Martingales

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Outline

1. Martingales

2. Doob decomposition

Definition

A sequence $Y = \{Y_t;\ t \geq 0\}$ is a (discrete) martingale with respect to the sequence $X = \{X_t;\ t \geq 0\}$ if for all $t \geq 0$

1. $E \mid Y_t \mid < \infty$
2. $E[Y_{t+1}|X_0, X_1, \ldots, X_t] = Y_t$

Example: simple random walk

Simple random walk: particle jumps to the right $(+1)$ with a probability p, and to the left (-1) with a probability $q = 1 - p$.

1. The location of the particle after t jumps satisfies $E \mid S_t \mid \leq t$.
2. $E[S_{t+1}|X_1, X_2, \ldots, X_t] = E[X_1 + \cdots + X_t|X_1, X_2, \ldots, X_t] + E[X_{t+1}|X_1, X_2, \ldots, X_t] = S_t + (p - q)$

We deduce that $Y_t = S_t - t(p - q)$ defines a martingale with respect to X. Indeed,
$E[Y_{t+1}|X_1, X_2, \ldots, X_t] = E[S_{t+1} - (t+1)(p-q)|X_1, X_2, \ldots, X_t] = S_t + (p - q) - (t + 1)(p - q) = Y_t$

Example: roulette strategy

Let a gambler with a large amount of money plays the roulette game. He chooses the following strategy.

He bets CHF 1 on red. If he loses, he bets an additional CHF 2 on red, doubling his initial investment. He goes on doubling the size of the gamble as long as he loses.

Thus, if he loses for t consecutive bets, he bets 2^t on red for the $(t+1)$th round.

This strategy always wins at some point in time. Indeed, if he wins the $(T+1)$th round, he gets $2^T - (1 + 2 + 2^2 + \cdots + 2^{T-1})$.

Example: roulette strategy (cont'd)

The cumulative gain process Y_t (sum of all bets) of this strategy is a martingale:

- if the gambler stops at time $t+1$ (red comes for the first time), we have $Y_{t+1} = Y_t$
- if the gambler goes on gambling, we have $Y_{t+1} = \begin{cases} Y_t - 2^t & \textit{with probability } \frac{1}{2} \\ Y_t + 2^t & \textit{with probability } \frac{1}{2} \end{cases}$ which gives $E[Y_{t+1}|Y_1, Y_2, \ldots, Y_t] = Y_t$.

The strategy is thus a martingale for the gain with respect to itself.

Example: roulette strategy (cont'd)

Even though the gambler is sure to win at some point in time, his expected loss is infinite.

Let T be a random variable indicating the time of the first win. This random variable has a probability mass function $P[T=t] = \left(\frac{1}{2}\right)^t$, and we have $\lim_{t\to\infty} P[T=t] = 0$ as expected (the gambler wins for sure).

Until this date however, the gambler loses an amount L, whose expected value is given by $E[L] = \sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^t (1+2+2^2+\cdots+2^{t-2}) = \infty$.

Therefore, the gambler loses an infinite amount of money on average, and he can expect the strategy to be beneficial only if his initial capital is extremely large. Additionally, casinos forbid such practices by imposing a maximum size of bets, which prevents any gambler to implement the strategy.

Example: De Moivre's martingale

The martingale characterization helps to solve the following question related to a player ruin.

We consider a simple random walk on $\{0, 1, \ldots, N\}$ that stops when it reaches absorbing barriers at 0 or N.

The question is to determine the probability of being absorbed at 0.

Example: De Moivre's martingale (cont'd)

Let $X_1, X_2, \ldots$ be the jumps of the random walk, with $p = P[X_t = 1]$ and $q = 1 - p = P[X_t = -1]$.

We denote by S_t the position after t jumps (the wealth after t jumps), with $S_0 = k$. We define $Y_t = \left(\frac{q}{p}\right)^{S_t}$ and have

$$
\begin{aligned}
E[Y_{t+1}|X_1, X_2, \ldots, X_t] &= E[\left(\frac{q}{p}\right)^{S_t + X_{t+1}} |X_1, X_2, \ldots, X_t] \\
&= \left(\frac{q}{p}\right)^{S_t} E[\left(\frac{q}{p}\right)^{X_{t+1}} |X_1, X_2, \ldots, X_t] \\
&= \left(\frac{q}{p}\right)^{S_t} \left[p \left(\frac{q}{p}\right)^{1} + q \left(\frac{q}{p}\right)^{-1} \right] = \left(\frac{q}{p}\right)^{S_t} \\
&= Y_t
\end{aligned}
$$

$Y_t = \left(\frac{q}{p}\right)^{S_t}$ is thus a martingale.

