Stochastic Processes in Finance
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HW 2

Markov Chains

Exercise 1

Show that if P is a stochastic matrix then P^h is also a stochastic matrix, $h \in \mathbb{N}^*$.

Exercise 2

Determine periodicity of the states for the following Markov chains.

1. $$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

2. $$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Exercise 3

Let a Markov chain be defined by the following transition matrix:

$$P = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}.$$

1. Show that state 1 is recurrent.
2. Show that state 2 is transitory.
3. Compute the mean time of recurrence of state 1.
4. Show that the states are aperiodic.

Exercise 4

Let a Markov chain be defined by the following transition matrix:

$$P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}.$$

1. Verify that:

$$
\begin{aligned}
f_{00}(n) &= p_{01}(p_{11})^{n-2} p_{10}, \quad n \geq 2, \\
f_{01}(n) &= (p_{00})^{n-1} p_{01}, \quad n \geq 1, \\
f_{10}(n) &= (p_{11})^{n-1} p_{10}, \quad n \geq 1, \\
f_{11}(n) &= p_{10}(p_{00})^{n-2} p_{01}, \quad n \geq 2.
\end{aligned}
$$

2. Compute f_{00}, f_{11}.

3. Show that the states of the following chains are recurrent.

$$
P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.25 & 0.75 \\ 0.25 & 0.75 \end{bmatrix}, \quad R = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}.
$$

Exercise 5

The wealth of a gambler is described by a random walk $\{S_t, t \geq 0\}$, $S_0 = 1$, with absorbing barriers in 0 and 3 (States are numbered from 0 to 3). The probability of winning one franc is denoted by p and the probability of losing one franc is q. The game ends when either $S_t = 0$ or $S_t = 3$.

1. Give the transition matrix.

2. Find the probabilities that the gambler wins or loses the game (i.e. that he is absorbed in 0 or 3).

3. Determine the average time for the gambler to go bankrupt.

Exercise 6

Let the stochastic matrix

$$
P = \begin{bmatrix}
\frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}
\end{bmatrix}.
$$

1. Draw the transition graph.

2. Determine which states are transitory, resp. recurrent.

3. Compute the mean time of recurrence of states 1, 4 and 6.

4. Show that by renumbering the states to group persistent states, resp. transitory states, the matrix P has the form $P = \begin{bmatrix} P_{1,1} & 0 \\ P_{2,1} & P_{2,2} \end{bmatrix}$.

5. Let $P_{1,1} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{bmatrix}$. Does the Markov chain associated to $P_{1,1}$ have a stationary distribution?

6. Compute the mean time of recurrence of persistent states of the Markov chain associated to $P_{1,1}$.