A.III. Asymptotic properties

Olivier Scaillet

University of Geneva and Swiss Finance Institute

Outline

Introduction

The *asymptotic properties* concern the properties of the estimators when the sample size is large, i.e. $T \to \infty$.

The estimators are usually convergent and asymptotically normal.

It comes from the application of the *law of large numbers* and of the *central limit theorem*.

Empirical Mean

1. Empirical mean:

$$\sqrt{T}(\hat{m} - m) \Rightarrow N(0, \Sigma)$$

The empirical mean when T becomes large follows a normal distribution, centered on the true unknown mean and with covariance matrix equal to the true unknown covariance matrix of the random variable.

Using an estimate $\hat{\Sigma}$ of Σ, for example the empirical covariance matrix, we may construct asymptotic confidence intervals.

An *asymptotic confidence interval* $\Im_{1-\alpha}$ of level $1 - \alpha$ is

$$\Im_{1-\alpha} = \left[\hat{m} - \frac{z_{\alpha/2}}{\sqrt{T}}\hat{\Sigma}^{1/2}, \hat{m} + \frac{z_{\alpha/2}}{\sqrt{T}}\hat{\Sigma}^{1/2}\right],$$

Empirical Mean

where $z_{\alpha/2}$ is the quantile of level $1-\alpha/2$ of a standard normal distribution

$$P\left[N(0,1) < z_{\alpha/2}\right] = 1-\alpha/2.$$

Asymptotically, the probability that the true unknown mean belongs to the interval $\Im_{1-\alpha}$ defined by the lower bound

$$\hat{m} - \frac{z_{\alpha/2}}{\sqrt{T}}\hat{\Sigma}^{1/2}$$

and the upper bound

$$\hat{m} + \frac{z_{\alpha/2}}{\sqrt{T}}\hat{\Sigma}^{1/2}$$

is equal to $1-\alpha$.

Empirical CDF

2. Empirical cumulative distribution function:

$$\sqrt{T}(\hat{F}_i(\zeta_i) - F_i(\zeta_i)) \Rightarrow N(0, F_i(\zeta_i)(1 - F_i(\zeta_i)))$$

The asymptotic variance $F_i(\zeta_i)(1 - F_i(\zeta_i)$ can be estimated by $\hat{F}_i(\zeta_i)(1 - \hat{F}_i(\zeta_i)$.

Kernel Density Estimator

3. Kernel density estimator:

For $h \to 0$ and $Th \to \infty$ when $T \to \infty$

$$\sqrt{Th}\left(\hat{f}_i(\zeta_i) - f_i(\zeta_i)\right) \Rightarrow N(0, f_i(\zeta_i) \int_{-\infty}^{+\infty} K^2(u)du)$$

For a Gaussian kernel we have:

$$\int_{-\infty}^{+\infty} K^2(u)du = \frac{1}{2\sqrt{\pi}}$$

Hence the asymptotic precision measured by the asymptotic variance $f_i(\zeta_i) \int_{-\infty}^{+\infty} K^2(u)du$ depends on the choice of the kernel and can be estimated by $\hat{f}_i(\zeta_i) \int_{-\infty}^{+\infty} K^2(u)du$.

Kernel Density Estimator

Note that kernel estimators converge at a slower rate due to the presence of smoothing.

Indeed since $h \to 0$, $\sqrt{T} > \sqrt{Th}$.

We need more data to compensate for the presence of $h \to 0$ to get the same level of asymptotic precision.