Stochastic Processes in Finance
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HW 3

Markov Chains: Applications

Exercice 1

Let $\{X_n, n \geq 1\}$ be a series of independent and identically distributed (iid) random variables (rvs) defined on $\mathbb{Z}$ (the set of integers).

Consider the following processes:

1. $S_n = \sum_{k=1}^{n} X_k$, $S_0 = 0$;
2. $Y_n = X_n + X_{n-1}$, $Y_0 = 0$, $X_0 = 0$;
3. $Z_n = \sum_{k=0}^{n} S_k$.

Are the processes S_n, Y_n, and Z_n Markov Chains?

Exercice 2

Discuss the *Random Walk Hypothesis*. Is such hypothesis valid in financial markets? If not, discuss empirical facts that lead you to this conclusion.

Exercice 3

Let $Z_1, Z_2, Z_3, \ldots$. be iid rvs, such that:

$P[Z_n = 1] = p$ and $P[Z_n = -1] = 1 - p, \quad n = 1, 2, \ldots$.

Let

$$X_n = a + \sum_{i=1}^{n} Z_i, \quad n = 1, 2, \ldots \quad ,$$

with $X_0 = a \in \mathbb{N}$.

1. Find the Expectation and the Variance of X_n.
2. Let $b > a$ and $P_b(a)$ denotes the probability that the process X_n reaches value b before it touches 0. Show that $P_b(a) = pP_b(a+1) + qP_b(a-1)$. Deduce value of $P_b(a)$.
3. Show that for any *discrete* rv $Y > 0$, we have:

$$\mathbb{E}[Y] = \sum_{k=0}^{\infty} P[Y > k] = \sum_{k=1}^{\infty} P[Y \geq k].$$

4. Let $T = \min\{n \mid X_n = b\}$. Show that the event $\{T \geq n\}$ is in $\sigma(Z_1, \ldots, Z_{n-1})$ (equivalently, the event is known at time $(n-1)$).

5. Suppose that $\mathbb{E}[T]$ is finite. We have $X_T - a = X_T - a = \sum_n Z_n 1_{\{T \geq n\}}$. Using 3 and 4, show that $b - a = \mathbb{E}Z_1 \cdot \mathbb{E}T$.

6. If $p \leq 1/2$, show that T does not have a finite expectation.

Exercice 4

The price of the stock is 30 CHF today. At maturity, the expected variation is +10% for the "high" state, and −8% for the "low" state. The risk-free rate is 3%.

1. Determine the risk-neutral probabilities of the states in the corresponding 1-period binomial model.

2. Using the 1-period binomial model, determine the price of an European call with 29 CHF strike.

3. Find the prices of the option from the previous question in the 2-period and 6-period models.

4. Consider the 2-period binomial model. Compute the price of a put option with 29 CHF strike.