B.X. Nonstationary Series

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Outline

Introduction

A weakly stationary time series has a constant mean, a constant variance and the covariance is independent of time

Quick rule of thumb:

- Plot the series against time
- If path crosses the sample mean many times, chances are that the variable is stationary
- Otherwise that is an indication of a persistent trend away from the mean of the series

Trend Stationary Variables

A *trend stationary variable* is a variable whose mean grows around a fixed trend

Concept covers economic time series that grow at a constant rate

Series tend to evolve around a steady state, upward sloping curve without big swings away from that curve

Detrending the series will give a stationary process

Trend Stationarity Example

$$Y_t = a + bt + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma^2)$$

The mean varies with time but the variance is constant

$$E[Y_t] = E[a + bt + \varepsilon_t] = a + bt,$$

$$V[Y_t] = E\left[(Y_t - E[Y_t])^2\right] = E\left[(a + bt + \varepsilon_t - (a + bt))^2\right] = \sigma^2$$

So, $Y_t^* = Y_t - (a + bt)$ will be stationary.

Unit Roots

An autoregressive process of order p has a unit root if the autoregressive polynomial in the lag operator L, $1 - \omega_1 L - \ldots - \omega_p L^p$, has a root equal to one.

The simplest example is the *random walk*, i.e. an AR(1) with a unit root

$$Y_t = Y_{t-1} + \varepsilon_t, (*)$$

using $Y_{t-1} = Y_{t-2} + \varepsilon_{t-1}$, replacing in (*) and iterating gives

$$Y_t = Y_0 + \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_{t-1} + \varepsilon_t.$$

Unit Roots

The mean will be

$$Y_t = E\left[Y_0 + \varepsilon_1 + \varepsilon_2 + ... + \varepsilon_{t-1} + \varepsilon_t\right] = Y_0.$$

The variance will be

$$Y_t = V\left[Y_0 + \varepsilon_1 + ... + \varepsilon_t\right] = V\left[\varepsilon_1\right] + ... + V\left[\varepsilon_t\right] = t\sigma^2$$

When we move further into the future ($t \rightarrow \infty$) the variance becomes infinite.

A unit root process only crosses the sample mean very infrequently, and the process experiences long positive and negative strays from the sample mean (explosive behavior)

The random walk does not exhibit a mean reverting behavior

Integrated Processes

- A process that has a unit root is also called *integrated of order one*, denoted I(1)
    - We need to differentiate once the process to get a new stationary process
- A stationary process is I(0)
    - We do not need to differentiate

ARIMA Processes

Take first differences of series and analyze the differenced process

ARIMA(p,d,q)

$$\Omega(L)(1-L)^d Y_{i,t} = \mu + \Theta(L)\varepsilon_t$$

with

$$\Omega(L) = 1 - \omega_1 L - ... - \omega_p L^p, \Theta(L) = 1 - \theta_1 L - ... - \theta_q L^q$$

and d is an integer.

Unit Root Tests

Standard t-tests cannot be applied for a process with a unit root

Example of AR(1):

$$Y_t = \rho Y_{t-1} + \varepsilon_t, \qquad \sqrt{T}\,(\hat{\rho} - \rho) \Rightarrow N(0, 1 - \rho^2)$$

If true value is $\rho = 1$, then the distribution is degenerate and standard testing procedures do not apply

It can be shown in that case that the convergence is in T (superconsistency) instead of usual $\sqrt{T}$ and the asymptotic distribution is nonnormal.

Importance of Detecting Unit Roots

Spurious regressions:
Granger and Newbold (1974)

$$X_t = X_{t-1} + u_t, Y_t = Y_{t-1} + v_t$$

with u_t and v_t independent

Regression of Y_t on X_t:

$$Y_t = a + bX_t + \varepsilon_t$$

OLS estimator $\hat{b}$ will not converge towards its true value zero (cf both series are independent) but to a random variable

$\Rightarrow$ Detecting presence of unit roots is key to avoid spurious results

Unit Root Testing Procedures

Testing procedures differ depending on the form of the null hypothesis:

- Either null hypothesis = non stationarity
- Or null hypothesis = stationarity

Since all statistics admit particular nonnormal distributions, their practical implementation needs use of tables containing tabulated critical values obtained by simulations.

These critical values are usually given for common significance levels: 1%, 5%, 10% and sample lengths: 25, 50, 100, 250, ∞.

Null Hypotheses of Non-Stationarity

Dickey-Fuller:

Regression:

$$\Delta X_t = d_t + \rho X_{t-1} + u_t$$

Hypothesis:

$$\begin{aligned} d_t =& a \ : H_0 = \{a = \rho = 0\} \\ d_t =& a + bt \ : H_0 = \{b = \rho = 0\} \end{aligned}$$

Implementation of test based on either standard t-statistic for significance of ρ ($\rho = 0$) or statistic $T\hat{\rho}$.

Null Hypotheses of Non-Stationarity

Augmented Dickey-Fuller:

Takes into account possibility of autocorrelated innovations instead of pure random walk with or without drift

Regression:

$$\Delta X_t = d_t + \sum_{i=1}^{p} \gamma_i \Delta X_{t-i} + \rho X_{t-1} + u_t$$

Hypothesis:

$$d_t = a \ : H_0 = \{a = \rho = 0\}$$

$$d_t = a + bt \ : H_0 = \{b = \rho = 0\}$$

Test statistic: $\frac{T\hat{\rho}}{1-\sum_{i=1}^{p}\hat{\gamma}_i}$

Order p should be large enough to eliminate autocorrelation of innovations.

Null Hypotheses of Non-Stationarity

Schmidt-Phillips:

Tests valid under presence of trend or not. The regression on detrended series:

$$\Delta X_t = a + \rho \hat{S}_{t-1} + u_t$$

with

$$\hat{S}_t = X_t - \hat{\psi} - \hat{\xi} t$$
$$\hat{\xi} = \frac{X_T - X_1}{T - 1}$$
$$\hat{\psi} = X_1 - \hat{\xi}$$

Implementation of test based on either standard t-statistic for significance of ρ ($\rho = 0$) or statistic $T\hat{\rho}$.

Null Hypotheses of Non-Stationarity

Elliott-Rothenberg-Stock:

Variant of Dickey-Fuller with better power properties.

Null Hypotheses of Stationarity

Kwiatkowski-Phillips-Schmidt-Shin:

Regress X_t on a constant or a trend and compute

$$\hat{S}_t = \sum_{\tau=1}^{t} \hat{e}_t$$

with estimated residuals $\hat{e}_t$ and variance $\hat{s}^2$ of residuals

Test statistic:

$$\frac{1}{\hat{s}^2} \frac{\sum_{t=1}^{T} \hat{S}_t^2}{T^2}$$