Point processes

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Outline

Introduction

In this chapter, we model a random distribution of points in a space, typically a subset of $[0, +\infty)$, $\mathbb{R}$, or $\mathbb{R}^d$.

We may model many natural phenomenons with this approach:

- The allocation and connection of groups of machines.
- The time and position of earthquakes in the 50 years to come.
- The location of oil resources in a given area.
- The location of troops in a battlefield.

Definition and basic notions

Let E be a subset of an Euclidean space. In the following, we consider a subset of $[0, +\infty)$, $\mathbb{R}$, or $\mathbb{R}^d$.

We distribute points randomly in E and consider a simple notation allowing to count the number of points that lie in a closed set A. If $E = \mathbb{R}^2$, A is a rectangle in $\mathbb{R}^2$.

Suppose that $\{X_t : t \geq 0\}$ stands for consecutive points in the state space E. If we define the discrete measure

$$\epsilon_{X_t}(A) = \begin{cases} 1 & \textit{if } X_t \in A \\ 0 & \textit{if } X_t \notin A \end{cases}$$

then summing on t, we get the total number of points X_t in A.

Definition and basic notions (cont'd)

If we define the counting measure N by $N(\cdot) = \sum_t \epsilon_{X_t}(\cdot)$, then $N(A) = \sum_t \epsilon_{X_t}(A)$ is the random number of points that lie in the set A.

N is called a point process and $\{X_t\}$ are called points.

The intensity of N, or mean measure of N is defined by $\mu(A) = E[N(A)]$, which corresponds to the expected number of points in A.

Poisson random measure

Let N be a point process defined on a state space E and with a Borel σ-algebra, that is, the σ-algebra Φ made of open subsets of E. Then, N is a Poisson process with mean measure μ, also called a Poisson random measure, if:

1. For any $A \in \Phi$, $P[N(A) = k] = \begin{cases} \frac{e^{-\mu(A)}\mu(A)^k}{k!} & \text{if } \mu(A) < \infty \\ 0 & \text{if } \mu(A) = \infty. \end{cases}$
2. If $A_1, A_2, \ldots, A_k$ are disjoint subsets of E in Φ, then $N(A_1), N(A_2), \ldots, N(A_k)$ are independent random variables.

Poisson random measure (cont'd)

Thus, N is Poissonian if the random number of points in a set A follows a Poisson distribution with parameter $\mu(A)$ and the number of points in disjoint regions are independent random variables.

If $E = \mathbb{R}$, the second condition is equivalent to the property of independence of increments for all $t_1 < t_2 < \cdots < t_k$, where $N([t_i, t_{i+1}])$ are independent variables.

Example: location of a competitor

Consider a banking agency willing to assess the average distance to any competitor if the concentration rate is $\alpha = 3$ agencies per square kilometer.

Let R be the distance to the closest competitor and let $d(r)$ be the disk with radius r centered on the agency. We have that $A = d(r)$ and $\mu(A) = \alpha\pi r^2$ as the surface of a circle with radius r is πr^2.

We get $P[r < R] = P[N(A) = 0] = e^{-\mu(A)} = e^{-\alpha\pi r^2}$.

Example: location of a competitor (cont'd)

The average distance to the closest competitor is given by $E[R] = \int_0^\infty P(r < R)dr = \int_0^\infty e^{-\alpha\pi r^2}dr$.

Let $\alpha\pi = \frac{1}{2\sigma^2}$, the integral can be written as $\int_0^\infty e^{-\frac{r^2}{2\sigma^2}}dr = \frac{1}{2}\int_{-\infty}^\infty e^{-\frac{r^2}{2\sigma^2}}dr = \frac{\sqrt{2\pi\sigma^2}}{2}\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{r^2}{2\sigma^2}}dr = \frac{\sqrt{2\pi\sigma^2}}{2}$ by using the fact that the normal density integrates to 1.

We obtain $E[R] = \frac{1}{2}\sqrt{\frac{1}{\alpha}}$, that is, 0.2887 kilometers for $\alpha = 3$.

Marked point processes

Given an initial Poisson process, we consider the circumstances in which we may increase the dimension of the points and still keep a Poissonian structure.

Let $\{X_t\}$ be a Poisson process with mean measure μ and defined on the state space E_1.

Let $\{J_t\}$ be a series of i.i.d. random variables in a second space E_2 with distribution function F.

