Last name: Geneva, Jan 11 2016.
First name:

Stochastic processes

- The exam lasts two hours.
- Do not forget to write your name on all pages.
- Any document or calculator is forbidden.
- Justify your answers and answer clearly.

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Question 1

We consider the arrival process of phone calls in a customer service center. We suppose that it follows a Poisson process with parameter $\lambda = 6$ per minute.

1. What is the probability to receive no call for a period of 30 seconds? What can we conclude on the probability that the time between two calls is more than 30 seconds? Justify.
2. What is the average number of calls in an hour?
3. What is the probability to receive more than 3 calls in an interval of 30 seconds?
4. For each call, the probability that it is a complaint is given by a Bernoulli distribution with parameter 0.6. What is the average number of complaints in an hour?

Reminder

Let X follow a Poisson distribution with parameter λ. Then:

$$P[X = x] = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2...$$

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Question 2

1. Draw the transition graph of the Markov chains (a) and (b) below.
2. Determine which states are recurrent and which states are transitory for both chains. States are numbered from 1 to 4.

   (a)
$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

   (b)
$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

3. For the chain (a), compute $f_{34}(n)$, the probability that the first visit of state 4 starting at state 3 needs n steps. Deduce that the probability of absorption in state 4 starting in state 3 is equal to $\frac{2}{3}$.
4. Explain what is the stationary distribution of a Markov chain and how you may compute it.

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Question 3

The price of share X is \$100 today ($t = 0$). We consider the binomial model for pricing a derivative product on X called "straddle". We expect a move per period of $+5\%$ in the up state and -5% in the down state. The risk-free rate is 0%. For an exercise price of E, the payoff of a straddle expiring at date t is given by

$$(X_t - E)_+ + (E - X_t)_+ \; where \, (\cdot)_+ = \max(\cdot, 0) \quad .$$

1. Explain what are risk-neutral probabilities.

2. Determine the value today of the straddle with exercise price $E = 105$ and maturity $t = 2$, using a two-period binomial tree. Explain.

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Question 4

Let $S(t)$ for any $t \geq 0$ be a set of i.i.d. standard Gaussian random variables.

1. What is the distribution of the increments of $B_t = \sqrt{t}S_t$?
2. Is B_t a martingale? Justify.
3. Is B_t a Brownian motion? Justify.

We consider the process $Y_t = W_t - \frac{1}{2}t$, where W_t is a standard Brownian motion.

4. Is $\exp(Y_t)$ a martingale? Show this point using conditional expectations (without Ito's lemma).

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Question 5

We consider the Vasicek model for the short rate, such that:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t$$

with $\kappa > 0$, $\theta \in \mathbb{R}$, $\sigma > 0$.

1. Compute $d\left(e^{\kappa t} r_t\right)$ using Ito's formula.
2. Compute the solution r_t of the stochastic differential equation by taking the integral of $d\left(e^{\kappa t} r_t\right)$.
3. What is the long-term average value of the process? Justify.