Stochastic Processes in Finance
Prof. Olivier Scaillet
TA Evgenii Anisimov

HW 1

Elementary Distributions

Exercise 1

A *discrete* random variable (rv) X follows a **Poisson distribution** $\mathcal{P}(\lambda)$ with the parameter $\lambda > 0$, if it takes values $k = 0, 1, \ldots$ with probability

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

1. Determine the expectation and the variance of X.
2. Let $\lambda = 2$. Compute the probability that $X \leq \mathrm{E}[X]$. What can you conclude about the distribution?

Exercise 2

A *continuous* rv X follows an **exponential distribution** $\mathcal{E}(\lambda)$ with the parameter $\lambda > 0$, if its probability density function is

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \lambda e^{-\lambda x}, & \text{otherwise.} \end{cases}$$

1. Determine the expectation and the variance of X.
2. Show that $\mathcal{E}(\lambda)$ verifies the *absence of memory* property:

$$\forall\, t, s > 0 : \; \mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t).$$

Exercise 3

A continuous rv X follows a **Gamma law** $\mathcal{G}(n, \beta)$ with the parameters $n \in \mathbb{N}$ and $\beta \in \mathbb{R}$, if its probability density function is

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \frac{\beta^n}{\Gamma(n)} x^{n-1} e^{-\beta x}, & \text{otherwise,} \end{cases}$$

where $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$. For $n \in \mathbb{N}$, $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(n) = (n-1)!$.

1. Show that the exponential distribution is a particular case of the Gamma law.
2. Compute the expectation of X.

A trader working on low liquidity assets estimates the time he would need to sell 1000 shares of a company i. He would like to sell all of his shares in two separate orders (400 and then 600). Let a rv X_1 (X_2) denotes the time needed the first (second) order to be filled. Assume the rvs X_1 and X_2 are independent and follow the $\mathcal{E}(\lambda)$ exponential law.

3.* Determine the law of a rv $Y = X_1 + X_2$ measuring the time the trader would need to sell his shares of title i.

4.* In general, show that the sum of two independent rvs $X \sim \mathcal{G}(a, \beta)$ and $Y \sim \mathcal{G}(b, \beta)$ follows the $\mathcal{G}(a + b, \beta)$ law.

Exercise 4

A continuous rv X follows a **uniform law** $\mathcal{U}[a, b]$ on the interval $[a, b]$ if its density function is

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise.} \end{cases}$$

1. Show that f is a probability density function.
2. Compute the expectation and the variance of X.
3. Let $a = 1$ and $b = 2$. Draw the density function and the distribution function of X.

Exercise 5

A continuous rv X follows a **normal distribution** $\mathcal{N}\left(\mu, \sigma^2\right)$ with the parameters μ and σ^2, if its density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

1. Determine the expectation and the variance of X.
2. Draw the density function of the *standard normal distribution* ($\mathcal{N}(0, 1)$).

The vector $X = (X_1, \ldots, X_k)'$ follows a multidimensional normal distribution $\mathcal{N}(m, \Omega)$ with the parameters $m \in \mathbb{R}^k$ and $\Omega \in M_{k \times k}$, if for $x \in \mathbb{R}^k$ its density function is given by

$$f(x) = \frac{1}{\left(\sqrt{2\pi}\right)^n} \frac{1}{\sqrt{\det(\Omega)}} \exp\left(-\frac{1}{2}(x-m)' \Omega^{-1} (x-m)\right).$$

We consider financial assets 1 and 2. A daily return on the asset i follows a normal distibution with m_i mean ($i = 1, 2$). The covariance matrix of the returns is $\Omega = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$. We build the portfolio $Y = a_1 X_1 + a_2 X_2$ and would like to find distribution of its returns.

3.* In the case of $\sigma_{12} = 0$, show that the return on the portfolio Y follows a normal distribution and determine its parameters.

4.* Given that the sum of two normal rvs follows a normal distribution, find the density function of the returns on the portfolio $Y = 0.3X_1 + 0.7X_2$, where $m = (3, 4)'$ and $\Omega = \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix}$.