Marked point processes (cont'd)

If $\{X_t\}$ and $\{J_t\}$ are defined on the same probability space and are independent, then the counting measure N, $N(\cdot) = \sum_t \epsilon_{(X_t, J_t)}(\cdot)$, defines a point process corresponding to a Poisson random process in $E_1 \times E_2$ and with mean measure $\mu \times F$, as if $A_i \subset E_i$, $i = 1, 2$, then $(\mu \times F)(A_1 \times A_2) = \mu(A_1)F(A_2)$.

We then say that we add the mark J_t to the point X_t and speak of marked point processes.

Marked point processes (cont'd)

- The compound Poisson process that we use to model sinisters in insurance is a particular case of marked point processes where the marks stand for the size of sinisters.
- If we relax the assumption of exponential durations between consecutive events, and/or the the independence between arrivals and marks, we obtain a much larger class of marked point processes but it requires a separate study of the behavior of arrivals and marks (notion of compensator).

Application to price quotes

High frequency price quotes are data sets including a full description of prices across time, bid-ask quotes, exchange prices and quantities, etc.

Some particular characteristics include:

- durations between observations are irregular,
- the market behavior is fully observable,
- the data have very particular characteristics,
- the unit of time is the second.

Application to price quotes (cont'd)

Most stock exchanges have their own data records (NYSE, NASDAQ, Paris Exchange, Madrid Exchange, ...) and share their database (NYSE TAQ database).

The exchanges may be either

- based on quotes,
- or based on order books.

For markets based on quotes, a market maker is responsible of the liquidity, and buys at the bid price and sells at the ask price.

The market maker gets profit through the bid-ask spread.

Application to price quotes (cont'd)

The market maker has private information as he knows the limit order book.

For markets based on order books, there is not necessarily a market maker.

Buy and sell orders or limit orders are directly added to the order book.

A transaction happens when there are compatible bids and asks.

There are priority rules in terms of price and time.

The spread is given by the difference between the best bid and the best ask.

Some exchange markets are hybrid (NYSE).

Application to price quotes (cont'd)

The theory related to markets microstructure tries to explain the behavior of agents and prices in a market following some set of rules.

This theory is based on

- trading systems,
- liquidity,
- the information asymmetry between agents,
- the information given by prices,
- ...

Application to price quotes (cont'd)

We answer questions such as:

- Can a market maker manage his inventory optimally?
- Does a market maker take into account the potential private information of buyers and sellers?
- Is there a relationship between the inventory and the quotes?

Application to price quotes (cont'd)

The important point in these models is that durations between transactions are no longer exogenous.

In particular, it implies that empirical studies based on price data only may lead to incorrect conclusions as they ignore the information given by the durations between transactions.

Application to price quotes (cont'd)

If durations are long, it means that agents - and in particular informed agents - have not changed much their expectations.

Short durations reveal that some agents have information, and the price is adjusted as the exchanges take place.

Also, the volume of exchanges gives information about informed agents.

An empirical analysis of the duration between transactions is thus primordial in order to understand the behavior and the impact of private information.

Application to price quotes (cont'd)

We use

- The duration between transactions to measure the trading intensity: time until next trade.
- The duration of prices to measure the volatility: time until a given cumulative price change.
- The duration of volumes to measure liquidity: time until a given cumulative volume change.

Each of them is a point process that is possibly marked if we add dimensions, for instance the duration between transactions (duration between events) and observed price changes and volume changes (marks).

Application to price quotes (cont'd)

We model durations $d_i = T_i - T_{i-1}$ between events as $d_i = g(z_i;\theta)\epsilon_i$, with g a function of exogenous variables (that could include past durations in a dynamic model).

The error term ϵ_i stands for a naive duration that follows a given distribution.

The factor $g(z_i;\theta)$ drives the acceleration and deceleration of the naive duration as a function of the values taken by the variables and the parameter θ.

Application to price quotes (cont'd)

The conditional density of durations as a function of the density $f_\epsilon(\cdot;\beta)$ of the naive duration with parameter β is given by $f_d(y|z;\theta,\beta) = \frac{1}{g(z;\theta)} f_\epsilon(\frac{y}{g(z;\theta)};\beta)$.

Econometric models such as ACD (autoregressive conditional duration) use a linear combination of past durations.

This kind of models helps to take into account the clustering in durations, that is, the fact that long (resp. short) durations have a tendency to be follows by long (resp. short) durations.

