Stochastic Processes in Finance
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HW 5

Martingales

Exercise 1

Let $\{N_t, t \geq 0\}$ be a Poisson process such that, on average, the number of events between 0 and t is equal to λt. Show that the process $M_t = N_t - \lambda t$ is a martingale. Identify the predictable part of N_t.

Exercise 2

Financial markets are influenced by good and bad news. Let N_t^G be the total number of arrivals of good news until time t and N_t^B the total number of arrivals of bad news until time t. N_t^G and N_t^B are independent Poisson processes with parameter λt. Show that the process $M_t{=}N_t^G - N_t^B$ is a martingale with respect to the filtrations

- $\mathcal{F}_t = \sigma(N_s^G, N_s^B,\, s \leq t)$,
- $\mathcal{G}_t = \sigma(M_s,\, s \leq t)$.

Exercise 3

Let $\{S_t, t \geq 0\}$ be a stochastic process with $S_0 = 0$ and independent and identically distributed (iid) increments: $S_t - S_s \sim \mathcal{N}(0, \sigma^2\,(t-s))$, $\forall\, t > s$. Are the following processes martingales?

a. $Z_t = S_t^2$

b. $Z_t = S_t^2 - \sigma^2 t$.

Exercise 4

Show that if Z_t is a martingale with respect to filtration $\mathcal{F}_t$, we have:

$$\forall t : \; E[Z_t] = k,$$

where k is constant.

Exercise 5

Let Z_1, Z_2, $Z_3, \ldots$ be iid random variables (rvs) such that

$$P\left[Z_n = 1\right] = P\left[Z_n = -1\right] = \frac{1}{2}, \quad n = 1, 2, \ldots.$$

Let

$$X_n = \sum_{i=1}^{n} Z_i, \quad n = 1, 2, \ldots,$$

with $X_0 = 0$. The process X_n is a *symmetric random walk.*

1. Show that the filtrations $\mathcal{F}_n = \sigma\left(X_0, X_1, \ldots, X_n\right)$ and $\mathcal{G}_n = \sigma\left(Z_0, Z_1, \ldots, Z_n\right)$ contain the same information.
2. Show that X_n is a martingale with respect to the filtration $\mathcal{F}_n$.
3. Show that $X_n^2 - n$ is a martingale with respect to the filtration $\mathcal{G}_n$.

Exercise 6

Let W_t be a standard Brownian motion. Are the following processes martingales?

1. W_t
2. $Z_t = W_t - \mu t$
3. $Z_t = W_t^2 - t$.

Exercise 7

Let X_t be a sequence of iid standard normal rvs. What is the distribution of the increments of $Z_t = \sqrt{t} X_t$? Is Z_t a Brownian motion?

Exercise 8

Show that $Z_t = -W_t$ is a Wiener process, if W_t is.

Exercise 9

Let W_t and $\widetilde{W}_t$ be two independent Brownian motions and ρ is a constant between -1 and 1. What is the distribution of $Z_t = \rho W_t + \sqrt{1 - \rho^2}\, \widetilde{W}_t$? Is Z_t a Brownian motion?