A.X. Applications

Olivier Scaillet

University of Geneva and Swiss Finance Institute

Outline

Introduction

Why determine sensitivity of VaR and ES w.r.t. modification of portfolio allocation ?

Risk management environment:

regulatory environment / control of risk
$\Rightarrow$ development of proprietary risk measurement models

VaR

VaR = *quantitative and synthetic measure of risk*

- set capital requirements in financial institutions
- regulate risks (traders and insurance writers)
- allocate internal resources
- management of risk limits based on incremental VaR
- large portfolios preclude online computations

$\Rightarrow$ need to avoid to recompute VaR when portfolio slightly modified.

- portfolio selection problems when VaR used instead of variance

$\Rightarrow$ need to know derivatives

VaR

Thus the knowledge of the sensitivity of VaR is crucial.

Notation:

portfolio allocation:

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} = (a_1, ..., a_i, ..., a_n)'$$

portfolio return:

$$a' Y_t = \sum_{i=1}^{n} a_i Y_{i,t}$$

VaR

Formal definition of Value at Risk (VaR)

$$P\left[a'Y_t + VaR(a,\alpha) < 0\right] = \alpha$$

$$\Leftrightarrow P\left[-a'Y_t > VaR(a,\alpha)\right] = \alpha$$

VaR

= return so that, when added to the portfolio return, the probability that global return is negative is equal to α

= quantile of the loss distribution

= function of the portfolio allocation and the loss probability level

Sensitivity of VaR

Sensitivity of VaR:
first derivative of $VaR(a, \alpha)$ w.r.t. portfolio allocation *vector a*

$$\frac{\partial VaR(a,\alpha)}{\partial a} = \begin{pmatrix} \frac{\partial VaR(a,\alpha)}{\partial a_1} \\ \vdots \\ \frac{\partial VaR(a,\alpha)}{\partial a_i} \\ \vdots \\ \frac{\partial VaR(a,\alpha)}{\partial a_n} \end{pmatrix}$$

Sensitivity of VaR

It can be shown that

$$\frac{\partial VaR(a,\alpha)}{\partial a} = E\left[-Y_t \,|-a'Y_t = VaR(a,\alpha)\right] = \begin{pmatrix} E\left[-Y_{1,t} \,|-a'Y_t = VaR(a,\alpha)\right] \\ \vdots \\ E\left[-Y_{i,t} \,|-a'Y_t = VaR(a,\alpha)\right] \\ \vdots \\ E\left[-Y_{n,t} \,|-a'Y_t = VaR(a,\alpha)\right] \end{pmatrix}$$

$\Rightarrow$interpretation of derivatives as conditional expectations

Example: Riskmetrics

Under the assumption of Gaussian returns

$$Y_t \sim N(\mu, \Sigma)$$

we get an explicit form for the VaR

$$VaR(a, \alpha) = -a'\mu + (a'\Sigma a)^{1/2} z_{1-\alpha}$$

as well as for its derivatives:

$$\begin{aligned}
\frac{\partial VaR(a, \alpha)}{\partial a} &= -\mu + \frac{\Sigma a}{(a'\Sigma a)^{1/2}} z_{1-\alpha} \\
&= -\mu + \frac{\Sigma a}{(a'\Sigma a)} \left(VaR(a, \alpha) + a'\mu \right) \\
&= -E\left[Y_t \middle| -a'Y_t = Var(a, \alpha) \right]
\end{aligned}$$

Estimation of VaR

Estimation:

- Gaussian case: Only need to replace the unknown mean and unknown covariance matrix by their empirical counterparts.
- General case: Do not make any parametric assumption on the distribution of returns.

Then we need to estimate nonparametrically
$\Rightarrow$ use of a kernel approach

Nonparametric Estimation of VaR

Recall the definition of the VaR

$$P\left[-a'Y_t > VaR(a,\alpha)\right] = \alpha$$

but

$$P\left[-a'Y_t > VaR(a,\alpha)\right] = \int_{VaR(a,\alpha)}^{+\infty} f(z)dz$$

where $f(z)$ = density of the loss portfolio return $Z_t = -a'Y_t$

Nonparametric Estimation of VaR

Thus we may replace the unknown $f(z)$ by its kernel estimate

$$\hat{f}(z) = \frac{1}{Th} \sum_{t=1}^{T} K\left(\frac{z_t - z}{h}\right)$$

computed from loss data $z_t = -a' y_t$, and solve to find estimate $\widehat{VaR}(a, \alpha)$: $\int_{\widehat{VaR}(a,\alpha)}^{+\infty} \hat{f}(z) dz = \alpha$

Remark: Gaussian kernel

$$\int_{\widehat{VaR}(a,\alpha)}^{+\infty} \hat{f}(z) dz = \frac{1}{T} \sum_{t=1}^{T} \Phi\left(\frac{z_t - \widehat{VaR}(a,\alpha)}{h}\right)$$