Application to price quotes (cont'd)

We estimate these models using a maximum likelihood estimator after the seasonality has been removed.

The intraday or intraweek seasonality is due to the typical behavior of agents (e.g., lunch time, rebalancing before the weekend, . . .).

The estimation of these models allows validating some economic theories empirically.

We also use these techniques for intraday trading or intraday risk management.

High frequency trading

HFT is a consequence of the modernisation of stock exchanges (natural evolution):

- Early 70s: "paper work crisis" leads to creation of the first electronic order routing system in 1976.
- 80s: development of program trading to buy bundles of assets to build portfolios (diversification).
- 90s: guarantee of anonymity of buyer and seller to avoid exploiting the info (manipulation and insider trading).

High frequency trading

- 2000: operational fairness and fair access (uniform rules); decimalisation to decrease bid-ask spreads (reduction of transaction costs).
- June 2005: introduction of National Market System rule 611 by SEC; transactions should be automatically executed at best available quote : NBBO (National Best Bid and Offer) thus explosion of algorithmic trading.

Academic evidence

Academic evidence: small volume transactions share more predictability (not true for large volume transactions). Explanation:

1. Optimal execution of an order: break large orders in small pieces to avoid too large impact thus serial dependence between successive orders.
2. Statistical arbitrage: herding effect thus serial dependence.

Importance in terms of daily transactions volume: US: 40% to 75%, Europe: 25%, Australia: 10%, FX (currency): 20% to 40%

Academic evidence (cont'd)

- HFT reduces supply and demand imbalances.
- HFT contributes to liquidity (provide liquidity, lower short term volatility, reduce bid-ask spreads) and market efficiency (reveil information, reduce information asymmetry).
- HFT is not a trading strategy per se but describes the use of sophisticated technology that implements traditional trading strategies.

Revenues for service (study in 2010 on HFT firms): 40 (2% of the 2000) firms operating in the US, 68% of Nasdaq volume, gross return (not net profit) = USD 2.8 billion = .75 cent per 100 dollars (need a lot of trades) = 1/7 of traditional market makers.

Regulation

US, NMS rule 611 : this is the "trade through" or "order protection" rule. Exchanges cannot trade at prices that are worse for their customers than the NBBO. If cannot, then orders are rerouted to another exchange or canceled.

Europe, MiFID(Markets in Financial Instruments Directive): principle-based best execution and not rule-based approach as in US.

Both approaches aim at improving fee and service competition.

Flash crash

Flash crash of May 6, 2010: in 30 minutes drop of 5%, USD 1 billion evaporated, 1/3 of volume on e-mini S&P500 futures by HFT:

1. Large initial buy (49% of total volume, single block of USD 4.1 billion) by HFT from a single fundamental seller (FS = mutual fund) between 13:45:13 and 13:45:27 (14 seconds),
2. then selling by HFT (removing inventories) and competition with FS to sell,
3. then hot potato effect (repeated buy and sell from each other),
4. then the FS left the market and Fundamental buyers stopped the fall after a circuit breaker was put in place.

Conclusion: HFT have contributed but are not responsible for the crash.

Flash crash (cont'd)

Solutions:

1. New circuit breakers on individual securities: halt of 5 minutes (Accenture USD 40 to USD 0.01, Sotheby's USD 34 to USD 99,999.99).
2. Forbid stub quotes (quotes far away from actual prices) : pre-defined bounds around the NBBO (notion of quote stuffing = flood the market place with bogus orders to distract rival trading firms).
3. Creation of a database "consolidated audit trail system" to track "the ultimate customer who generated the order" and of a large trader reporting system to identify large traders.

Race to zero

Super human speed: 50 to 55 microsecond (one millionth of a second) for data to travel one way on 10 km of optical fiber = at a pace 100 times faster that a human trader to blink (human reaction time of the order of 200 to 400 millisecond (one thousandth of a second).

Analogy: we can do 40,000 trades back-to-back in the blink of an eye; average household could complete its shopping for a lifetime in under a second if they ran HFT programs in supermarkets.

We observe several submissions and cancelations of orders with 10 milliseconds thus no intention of signaling anything to human traders but of triggering responses from other algorithms (operate at their own pace).

Debate

NYSE specialists (market makers) were obliged to stabilise prices and maintain a continuous presence; HFT are not, and may walk away and cause fragility.

This belongs to the debate on large socio-technical systems (interactions between technology and humans).