B.V. Conditional VaR and ES

Olivier Scaillet

University of Geneva and Swiss Finance Institute

Outline

Conditional VaR

VaR may be made time dependent.

We will condition on past values

$$P\left[-a'Y_t > VaR_t(a,\alpha)\,\middle|\,a'Y_{t-1},...,a'Y_{t-p}\right] = \alpha.$$

We need to estimate the conditional pdf.

Nonparametric Estimation of Conditional VaR

The conditional pdf may be estimated nonparametrically

$$\hat{f}(z\,|a'Y_{t-1} = y_1\ ,\ldots, a'Y_{t-p} = y_p)$$

$$= \frac{\frac{1}{Th^{p+1}} \sum_{t=1}^{T} K\left(\frac{-a'y_t - z}{h}\right) \prod_{j=1}^{p} K\left(\frac{a'y_{t-j} - y_j}{h}\right)}{\frac{1}{Th^{p}} \sum_{t=1}^{T} \prod_{j=1}^{p} K\left(\frac{a'y_{t-j} - y_j}{h}\right)}$$

We then solve to find estimate $\widehat{VaR_t}(a, \alpha)$

$$\int_{\widehat{VaR_t}(a,\alpha)}^{+\infty} \hat{f}(z\,|a'Y_{t-1} = y_1\ ,\ldots, a'Y_{t-p} = y_p)dz = \alpha$$

Nonparametric Estimation of Conditional VaR

Remark: Using the Gaussian kernel we get

$$\int_{\hat{V}aR_t(a,\alpha)}^{+\infty} \hat{f}(z\,|a'Y_{t-1} = y_1 \,, \ldots, a'Y_{t-p} = y_p)dz$$

$$= \frac{\frac{1}{Th^p}\sum_{t=1}^{T}\Phi\left(\frac{-a'y_t - \hat{V}aR_t(a,\alpha)}{h}\right)\prod_{j=1}^{p}\varphi\left(\frac{a'y_{t-j}-y_j}{h}\right)}{\frac{1}{Th^p}\sum_{t=1}^{T}\prod_{j=1}^{p}\varphi\left(\frac{a'y_{t-j}-y_j}{h}\right)}$$

Conditional ES

ES may also be made time dependent

$$ES_t(a,\alpha) = E\left[-a'Y_t \,\middle|\, -a'Y_t > VaR_t(a,\alpha), a'Y_{t-1}, ..., a'Y_{t-p}\right]$$

Kernel estimator:

$$\hat{E}S_t(a,\alpha) = \frac{\frac{1}{Th^{p+1}} \sum_{t=1}^{T} (-a'y_t) \int_{\hat{V}aR_t(a,\alpha)}^{\infty} K\left(\frac{-a'y_t - z}{h}\right) dz \prod_{j=1}^{p} K\left(\frac{a'y_{t-j} - y_j}{h}\right)}{\alpha \frac{1}{Th^p} \sum_{t=1}^{T} \prod_{j=1}^{p} K\left(\frac{a'y_{t-j} - y_j}{h}\right)}$$

Conditional ES

Remark: In case a Gaussian kernel is used

$$\hat{E}S_t(a,\alpha) = \frac{\frac{1}{Th^p}\sum_{t=1}^{T}(-a'y_t)\Phi\left(\frac{-a'y_t-\hat{V}aR_t(a,\alpha)}{h}\right)\prod_{j=1}^{p}\varphi\left(\frac{a'y_{t-j}-y_j}{h}\right)}{\alpha\frac{1}{Th^p}\sum_{t=1}^{T}\prod_{j=1}^{p}\varphi\left(\frac{a'y_{t-j}-y_j}{h}\right)}$$

Empirical Illustration

Daily Return of Stock Indices

- Sample:
  - Indices: CAC40, DAX30, S&P500, DJI, Nikkei225
  - From 03/01/1994 to 07/07/2000, 1700 observations
- 90% pointwise confidence bands
- Computed by block bootstrap (length = 11, data = $T^{1/3}$)

Block Bootstrap

Idea of block bootstrap:

1. Divide sample into blocks of given length.
2. Resample blocks to build new data sample of length T (concatenate the blocks) and reestimate quantities of interest (here VaR or ES).
3. Repeat (2) a sufficiently large number of times.
4. Compute quantiles of level $\alpha/2$ and $1-\alpha/2$ (ex. 5% and 95%) of empirical distribution of estimates to build confidence intervals of level $1-\alpha$ (ex. 90%).