Estimation of VaR Sensitivity

$$VaR^{(1)}(a,\alpha) = \frac{\partial VaR(a,\alpha)}{\partial a} = E\left[-Y_t \,\middle|\, -a'Y_t = VaR(a,\alpha)\right]$$

- Gaussian case: Again, we only need to replace the unknown mean and unknown covariance matrix by their empirical counterparts.
- General case:

$$\widehat{VaR}^{(1)}(a,\alpha) = \frac{\frac{1}{Th}\sum_{t=1}^{T}(-y_t)\,K\left(\frac{-a'y_t-\hat{V}aR(a,\alpha)}{h}\right)}{\frac{1}{Th}\sum_{t=1}^{T}K\left(\frac{-a'y_t-\hat{V}aR(a,\alpha)}{h}\right)}$$

cf. estimate of a conditional mean

Empirical Illustration

Empirical illustration:

French stock data from CAC 40

- Thomson-CSF (electronic devices)
- L'Oreal (cosmetics)

Daily returns: 04/01/97 to 05/04/99 (546 obs.)

Empirical Results

Empirical results:

- Standard normal VaR underestimate (skewness and kurtosis)
- Smoother patterns for kernel estimates
- Nonmonotonicity of sensitivities
- VaR symmetry lost
- VaR efficient portfolio = tangency points of $a_1 \hat{\mu}_1 + a_2 \hat{\mu}_2 = cst$ with isoVaR curves

Definition of Expected Shortfall

Expected Shortfall (ES):

$$ES(a,\alpha) = E\left[-a'Y_t \left| -a'Y_t > VaR(a,\alpha)\right.\right]$$

ES = expected loss knowing that losses are above VaR

Advantages over VaR:

- Subadditive risk measure

$$ES(a_1 + a_2, \alpha) \leq ES(a_1, \alpha) + ES(a_2, \alpha)$$

The total risk on a portfolio should not be greater than the sum of individual risks (diversification).

VaR is *not* subadditive

- VaR tells us nothing about size of potential loss

Sensitivity of ES

First derivative of $ES(a, \alpha)$ w.r.t. portfolio allocation a

$$\frac{\partial ES(a,\alpha)}{\partial a} = \begin{pmatrix} \frac{\partial ES(a,\alpha)}{\partial a_1} \\ \vdots \\ \frac{\partial ES(a,\alpha)}{\partial a_n} \end{pmatrix} = E\left[-Y_t \left| -a' Y_t > VaR(a,\alpha) \right.\right] = \begin{pmatrix} E\left[-Y_{1,t} \left| -a' Y_t > VaR(a,\alpha) \right.\right] \\ \vdots \\ E\left[-Y_{n,t} \left| -a' Y_t > VaR(a,\alpha) \right.\right] \end{pmatrix}$$

$\Rightarrow$again interpretation of derivatives as conditional expectations

Example

Example: (Riskmetrics)
Gaussian return assumption: $Y_t \sim N(\mu, \Sigma)$
Explicit forms:

$$ES(a, \alpha) = -a'\mu + (a'\Sigma a)^{1/2}\varphi(z_{1-\alpha})/\alpha$$

$$\begin{aligned}
\frac{\partial ES(a,\alpha)}{\partial a} &= -\mu + \frac{\Sigma a}{(a'\Sigma a)^{1/2}}\varphi(z_{1-\alpha})/\alpha \\
&= -\mu + \frac{\Sigma a}{(a'\Sigma a)}(ES(a,\alpha) + a'\mu) \\
&= -\mu + \frac{\Sigma a}{(a'\Sigma a)^{1/2}}\varphi\left(\frac{VaR(a,\alpha)+a'\mu}{(a'\Sigma a)^{1/2}}\right)/\alpha
\end{aligned}$$

Estimation of ES

Estimation:

- Gaussian case: Replace unknown mean and unknown covariance matrix by their empirical counterparts.
- General case: No parametric assumption on the distribution of returns, i.e., estimate nonparametrically
  $\Rightarrow$ use of a kernel approach

Nonparametric Estimation of ES

Recall definition of ES

$$\begin{aligned} ES(a,\alpha) &= E\left[-a'Y_t \,|-a'Y_t > VaR(a,\alpha)\right] \\ &= \frac{E\left[-a'Y_t 1_{-a'Y_t > VaR(a,\alpha)}\right]}{P[-a'Y_t > VaR(a,\alpha)]} \\ &= \frac{E\left[-a'Y_t 1_{-a'Y_t > VaR(a,\alpha)}\right]}{\alpha} \end{aligned}$$

= expectation of losses above VaR divided by their probability of occurrence. Estimated nonparametrically by

$$\hat{E}S(a,\alpha) = \frac{\frac{1}{Th}\sum_{t=1}^{T}(-a'y_t)\int_{\hat{V}aR(a,\alpha)}^{+\infty} K\left(\frac{-a'y_t-u}{h}\right)du}{\alpha}$$

