Stochastic Processes in Finance
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HW 6

Ito's Lemma

Exercise 1

Let W_t be a Wiener process and $t > s$. Compute:

a. $\mathbb{E}\left[W_t \mid W_s\right]$

b. $\mathbb{E}\left[W_s^2 W_t\right]$.

Exercise 2

Prove that $Z_t = \exp\left[-\theta W_t - \frac{1}{2}\theta^2 t\right]$ is a martingale, where W_t is a Wiener process.

Exercise 3

Let W_t be a Wiener process. Compute the stochastic differential equation (i.e., dZ_t) followed by:

a. $Z_t = (X_t)^2$, where $dX_t = \mu X_t dt + \sigma X_t dW_t$

b. $Z_t = 3 + t + e^{W_t}$

c. $Z_t = e^{\alpha t}$

d. $Z_t = \int_0^t g(s) dW_s$

e. $Z_t = e^{\alpha W_t}$

f. $Z_t = e^{\alpha X_t}$, where $dX_t = \mu dt + \sigma dW_t$.

Exercise 4

Let W_t be a Wiener process. Applying the Ito's lemma, compute dX_t^2 for:

a. $dX_t = dt + X_t dW_t$

b. $dX_t = dW_t$.

Exercise 5

Let W_t be a Wiener process. Show that:

a. $\mathbb{E}\left[\int_0^t dW_s\right] = 0$

b. $\mathrm{Var}\left[\int_0^t dW_s\right] = t$.

Exercise 6

Show that:

a. $\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t$

b. $\int_0^t s dW_s = tW_t - \int_0^t W_s ds$

c. $\int_0^t W_s^2 dW_s = \frac{1}{3}W_t^3 - \int_0^t W_s ds$,

where W_t is a Wiener process.

Exercise 7

Consider the following stochastic differential equation:

$$dX_t = -\beta X_t dt + \sigma dW_t,$$

where $\beta > 0$, $X_0 = x$, and W_t is a Wiener process.

a. What is the stochastic differential equation of $Y_t = e^{\beta t} X_t$?

b. Given a deterministic function $g(t)$, show that:
$\int_0^t g(s) dW_s$ is the $\mathcal{N}\left(0, \int_0^t g^2(s) ds\right)$ Gaussian random variable.

c. What is the distribution of X_t?

d. Find the limit distribution of X_t, i.e. $t \to \infty$.