right]$$

or

$$E_t\left[Y_{i,t+1}\right] - Y_{0,t+1} = -\frac{Cov_t\left[u'(c_{t+1}), Y_{i,t+1}\right]}{E_t[u'(c_{t+1})]}.$$

If an asset is negatively correlated with the marginal utility of consumption its expected excess return will be high.

This asset doesn't insure well against utility variations and thus investors require a high return on it as compensation.

CCAPM

The last equation can also be written

$$E_t\left[Y_{i,t+1}\right] - Y_{0,t+1} = \left(\frac{Cov_t\left[u'(c_{t+1}), Y_{i,t+1}\right]}{V_t[u'(c_{t+1})]}\right)\left(-\frac{V_t[u'(c_{t+1})]}{E_t\left[u'(c_{t+1})\right]}\right)$$

or

$$E_t\left[Y_{i,t+1}\right] - Y_{0,t+1} = \beta_{i,t}\lambda_t$$

where $\beta_{i,t} = \frac{Cov_t\left[u'(c_{t+1}), Y_{i,t+1}\right]}{V_t[u'(c_{t+1})]}$ is the *consumption beta* and $\lambda_t = -\frac{V_t[u'(c_{t+1})]}{E_t[u'(c_{t+1})]}$ is the *risk premium.*

At optimum excess returns are proportional to their betas.

Econometrics of the CCAPM

Econometrics of CCAPM:

Two objectives:

1. Estimate parameter of the utility function, for example degree of risk aversion,
2. Test of validity of CCAPM,i.e. test whether it is is a good model to describe how stock prices behave.

Nonlinear Least Squares

Nonlinear least squares:
Usually the stochastic discount factor is parametrised $M_{t+1}(\theta)$, due to the specification of a *parametric utility function.*

For example we may use a time-separable power utility function

$$u(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta}$$

where θ is the coefficient of risk aversion.

Then we could think of replacing the *theoretical* Euler equation

$$1 = E_t \left[M_{t+1}(\theta)(1 + Y_{i,t+1}) \right]$$

by the nonlinear regression

$$1 = M_{t+1}(\theta)(1 + Y_{i,t+1}) + \varepsilon_{t+1}$$

where ε_{t+1} is an error term satisfying $E_t \varepsilon_{t+1} = 0$.

Nonlinear Least Squares

The parameter θ can be estimated by minimizing the nonlinear least squares criterion:

$$\hat{\theta} = \arg\min \sum_{t=1}^{T} \left(1 - M_{t+1}(\theta)(1 + Y_{i,t+1})\right)^2 .$$

However this strategy is usually not adopted since the regressors involved in $M_{t+1}(\theta)(1 + Y_{i,t+1})$ and the error term ε_{t+1} have no a priori reason to be uncorrelated.

This condition is necessary to get consistency of $\hat{\theta}$.

GMM

Hansen (1982) Generalised Method of Moment (GMM):

From the properties of the conditional expectation:

If $E[Y|X] = 0$ then $E[Yf(X)] = 0$ for any f.

Using the Euler equation, this leads for any X_t, which belongs to the information set available to the agent at date t, to

$$E[(M_{t+1}(\theta)(1 + Y_{i,t+1}) - 1) X_t] = 0.$$

GMM

The idea behind GMM is to use the empirical counterpart of the expectation, i.e. the function

$$m(\theta) = \frac{1}{T}\sum_{t=1}^{T}\left(M_{t+1}(\theta)(1+Y_{i,t+1})-1\right)X_t$$

and find the value of the parameter so that this quantity is close to zero using a quadratic norm indexed by a matrix Ω.

$$\hat{\theta} = \arg\min m(\theta)'\Omega m(\theta)$$

This estimator is consistent and asymptotically normal.

GMM

There exists also an optimal choice for the matrix Ω in the sense of having the best attainable precision (smallest asymptotic variance).

Since the GMM criterion is usually highly nonlinear, the estimator $\hat{\theta}$ does not admit a closed form solution obtained through solving the first order condition.

N*umerical optimization algorithms* are called for.

Equity Premium Puzzle

Empirical studies based on aggregate consumption data and stock index returns have shown that the CCAPM does not fully explain the difference between stock returns and the risk free rate.

This difference is in practice large and can only be justified by the CCAPM if very high values for the risk aversion are used.

This phenomenon is known as the

equity risk premium puzzle (Mehra and Prescott (1985))

Several extensions have been proposed in order to achieve better life cycle models, mainly through use of more general utility functions.