Remark: Gaussian kernel

$$\hat{E}S(a,\alpha) = \frac{\frac{1}{T}\sum_{t=1}^{T}(-a'y_t)\,\Phi\left(\frac{-a'y_t-\hat{V}aR(a,\alpha)}{h}\right)}{\alpha}$$

only need weighted empirical average

Nonparametric Estimation of ES Sensitivity

$$ES^{(1)}(a,\alpha) = \frac{\partial ES(a,\alpha)}{\partial a} = -E\left[Y_t \middle| -a'Y_t > VaR(a,\alpha)\right]$$

- Gaussian case: Replace unknown mean and unknown covariance matrix by their empirical counterparts
- General case:

$$\hat{E}S^{(1)}(a,\alpha) = \frac{\frac{1}{Th}\sum_{t=1}^{T}(-y_t)\int_{\hat{V}aR(a,\alpha)}^{+\infty} K\left(\frac{-a'y_t-u}{h}\right)du}{\alpha}$$

Remark: Gaussian kernel

$$\hat{E}S^{(1)}(a,\alpha) = \frac{\frac{1}{T}\sum_{t=1}^{T}(-y_t)\,\Phi\left(\frac{-a'y_t-\hat{V}aR(a,\alpha)}{h}\right)}{\alpha}$$

Empirical Illustration

Empirical illustrations:

1. Finance : Thomson-CSF and L'Oreal
2. Insurance : Fire insurance loss data

Danish data on total losses

$$
\begin{aligned}
&= \text{damage to buildings} \\
&+ \text{damage to furniture and personal property} \\
&+ \text{loss of profits}
\end{aligned}
$$

1794 losses over 1 million DKK (1980 to 1990)

Empirical Illustration

- Mean: 4,235,299.9, st. dev.: 9,256,401.5
- Minimum loss: 325,000. Maximum loss: 200,700,000
- Skewness: 13.629, kurtosis: 251.336
- Estimated expected shortfall at $\alpha = 1\%$
- DKK 66.96 million by kernel approach $h = \hat{\sigma} T^{-1/5}$
- DKK 58.69 and 69.59 million by EVT
- (POT method: thresholds at 10 and 20 million)

Extreme Value Theory

Remark: Extreme Value Theory

- It can be shown that no matter the shape in the center of the distribution the shape of the tail takes always a very particular form when we are far enough in the tail.
- For a very high threshold (very large loss), data above this threshold follows a *generalized pareto distribution* (GPD)

Idea:

- Fit parameters of GPD to data above a given threshold u and compute VaR and ES using their associated explicit forms.

Estimating the GPD

- GPD depends on parameters ξ, σ, estimated using the empirical mean $\hat{m}$ and variance S^2 on portfolio returns $> u$:

$$\hat{\xi} = \frac{1}{2}\left(1 - \frac{(\hat{m}-u)^2}{S^2}\right), \qquad \hat{\sigma} = \frac{\hat{m}-u}{2}\left(\frac{(\hat{m}-u)^2}{S^2} + 1\right)$$

- Then we may compute the VaR and ES with $\widehat{VaR}(a,\alpha) = \tilde{\mu} + \frac{\tilde{\sigma}}{\hat{\xi}}(\alpha^{-\hat{\xi}} - 1)$, and $\widehat{ES}(a,\alpha) = \hat{V}aR(a,\alpha) - \frac{\tilde{\sigma}}{\hat{\xi}-1}\alpha^{-\hat{\xi}}$

where $\tilde{\sigma} = \hat{\sigma}\left(\frac{N}{T}\right)^{\hat{\xi}}$, $\tilde{\mu} = u - \frac{\tilde{\sigma}}{\hat{\xi}}\left(\left(\frac{N}{T}\right)^{-\hat{\xi}} - 1\right)$, $N = \#$ points above u, $N/T=$ ratio of points *above u*

Incremental VaR and ES

- VaR and ES are homogeneous functions of degree one in the portfolio allocation, i.e. if portfolio allocation is doubled, tripled,. . . so will be VaR and ES
- For such functions, Euler's theorem implies that the function may be rewritten as a linear combination of its derivatives. Hence we get

$$VaR(a,\alpha) = a'\frac{\partial VaR(a,\alpha)}{\partial a} = \sum_{i=1}^{n} a_i \frac{\partial VaR(a,\alpha)}{\partial a_i}$$

$$ES(a,\alpha) = a'\frac{\partial ES(a,\alpha)}{\partial a} = \sum_{i=1}^{n} a_i \frac{\partial ES(a,\alpha)}{\partial a_i}$$

Incremental VaR and ES

- The quantities $a_i \frac{\partial VaR(a,\alpha)}{\partial a_i}$ and $a_i \frac{\partial ES(a,\alpha)}{\partial a_i}$ are the contributions of asset i to the global risk of the portfolio measured by VaR and ES, respectively.
- They are called *incremental VaR* and *incremental ES*.
- This allows ranking the assets by their risk contributions.
- These contributions can be estimated using the aforementioned parametric and nonparametric estimators of VaR and ES derivatives.