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A.IV. Linear regression

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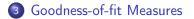
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Outline









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Intro	duction		

A linear regression is defined by

$$Y_t = X_t\beta + \varepsilon_t$$
$$= \sum_{k=1}^{K} X_{k,t}\beta_k + \varepsilon_t$$

where

 ε_t = noise with mean zero, i.e., innovation or error term,

$$\begin{aligned} X_t \beta &= E\left[Y_{i,t} \left| X_t\right] \right] \\ &= \text{conditional mean of } Y_t \\ &= \text{linear function of } X_t \end{aligned}$$

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 β = parameter (estimation by OLS)

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OLS	5 Estimator		

The OLS estimator is the value of β which minimizes the sum of squared residuals

$$\hat{\beta} = \arg\min \sum_{t=1}^{T} (Y_t - X_t \beta)^2$$
$$= \left[\sum_{t=1}^{T} X'_t X_t\right]^{-1} \sum_{t=1}^{T} X'_t Y_t$$

The OLS residuals are given by $\hat{\varepsilon}_t = Y_t - X_t \hat{\beta}$

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OL	S Estimator		

Matrix notation:

$$\begin{array}{l}
\mathbf{Y} \\
\mathbf{Y} \\
(T \times 1) \\
\begin{pmatrix}
\mathbf{Y}_{1} \\
\mathbf{Y}_{2} \\
\vdots \\
\mathbf{Y}_{T}
\end{pmatrix}, \\
\begin{pmatrix}
\mathbf{X} \\
(T \times K) \\
(T \times K) \\
\begin{pmatrix}
\mathbf{X}_{1} \\
\mathbf{X}_{2} \\
\vdots \\
\mathbf{X}_{T}
\end{pmatrix}, \\
\begin{pmatrix}
\varepsilon \\
(T \times 1) \\
\varepsilon \\
(T \times 1) \\
\varepsilon \\
\varepsilon \\
\varepsilon \\
\end{bmatrix}, \\
\begin{pmatrix}
\varepsilon \\
\varepsilon \\
\varepsilon \\
\varepsilon \\
\varepsilon \\
\end{bmatrix}, \\
\hat{\beta} \\
= (X'X)^{-1}X'Y \\
\hat{\varepsilon} \\
= \mathbf{Y} - X\hat{\beta}
\end{array}$$

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Inter	rpretation		

Interpretation:

OLS consists in decomposing Y into two *orthogonal projections*, one on the space spanned by the columns of X, and the other on the space of innovations.

Let $P_X = X(X'X)^{-1}X'$ be the orthogonal projection matrix associated with the regressors, and

$$M_X = Id - P_X,$$

then

$$Y = P_X Y + M_X Y$$

= $X(X'X)^{-1}X'Y + (Id - X(X'X)^{-1}X')Y$
= $X\hat{\beta} + (Y - X\hat{\beta})$
= $X\hat{\beta} + \hat{\varepsilon}$

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Goodness-	of-fit Measures		

When there is a constant in the model (first element of X_t is 1 for each t), a measure of goodness-of-fit is

$$R^{2} = \frac{\sum_{t=1}^{T} \left(\hat{Y}_{t} - \bar{Y}\right)^{2}}{\sum_{t=1}^{T} \left(Y_{t} - \bar{Y}\right)^{2}} = \frac{\text{EXPLAINED VARIANCE}}{\text{TOTAL VARIANCE}}$$

It satisfies $0 \le R^2 \le 1$.

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Statistical Inference

Statistical inference:

H1: a) X_t is deterministic b) ε_t is i.i.d. with mean 0 and variance σ^2 c) ε_t is Gaussian

$\hat{\beta}$ is normally distributed with mean β and covariance matrix $\sigma^2 (X'X)^{-1}$

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Fit Measures

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Gauss-Markov Theorem

Gauss Markov theorem: OLS estimator

= Best Linear Unbiased Estimator (BLUE)

OLS estimator has the smallest variance (largest precision) among all unbiased estimators linear in Y.

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t-Tests:

$$\begin{cases} H_0: \beta_k = b \\ H_1: \beta_k \neq b \end{cases}$$

For example, take b = 0 to obtain the test of the significance of the presence of the *k*-th regressor in explaining *Y*.

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t-Statistic			

t-Statistic:
$$t = \frac{\hat{eta}_k - b}{\hat{\sigma}_{\hat{eta}_k}}$$
, where

$$\hat{\sigma}_{\hat{\beta}_k} = \sqrt{s^2 \varsigma_{ii}},$$

with

$$s^{2} = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{T-K}$$

 $\varsigma_{ii} = \text{i-th diagonal element of } (X'X)^{-1}$

Reject the null hypothesis H_0 at level α if

$$|t| > t_{1-\alpha/2}$$

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where $t_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of a student distribution with swiss:fin Theerma Kutedegrees of freedom. Geneva Finance REFEARCH INSTITUTE

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E tests			

F tests:

$$\begin{cases} H_0: R\beta = r \\ H_1: R\beta \neq r \end{cases}$$

Test of *m* linear restrictions on β

$$R = (m imes K)$$
 matrix, $r = (m imes 1)$ vector

F statistic:

$$F = \frac{(R\hat{\beta} - r) \left(R(X'X)^{-1}R' \right)^{-1} (R\hat{\beta} - r)}{s^2 m},$$

Reject the null hypothesis H_0 at level α if

 $F > F_{1-\alpha}$ where $F_{1-\alpha}$ is the quantile of level $1 - \alpha$ of a Fisher variable with m and T - K degrees of freedom. Intro

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Other Sets of Assumptions

Other sets of assumptions: H2: a) X_t is stochastic and independent of ε_s , $\forall s$ b) ε_t is i.i.d. with mean 0 and variance σ^2 c) ε_t is Gaussian

 $\hat{\beta}$ is normally distributed with mean β and covariance matrix $\sigma^2(X'X)^{-1}$, but conditionally to X (when X is treated as fixed). Unconditionally, $\hat{\beta}$ is no longer normally distributed. However the distributions for t and F remain valid.

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Remark			

Remark: For large T (asymptotic theory), in H2, we may drop the assumption of Gaussian innovations, but the distributions for t and F become

- a) standard normal for t
- b) chi-square with m degrees of freedom for F