B.III. ACF, PACF and ARMA models

Olivier Scaillet

University of Geneva and Swiss Finance Institute

Outline

1. Dependence
2. Autocorrelation & Wold Theorem
3. The ARMA Process
4. Partial Autocorrelation Function
5. Determine ARMA order
6. Residuals Analysis
7. Unit Root Problems

Notion of Dependence

Covariance (or correlation)
= measure of *linear dependence* between two random variables

Will be used to measure dependence between current observations and past observations

Current return: Y_t
Return with lag τ: $Y_{t-\tau}$

The covariance between current and lagged return

$$\gamma(\tau) = Cov(Y_t, Y_{t-\tau})$$

is called the *autocovariance of order* τ.

Autocorrelation

If the random process is stationary, the autocovariance only depends on lag τ.

Autocovariance of order 0 = Variance
Autocorrelation of order τ

$$\rho(\tau) = \frac{Cov(Y_t, Y_{t-\tau})}{V(Y_t)} = \frac{\gamma(\tau)}{\gamma(0)}$$

The *autocorrelation function (ACF)*

$$= \rho(\tau), \quad \tau = 1, 2, \ldots \text{ of successive orders}$$

will allow to detect linear temporal dependence.

Remark:

$$\rho(0) = \frac{\gamma(0)}{\gamma(0)} = 1$$

Wold Theorem

Wold theorem:

A stationary process Y_t may always be represented as a linear combination of a random noise ε_t and its past realizations $\varepsilon_{t-1}, \varepsilon_{t-2}, \dots$

$$Y_t = \mu + \varepsilon_t + a_1\varepsilon_{t-1} + \dots + a_j\varepsilon_{t-j} + \dots$$

Using the backward (or lag) operator

$$L\varepsilon_t = \varepsilon_{t-1}, \quad L^2\varepsilon_t = LL\varepsilon_t = \varepsilon_{t-2}, \dots.$$

this can be rewritten more compactly

$$Y_t = \mu + \sum_{j=0}^{\infty} a_j L^j \varepsilon_t = \mu + A(L)\varepsilon_t$$

ARMA Process

Idea: approximate the sequence $A(L) = \sum_{j=0}^{\infty} a_j L^j$ by a finite polynomial (truncation of the infinite sum)

$$Y_t = \mu + \omega_1 Y_{t-1} + \ldots + \omega_p Y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \ldots - \theta_q \varepsilon_{t-q},$$

or more compactly

$$\Omega(L) Y_t = \mu + \Theta(L) \varepsilon_t,$$

with

$$\Omega(L) = 1 - \omega_1 L - \ldots - \omega_p L^p, \Theta(L) = 1 - \theta_1 L - \ldots - \theta_q L^q.$$

This is called an ARMA (autoregressive moving average) process of order p , q.

Properties

Pure autoregressive process: AR(p)

$$Y_t = \mu + \omega_1 Y_{t-1} + ... + \omega_p Y_{t-p} + \varepsilon_t$$

Pure moving average process: MA(q)

$$Y_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1} - ... - \theta_q \varepsilon_{t-q}$$

Identification of order through analysis of autocorrelations:
if pure MA(q), autocorrelations $\rho(\tau)$ are zero after order q:

$$\rho(\tau) = 0, \quad \tau > q$$

if pure AR(p), partial autocorrelations $r(\tau)$ are zero after order p:

$$r(\tau) = 0, \quad \tau > p$$

PACF

The partial autocorrelation $r(\tau)$ is defined as the last coefficient in the autoregression of order τ

$$Y_t = \mu + \omega_1 Y_{t-1} + \ldots + \omega_\tau Y_{t-\tau} + \varepsilon_t,$$

i.e., $r(\tau) = \omega_\tau$.

Partial autocorrelation function (PACF)

$= r(\tau), \quad \tau = 1, 2, \ldots$ of successive orders

Empirical counterpart of ACF and PACF

$=$ *correlogram* and *partial correlogram*

$$\hat{\rho}_i(\tau), \quad \tau = 1, 2, \ldots; \; \hat{r}_i(\tau), \quad \tau = 1, 2, \ldots$$

Identifying Orders

Identification of order p and q:
By examining on correlograms and partial autocorrelograms whether autocorrelations and partial autocorrelations are significantly different from zero i.e. outside the confidence intervals.

Example:
dollar/sterling exchange rate
from January 1974 and December 1994
(5192 observations)

Estimation of ARMA models:
usually by maximum likelihood.
pure autoregressive process also by OLS

Residuals Analysis

After estimation, residuals should be white noise

Test based on Portmanteau statistics (*Box-Pierce or Ljung-Box statistics*)
= sum of autocorrelations of residuals

Up to lag h

$$Q(H) = T \sum_{h=1}^{H} \hat{\rho}(h)^2$$

If residuals are white noise (no temporal dependence), $Q(H)$ should be close to zero.

Unit Root Problems

AR(1) process:

$$Y_t = \mu + \omega_1 Y_{t-1} + \varepsilon_t$$

We may be tempted to test whether coefficient $\omega_1 = 1$ (random walk hypothesis) using standard t-test

This is false since the process is non stationary.

The t-statistics does not follow a student distribution under the null hypothesis of a unit root.

The distribution is non standard and critical values have been tabulated (*Dickey-Fuller* table).