Example: De Moivre's martingale (cont'd)

Using iterated expectations, we have

$$\begin{aligned} E[Y_t] &= E[E[Y_t|X_1, X_2, \ldots, X_{t-1}]] \\ &= E[Y_{t-1}] = E[E[Y_{t-1}|X_1, X_2, \ldots, X_{t-2}]] \\ &= E[Y_{t-2}] = \cdots = E[Y_0] = Y_0 = \left(\frac{q}{p}\right)^{S_0} \\ &= \left(\frac{q}{p}\right)^{k} \end{aligned}$$

The relationship $E[Y_t] = Y_0 = \left(\frac{q}{p}\right)^k$ holds for any t. In particular, if we denote the number of jumps until absorption (either at 0 or at N) by T and suppose that $T < \infty$, we have $E[Y_T] = \left(\frac{q}{p}\right)^k$.

Example: De Moivre's martingale (cont'd)

This expectation also gives $E[Y_T] = E[\left(\frac{q}{p}\right)^{S_T}] =$
$P[absorbed\ at\ 0]\left(\frac{q}{p}\right)^0 + P[absorbed\ at\ N]\left(\frac{q}{p}\right)^N$ with
$P[absorbed\ at\ 0] = 1 - P[absorbed\ at\ N]$.

Using the two expressions of the expectation $E[Y_T]$ and defining $\rho = \frac{q}{p}$, we deduce that $P[absorbed\ at\ 0] = \frac{\rho^k - \rho^N}{1-\rho^N}$, which characterizes the ruin given an initial wealth of $S_0 = k$.

Information and filtration

The martingale concept may be generalized to a probability space (Ω, A, P) where $F = \{F_0, F_1, \dots\}$ is a sequence of sub-σ-algebra of A as $F_t \subseteq F_{t+1}$, $\forall t$.

We say that F is a filtration. The sequence $F = \{F_0, F_1, \dots\}$ describes the evolution of information across time.

As soon as an event is known at time t, that is, belongs to the information available at time t, it is also known at time $t+1$ as it belongs to the information available at time $t+1$.

Adapted process

A sequence (or process) $Y = \{Y_t; t \geq 0\}$ is called adapted if it is F_t-measurable for any t, that is, if knowing F_t defines the path of the process until t.

The realization Y_t of the process Y is a function of the information available at time t. In this context, we say that Y is a martingale with respect to filtration F if for all $t \geq 0$

1. $E \mid Y_t \mid < \infty$
2. $E[Y_{t+1} \mid F_t] = Y_t$.

Submartingale and supermartingale

There exist many cases where the equality $E[Y_{t+1} \mid F_t] = Y_t$ defining a martingale does not hold, but may be replaced by an inequality.

We say that Y is a submartingale with respect to the filtration F if for all $t \geq 0$

1. $E[\max(0, Y_t)] < \infty$
2. $E[Y_{t+1}|F_t] \geq Y_t$.

The process has a larger value tomorrow than its value today, on average.

Similarly, Y is a supermartingale with respect to the filtration F if for all $t \geq 0$

1. $E[-\min(0, Y_t)] < \infty$
2. $E[Y_{t+1}|F_t] \leq Y_t$.

Doob decomposition

Doob decomposition allows to characterize any submartingale as the sum of two terms, one predictable and increasing, the other is a martingale: $Y_t = S_t + M_t$.

The sequence $S = \{S_t; t \geq 0\}$ is said to be predictable if it is F_{t-1}-measurable. It is called increasing if $S_0 = 0$ and $P[S_t \leq S_{t+1}] = 1$ for all t.

S_t is the part of Y_t that can be predicted using the information available at time $t-1$ and is called compensator of the submartingale.

Doob decomposition (cont'd)

We compensate Y_t using S_t such that $Y_t - S_t = M_t$ becomes a martingale.

The martingale M_t is the part that we cannot predict, that is, the innovation or error term.

We may show that this decomposition is unique.

Doob decomposition (cont'd)

This concept is used in finance to decompose the evolution of a portfolio of financial assets in a predictable part and an unpredictable part.

The unpredictable part has mean zero. Investors generally study the evolution of the predictable part in order to determine an optimal allocation with a given expected return.

This use of the Doob decompostion sets the grounds of the dynamic valuation principle in the context of the binomial model in which the price of an asset is equal to the risk-neutral expectation of the discounted cashflows.